Zero-Sum Flows in Regular Graphs

S. Akbari[1,5], A. Daemi[2], O. Hatami[1], A. Javanmard[3], A. Mehrabian[4]

[1]*Department of Mathematical Sciences*

Sharif University of Technology

Tehran, Iran

[2]*Department of Mathematics*

Harvard University

Cambridge, USA

[3]*Department of Electrical Engineering*

Stanford University

Stanford, USA

[4]*Department of Combinatorics and Optimization*

University of Waterloo

Waterloo, Canada

[5]*School of Mathematics*

Institute for Research in Fundamental Sciences (IPM)

Tehran, Iran

s_akbari@sharif.edu adaemi@math.harvard.edu omidhatami@math.sharif.edu

adelj@stanford.edu amehrabi@uwaterloo.ca *†‡

Abstract

For an undirected graph G, a *zero-sum flow* is an assignment of non-zero real numbers to the edges, such that the sum of the values of all edges incident with each vertex is zero. It has been conjectured that if a graph G has a zero-sum flow, then it has a zero-sum 6-flow. We prove this conjecture and Bouchet's Conjecture for bidirected graphs are equivalent. Among other results it is shown that if G is an r-regular graph ($r \geq 3$), then G has a zero-sum 7-flow. Furthermore, if r is divisible by 3, then G has a zero-sum 5-flow. We also show a graph of order n with a zero-sum flow has a zero-sum $(n+3)^2$-flow. Finally, the existence of k-flows for small graphs is investigated.

*Keywords: regular graph, bidirected graph, zero-sum flow.

†AMS (2010) *Subject classification*: 05C21, 05C50.

‡Corresponding author: S. Akbari.

1 Introduction

Throughout this paper, all terminologies and notations on graph theory can be referred to the textbook by D. B. West [9]. Let G be a directed graph. A *k-flow* on G is an assignment of integers with maximum absolute value $k-1$ to each edge such that for every vertex, the sum of the values of incoming edges is equal to the sum of the values of outgoing edges. A *nowhere-zero k-flow* is a k-flow with no zero.

A celebrated conjecture of Tutte says that:

Conjecture. (Tutte's 5-flow Conjecture [8]) *Every bridgeless graph has a nowhere-zero 5-flow.*

Jaeger showed that every bridgeless graph has a nowhere-zero 8-flow, see [6]. Next Seymour proved that every bridgeless graph has a nowhere-zero 6-flow, see [7]. For a thorough account on the above conjecture and subsequent results, see [4].

Let $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. The *incidence matrix* of G, denoted by $W(G)$, is an $n \times m$ matrix defined as

$$W(G)_{i,j} = \begin{cases} +1 & \text{if } e_j \text{ is an incoming edge to } v_i, \\ -1 & \text{if } e_j \text{ is an outgoing edge from } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

The flows of G are indeed the elements of the null space of $W(G)$. If $[c_1, \ldots, c_m]^T$ is an element of the null space of $W(G)$, then we can assign value c_i to e_i and consequently obtain a flow. Therefore, in the language of linear algebra, Tutte's 5-flow Conjecture says that if G is a directed bridgeless graph, then there exists a vector in the null space of $W(G)$, whose entries are non-zero integers with absolute value less than 5.

One may also study the elements of null space of the incidence matrix of an undirected graph. For an undirected graph G, the incidence matrix of G, $W(G)$, is defined as follows:

$$W(G)_{i,j} = \begin{cases} 1 & \text{if } e_j \text{ and } v_i \text{ are incident,} \\ 0 & \text{otherwise.} \end{cases}$$

An element of the null space of $W(G)$ is a function $f : E(G) \longrightarrow \mathbb{R}$ such that for all vertices $v \in V(G)$ we have

$$\sum_{u \in N(v)} f(uv) = 0,$$

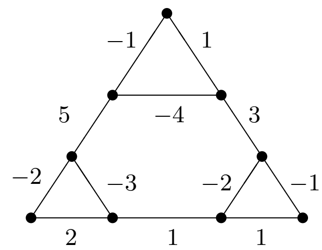


Figure 1: A graph with a zero-sum 6-flow.

where $N(v)$ denotes the set of adjacent vertices to vertex v. If f never takes the value zero, then it is called a *zero-sum flow* on G. A *zero-sum k-flow* is a zero-sum flow whose values are integers with absolute value less than k. Figure 1 shows a zero-sum 6-flow. There is a conjecture for zero-sum flows similar to the Tutte's 5-flow Conjecture for nowhere-zero flows:

Let G be an undirected graph with incidence matrix W. If there exists a vector in the null space of W whose entries are non-zero real numbers, then there also exists a vector in that space, whose entries are non-zero integers with absolute value less than 6, or equivalently,

Zero-Sum Conjecture [1]. *If G is a graph with a zero-sum flow, then G admits a zero-sum 6-flow.*

Using the graph given in Figure 1, one can see that 6 can not be replaced with 5 (this is proved in Section 5).

The following theorem determines all graphs having a zero-sum flow.

Theorem A [1]. *Let G be a connected graph. Then the following hold:*

(i) *If G is bipartite, then G has a zero-sum flow if and only if it is bridgeless.*

(ii) *If G is not bipartite, then G has a zero-sum flow if and only if removing any of its edges does not make any bipartite connected component.*

Let G be a bipartite graph with parts X and Y. Orient each edge from X to Y. It is easy to verify that a nowhere-zero flow for the oriented version of G gives a zero-sum flow for G and conversely. So Theorem A and Seymour's 6-flow theorem imply that:

Corollary 1.1. *If a bipartite graph has a zero-sum flow, then it has a zero-sum* 6*-flow.*

The goal of this paper is to provide a deeper study of the concept of zero-sum flow. In Section 2 we prove that Zero-Sum Conjecture and Bouchet's Conjecture for bidirected graphs are equivalent. In Section 3 we study zero-sum flows in regular graphs. We will show that every graph of order n has a zero-sum $(n+3)^2$-flow in Section 4. Finally, in Section 5, we investigate the existence of k-flows in small graphs, for some values of k.

2 Equivalence of Zero-Sum Conjecture and Bouchet's Conjecture

A *bidirected graph* G is a graph with vertex set $V(G)$ and edge set $E(G)$ such that each edge is oriented as one of the four possibilities: $\bullet\!-\!\blacktriangleleft\!-\!\blacktriangleright\!-\!\bullet$, $\bullet\!-\!\blacktriangleright\!-\!\blacktriangleright\!-\!\bullet$, $\bullet\!-\!\blacktriangleright\!-\!\blacktriangleleft\!-\!\bullet$, $\bullet\!-\!\blacktriangleleft\!-\!\blacktriangleleft\!-\!\bullet$. An edge with orientation $\bullet\!-\!\blacktriangleright\!-\!\blacktriangleleft\!-\!\bullet$ (resp., $\bullet\!-\!\blacktriangleleft\!-\!\blacktriangleright\!-\!\bullet$) is called an *in-edge* (resp., *out-edge*). An edge that is neither an in-edge nor an out-edge is called an *ordinary edge*. Let G be a bidirected graph. For any $v \in V(G)$, the set of all edges with tails (resp. heads) at v is denoted by $E^+(v)$ (resp. $E^-(v)$). Function $f : E(G) \longrightarrow \mathbb{R}$ is a *bidirected flow* of G if for every $v \in V(G)$, we have

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e).$$

If f takes values from the set $\{\pm 1, \ldots, \pm(k-1)\}$, then it is called a *nowhere-zero bidirected k-flow.*

Nowhere-zero bidirected flows generalize the concepts of nowhere-zero and zero-sum flows. For if we orient all edges of G to be ordinary, then a nowhere-zero bidirected flow in this setting corresponds to a usual nowhere-zero flow, and if we orient all edges of G to be in-edge, then a bidirected flow corresponds to a zero-sum flow.

In 1983, Bouchet proposed the following interesting conjecture.

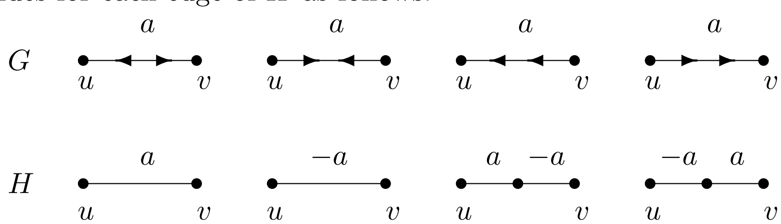
Bouchet's Conjecture [3, 10]. *Every bidirected graph which admits a nowhere-zero bidirected flow will admit a nowhere-zero bidirected 6-flow.*

The next theorem shows that for proving Bouchet's Conjecture it is enough to show that Zero-Sum Conjecture is true.

Theorem 2.1. *Bouchet's Conjecture and Zero-Sum Conjecture are equivalent.*

Proof. First suppose Bouchet's Conjecture is true. Let H be an undirected graph and f be a zero-sum flow on H. Construct a bidirected graph G from H such that all edges are in-edge. It can be seen that G admits a nowhere-zero flow by the same values of f. So by Bouchet's Conjecture there exists a nowhere-zero 6-flow, say f', for G. Therefore f' is a zero-sum 6-flow for H, and Zero-Sum Conjecture is true.

Next we will show that if Zero-Sum Conjecture holds, then Bouchet's Conjecture is true. Let G be a bidirected graph and g be a nowhere-zero bidirected flow for G. Remove all directions of edges of G and denote the resultant graph by H. Replace those edges of H that correspond to the ordinary edges of G with a path of length 2. Now, consider the following values for each edge of H as follows:



It can be easily seen that these values define a zero-sum k-flow for H if and only if g is a nowhere-zero k-flow for G. Now, since Zero-Sum Conjecture holds, we conclude that H has a zero-sum 6-flow. Hence G has a nowhere-zero bidirected 6-flow and Bouchet's Conjecture is true. $\square$

3 Zero-sum 7-flows for regular graphs

An *r-regular* graph is a graph all of whose degrees are r. It has been proved that every $2r$-regular graph, $r > 1$, has a zero-sum 3-flow, see [1].

In this section we want to study the existence of zero-sum flows for r-regular graphs.

Theorem 3.1. *Let G be an r-regular graph ($r \geq 3$). Then G has a zero-sum 7-flow. If $3 \,|\, r$, then G has a zero-sum 5-flow.*

Proof. First we define a new graph G'. Suppose that $V(G) = \{1, \ldots, n\}$ and let G' be a bipartite graph with two parts $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$. Join u_i to v_j if and only if the two vertices i and j are adjacent in G.

Assume that G' has a zero-sum k-flow, and a_{ij} is the value of edge u_iv_j. If $a_{ij} + a_{ji} \neq 0$ for all i,j, then we construct a zero-sum $(2k-1)$-flow for G, in the following way. For two adjacent vertices i, j in G, let e_{ij} be the edge between them. Assign $b_{ij} = a_{ij} + a_{ji}$ to e_{ij}. By our assumption, $b_{ij} \in \{\pm 1, \dots, \pm(2k-2)\}$. We have

$$\sum_{v_j \in N(u_i)} a_{ij} = 0 \ , \quad \sum_{u_j \in N(v_i)} a_{ji} = 0,$$

thus we find:

$$\sum_{j \in N(i)} b_{ij} = \sum_{v_j \in N(u_i)} a_{ij} + \sum_{u_j \in N(v_i)} a_{ji} = 0.$$

This means that the numbers b_{ij}, define a zero-sum $(2k-1)$-flow for G.

The graph G' is a bipartite r-regular graph. So by Hall's Marriage Theorem [5], all edges of G' can be partitioned into r perfect matchings. Let $E_1, \dots, E_r$ be the set of edges of these matchings. If $3 \,|\, r$, then let $s = r/3$ and define

$$f(e) = \begin{cases} 2 & e \in E_1 \cup E_2 \cup \ldots \cup E_s \\ -1 & e \in E_{s+1} \cup \ldots \cup E_{3s}. \end{cases}$$

Obviously, f is a zero-sum 3-flow for G' that satisfies $a_{ij} + a_{ji} \neq 0$. Hence G has a zero-sum 5-flow.

If $r \equiv 1 \pmod 3$, then we may assume that $r = 3s + 4$, and

$$f(e) = \begin{cases} 2 & e \in E_1 \cup E_2 \cup \ldots \cup E_s, \ s > 0 \\ -1 & e \in E_{s+1} \cup \ldots \cup E_{3s}, \ s > 0 \\ 3 & e \in E_{3s+1} \\ -1 & e \in E_{3s+2} \cup E_{3s+3} \cup E_{3s+4} \end{cases}$$

is a zero-sum 4-flow for G' that gives a zero-sum 7-flow for G by the above discussion.

If $r \equiv 2 \pmod 3$, then we may assume that $r = 3s + 5$, and

$$f(e) = \begin{cases} 2 & e \in E_1 \cup E_2 \cup \ldots \cup E_s, \ s > 0 \\ -1 & e \in E_{s+1} \cup \ldots \cup E_{3s}, \ s > 0 \\ -3 & e \in E_{3s+1} \cup E_{3s+2} \\ 2 & e \in E_{3s+3} \cup E_{3s+4} \cup E_{3s+5} \end{cases}$$

is a zero-sum 4-flow that gives a zero-sum 7-flow for G. □

Now, we propose the following conjecture.

Conjecture 1. *Every r-regular graph ($r \geq 3$) has a zero-sum 5-flow.*

Let v be an arbitrary vertex of a graph G with degree k. An *m-blowing* of vertex v ($1 \leq m \leq k$) is the replacement of v with the two vertices $\{v_1, v_2\}$, and joining v_1 to m arbitrary neighbors of v and v_2 to the remaining $k - m$ neighbors of v.

Theorem 3.2. *If every r-regular graph has a zero-sum k-flow, then every graph whose degrees are divisible by r, has a zero-sum k-flow.*

Proof. Let G be a graph, $v \in V(G)$ and $deg(v) = \ell r$ ($\ell > 1$). By an r-blowing of v we obtain two vertices, v_1 of degree r and v_2 of degree $(\ell - 1)r$. By repeating this procedure, v is replaced with ℓ vertices $\{u_1, \ldots, u_\ell\}$ of degree r. We denote the resultant graph by G'. Obviously, there is a correspondence between incident edges to $\{u_1, \ldots, u_\ell\}$ and incident edges to v. So any zero-sum k-flow for G' provides a zero-sum k-flow for G. For any vertex in G' of degree greater than r, we do the above process. Then we obtain an r-regular graph and by assumption, this graph has a zero-sum k-flow. Thus G has a zero-sum k-flow. $\square$

Now, as a consequence of the previous theorem, we have the following corollaries.

Corollary 3.1. *Let G be a graph with $gcd(deg(v_1), deg(v_2), \ldots, deg(v_n)) > 2$. Then G has a zero-sum 7-flow.*

Corollary 3.2. *Let G be a graph all of whose degrees are divisible by 3. Then G has a zero-sum 5-flow.*

4 Existence of zero-sum $(n+3)^2$-flows in graphs of order n

Let G be a graph with n vertices that has a zero-sum flow. The purpose of this section is to prove that if n is even then G has a zero-sum $(n+3)^2$-flow, and if n is odd then G has a zero-sum $(n+2)^2$-flow.

Lemma 4.1. *Let G be a $(2k+1)$-regular graph ($k \geq 1$). Then G has a zero-sum $(2k+3)$-flow f such that for each edge e, $f(e) \in \{-2k, 1, 2k+2\}$.*

Proof. Construct G' as in the proof of Theorem 3.1. By Hall's Marriage Theorem the edges of G' can be partitioned into $2k+1$ perfect matchings. Select k of them, assign $k+1$ to their edges, and assign $-k$ to the rest of edges. Now, as we did in the proof of

Theorem 3.1, for each edge e, $f(e) \in \{(k+1)-k, (k+1)+(k+1), -k-k\} = \{-2k, 1, 2k+2\}$.
$\square$

Corollary 4.1. *Let G be a graph and k be a positive integer. If $2k+1 \,|\, deg(v)$ for each $v \in V(G)$, then G has a zero-sum $(2k+3)$-flow f, such that for each edge e, $f(e) \in \{-2k, 1, 2k+2\}$.*

Proof. The proof uses the same idea as the proof of Theorem 3.2. $\square$

For an undirected graph G and an abelian group $(\Gamma, +)$, a Γ-*flow* on G is a function $f : E(G) \longrightarrow \Gamma$ such that for every $v \in V(G)$, the sum of values of edges incident to v is zero in Γ. A *zero-sum* Γ-*flow* is an Γ-flow that is nonzero on every edge. A zero-sum $\mathbb{R}$-flow is simply a zero-sum flow.

Theorem 4.1. *If G has a zero-sum $\mathbb{Z}_{2k+1}$-flow, then G has a zero-sum $(2k+1)^2$-flow.*

Proof. Let f be a zero-sum $\mathbb{Z}_{2k+1}$-flow for G. Consider a function $g : E(G) \longrightarrow \{1, \ldots, 2k\}$ in such a way that for each edge e, $g(e) = f(e) \pmod{2k+1}$. We construct a new graph G_1 from G. Let $V(G_1) = V(G)$ and for each $e = v_iv_j \in E(G)$, join v_i to v_j by $g(e)$ edges in G_1 and denote the set of these edges by $Q(e)$. All of the degrees of vertices of G_1 are divisible by $2k+1$, so by Corollary 4.1, G_1 has a zero-sum $(2k+3)$-flow, say $f_1 : E(G_1) \longrightarrow \{-2k, 1, 2k+2\}$. Now, for each edge e, let

$$h(e) = \sum_{e_1 \in Q(e)} f_1(e_1).$$

Obviously, $1 \leq |g(e)| \leq 2k$ and $|f_1(e_i)| \leq 2k+2$. Therefore

$$|h(e)| \leq \; 2k(2k+2) = (2k+1)^2 - 1.$$

Also, for every edge e_1 of G_1, $f_1(e_1) = 1 \pmod{2k+1}$, so for every edge e of G,

$$h(e) = g(e) \neq 0 \pmod{2k+1}.$$

Hence h is a zero-sum $(2k+1)^2$-flow for G. $\square$

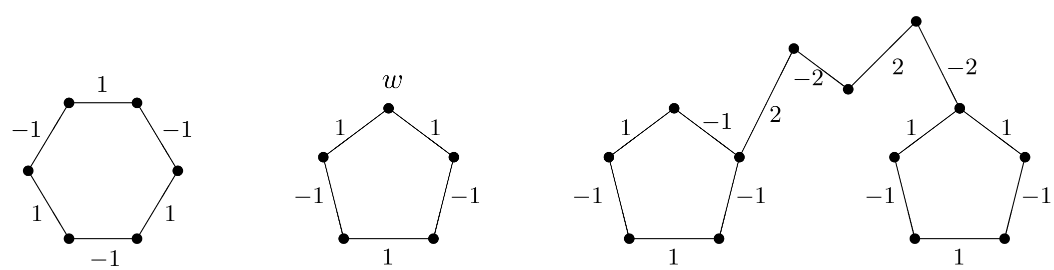
Remark 1. There are some graphs that have zero-sum $\mathbb{Z}_{2k}$-flow but no zero-sum flow. Consider the odd cycle C_{2n+1}. By assigning k to each edge of C_{2n+1}, we obtain a zero-sum $\mathbb{Z}_{2k}$-flow, but it has no zero-sum flow.

By an *even vertex* in a graph, we mean a vertex with even degree. A graph is called *Eulerian* if all of its vertices are even. An *even circuit* in a graph G is a sequence $C = (w_0, w_1, \ldots, w_{2k})$ of not necessarily distinct vertices of G, where $w_0 = w_{2k}$ and $w_i w_{i+1}$ are distinct edges of G, for $i = 0, 1, \ldots, 2k-1$. We denote by $E(C)$ the set of $2k$ edges of C. An odd circuit is defined similarly.

Lemma 4.2. *The following hold:*

- *a) Every even circuit has a zero-sum 2-flow.*
- *b) Let C be an odd circuit and w be a vertex of C. Then there exists a function $f : E(C) \longrightarrow \{\pm 1\}$, such that $\sum_{u \in N(v)} f(uv) = 0$, for each $v \in V(C) \setminus \{w\}$ and $\sum_{u \in N(w)} f(uw) = 2$.*
- *c) Let C_1 and C_2 be odd circuits, $v_1 \in C_1, v_2 \in C_2$ and P be a path disjoint from $E(C_1)$ and $E(C_2)$ that connects v_1 to v_2. Then there exists a zero-sum 3-flow with values ± 1 on $E(C_1) \cup E(C_2)$, and ± 2 on $E(P)$.*

Proof. The proofs are simple, and are illustrated in the following figures. Note that for Part (c), there are two slightly different cases depending on the parity of the length of P, and only one case is shown in the figure. $\square$



(a) (b) (c)

By a *dumbbell* in a graph, we mean two edge-disjoint odd cycles, C_1 and C_2 which are connected to each other by a path P, which is edge-disjoint from C_1 and C_2.

The following lemma was proved in [3].

Lemma 4.3. *If G is a graph with a zero-sum flow, then every edge of G is contained in an even cycle or a dumbbell.*

A graph G is called $\mathbb{Z}_k$-*strong* if for any function $\phi : E(G) \longrightarrow \mathbb{Z}_k$, there exists a function $f : E(G) \longrightarrow \mathbb{Z}_k$ such that $f(e) \neq \phi(e)$ for every edge e, and the sum of values of edges incident with each vertex is zero. A graph with no edge is obviously $\mathbb{Z}_k$-strong for every k. One can easily see that by setting $\phi \equiv 0$, a $\mathbb{Z}_k$-strong graph has a zero-sum $\mathbb{Z}_k$-flow.

Let k be a positive integer and G be a graph. A k-*push operation* on G is inserting some new edges to G, such that first, the number of inserted edges is less than k, and second, there exists an even circuit or a dumbbell in the new graph, containing the inserted edges.

Lemma 4.4. *Let G' be a $\mathbb{Z}_{2k+1}$-strong graph. Suppose that one performs a $(2k+1)$-push operation on G' and obtains a new graph G. Then G is $\mathbb{Z}_{2k+1}$-strong.*

Proof. Let I be the set of edges inserted to G', C be an even circuit or dumbbell that contains these edges, and $\phi : E(G) \longrightarrow \mathbb{Z}_{2k+1}$ be an arbitrary function.

By Lemma 4.2, there exists a zero-sum 3-flow on C, say h. Define $h_i = ih \pmod{2k+1}$ for $1 \leq i \leq 2k+1$. It can be easily seen that each h_i is a $\mathbb{Z}_{2k+1}$-flow on C. For an inserted edge e, we have $h(e) \in \{-2, -1, 1, 2\}$. Thus $h(e)$ and $2k+1$ are coprime. So there exists exactly one i for which $h_i(e) = ih(e) = \phi(e) \pmod{2k+1}$. Since the number of inserted edges is less than $2k+1$, there exists j, $1 \leq j \leq 2k+1$ for which we have $h_j(e) \neq \phi(e) \pmod{2k+1}$ for every $e \in I$.

We define a new function $\phi' : E(G') \longrightarrow \mathbb{Z}_{2k+1}$, which is equal to $\phi(e) - h_j(e)$ when $e \in C \cap E(G')$ and $\phi(e)$ elsewhere. Since G' is $\mathbb{Z}_{2k+1}$-strong, there exists a zero-sum $\mathbb{Z}_{2k+1}$-flow g' which is different from ϕ' on every edge of G'. Now, we define the following zero-sum $\mathbb{Z}_{2k+1}$-flow on G:

$$g(e) = \begin{cases} h_j(e) & e \in I \\ g'(e) + h_j(e) & e \in C \backslash I \\ g'(e) & e \notin C \end{cases}$$

It is easy to check that $g(e) \neq \phi(e)$, for every $e \in E(G)$, and the proof is complete. $\square$

Lemma 4.5. *Let G be a graph of order n and H be a subgraph of G. If G has a zero-sum flow, then G can be obtained from H via a sequence of $(n+2)$-push operations.*

Proof. We use induction on $|E(G)| - |E(H)|$. The base case is $H = G$, which is trivial. Assume that $|E(H)| < |E(G)|$, and take any $e \in E(G) \backslash E(H)$. By Lemma 4.3, there exists an even cycle or a dumbbell C containing e. Clearly C has at most $n+1$ edges. Hence adding the edges of $E(C)\backslash E(H)$ to H is a $(n+2)$-push operation. This operation results in graph H' which is again a subgraph of G but has more edges than H. By induction hypothesis, G can be obtained from H' via a sequence of $(n+2)$-push operations. Thus G can be also obtained from H via a sequence of $(n+2)$-push operations. $\square$

Theorem 4.2. *Let G be a graph of order n with a zero-sum flow. If n is odd, then G has a zero-sum $(n+2)^2$-flow. If n is even, then G has a zero-sum $(n+3)^2$-flow.*

Proof. First assume that n is odd. Let H be the graph with vertex set $V(G)$ and no edge. Then H is $\mathbb{Z}_{n+2}$-strong. By Lemma 4.5, G can be obtained from H via a sequence of $(n+2)$-push operations. So, by Lemma 4.4, G is $\mathbb{Z}_{n+2}$-strong as well. Hence by Theorem 4.1, G has a zero-sum $(n+2)^2$-flow.

If n is even, add a new isolated vertex to G. By the first part, the new graph has a zero-sum $(n+3)^2$-flow. So G has a zero-sum $(n+3)^2$-flow, too. $\square$

5 Zero-sum k-flows in small graphs

In this section, we study the graphs with minimum number of vertices having a zero-sum flow, but no zero-sum k-flow, for a pre-assumed k. Many of these results are found by a computer program search. We confine ourselves to those graphs having a zero-sum flow.

All graphs with less than 5 vertices have zero-sum 3-flow. There are exactly two graphs of order 5 with no zero-sum 3-flow. They have size 8 and are given in Figure 2. There are exactly 15 graphs of order 6 with no zero-sum 3-flow.

Figure 2: The smallest graphs with no zero-sum 3-flow.

All graphs with less than 6 vertices have zero-sum 4-flow. There is exactly one graph of order 6 and exactly one graph of order 7 with no zero-sum 4-flow. They are shown in the following.

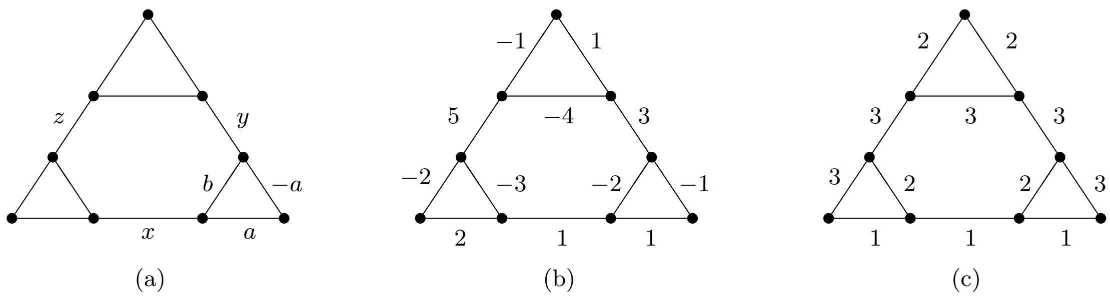


Figure 3: (a) The smallest graph with no zero-sum 5-flow. (b) A zero-sum 6-flow for (a). (c) A zero-sum $\mathbb{Z}_4$-flow for (a).

Remark 2. It first seems that all graphs have zero-sum 5-flow. In fact this is true for graphs up to 8 vertices. The smallest counterexample is the graph T, which is shown in Figure 3(a). Indeed if there exists a zero-sum 5-flow for T, we have $x + a + b = 0$ and $y + b - a = 0$. This implies that $x + y = -2b$ and $x - y = -2a$ and so $|x| \neq |y|$ and $x \equiv y \pmod 2$. Similarly, we find that $x \equiv y \equiv z \pmod 2$ and $|x|$, $|y|$ and $|z|$ should be distinct. There are no x, y, z in $\{\pm 1, \pm 2, \pm 3, \pm 4\}$, with desired property. So T has no zero-sum 5-flow. Using a similar argument it can be proved that it has no zero-sum $\mathbb{Z}_5$-flow. But it has zero-sum 6-flow, as shown in Figure 3(b). It has been proved that a graph has a nowhere-zero $\mathbb{Z}_k$-flow if and only if it has a nowhere-zero k-flow, see [8]. This example also shows that this is fact is not true for zero-sum flows, as T has a zero-sum $\mathbb{Z}_4$-flow, (Figure 3(c)) but no zero-sum 4-flow.

Remark 3. There exist an infinite number of graphs G, with $\Theta(|V(G)|^2)$ edges, having no zero-sum 5-flow. Let H be a bipartite, 2-edge-connected graph with n vertices and $\Theta(n^2)$ edges. Let G be the graph shown in Figure 4(a) in which the vertices shown in H, are from different parts. Since H is 2-edge-connected, by Theorem A, G has a zero-sum flow. Since H is bipartite, $a = b$. If $0 < |x| < 5$, then $0 < |y + a| < 5$, since $y + a = -x$. Using a zero-sum 5-flow for G, we can define a zero-sum 5-flow for the graph shown in Figure 4(b), a contradiction. Hence G has no zero-sum 5-flow.

There is exactly one graph of order 9 and exactly one of order 10 with no zero-sum 5-flow. They are shown in Figure 5.

There are exactly seven graphs of order 11 with no zero-sum 5-flow. They are shown in Figure 6.

All bipartite graphs with less than 13 vertices have zero-sum 4-flow. There is just one graph of order 13 and one graph of order 14 with no zero-sum 4-flow. They both have zero-sum 5-flow, and are shown in Figure 7.

We have summarized our results in the following tables.

n	Total number of non-isomorphic connected graphs of order n	with no zero-sum 4-flow	size
6	112	1	10
7	853	1	10

n	Total number of non-isomorphic connected graphs of order n	with no zero-sum 5-flow	size
9	261080	1	12
10	11716571	1	13
11	1006700565	7	14 or 15

n	Total number of non-isomorphic connected bipartite graphs of order n	with no zero-sum 4-flow	size
13	2241730	1	18
14	31193324	1	19
15	575252112	5	20 or 21

In Figure 8, a graph with no zero-sum $\mathbb{Z}_3$-flow has been given. By Theorem A, the graph has a zero-sum flow. To prove that it has no zero-sum $\mathbb{Z}_3$-flow, we note that the sum of any three non-zero elements of $\mathbb{Z}_3$ is zero if and only if they are the same. So, if there is a zero-sum $\mathbb{Z}_3$-flow, then all of the edges must have the same value. Thus the sum of values of two edges incident with the vertex of degree 2 can not be zero.

Now, we close the paper with the following remark.

Remark 4. Let $E(G) = \{e_1, \ldots, e_m\}$. Define $f(e_i) = x_i$, where x_i is an indeterminate

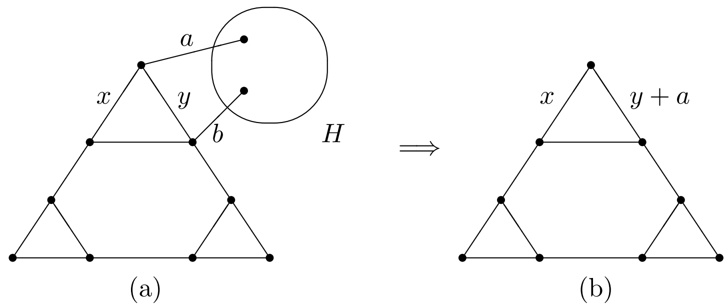


Figure 4: An infinite family of graphs with no zero-sum 5-flow.


Figure 5: Graphs of orders 9 and 10 with no zero-sum 5-flow.

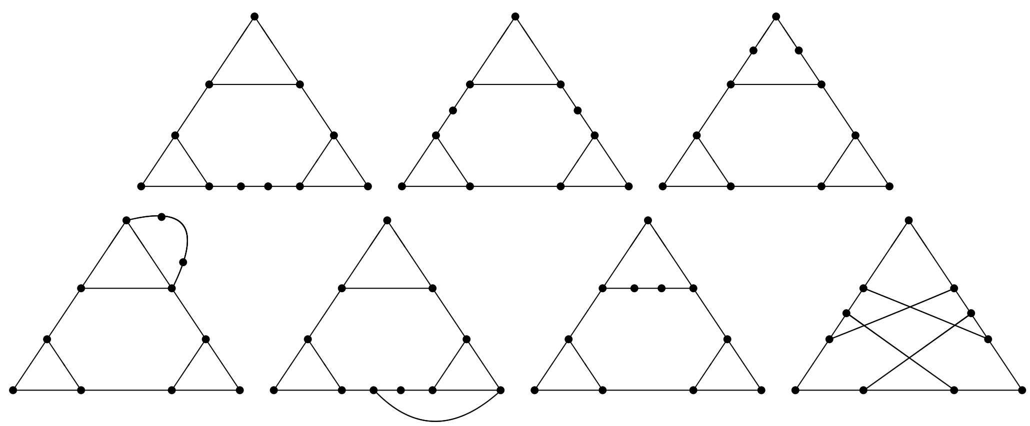


Figure 6: Graphs of order 11 with no zero-sum 5-flow.

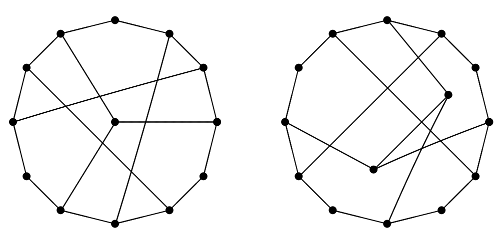


Figure 7: Bipartite graphs of orders 13 and 14 with no zero-sum 4-flow.

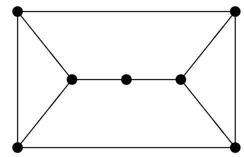


Figure 8: A graph with no zero-sum $\mathbb{Z}_3$-flow.

variable. Define polynomial $g(x_1, \ldots, x_m)$ as follows:

$$g(x_1, \ldots, x_m) = \prod_{v \in V(G)} \left(\left(\sum_{e_i \text{ and } v \text{ are incident}} x_i \right)^{p-1} - 1 \right).$$

Let $S = \{1, \ldots, p-1\}$. For every $x \in \mathbb{Z}_p \backslash \{0\}$, we have $x^{p-1} = 1$. It is easy to see that G has a zero-sum $\mathbb{Z}_p$-flow if and only if there exists an element $a = (a_1, \ldots, a_m) \in S^m$ such that $g(a) \neq 0$. Thus if we reduce g module of the ideal $(x_1^{p-1} - 1, \ldots, x_m^{p-1} - 1)$ and call it $\overline{g}$, then $g = \overline{g}$ over S^m. But $deg_{x_i}(\overline{g}) \leq p-2$, for $i = 1, \ldots, m$. Thus by Combinatorial Nullstellensatz Theorem [2], G has a zero-sum $\mathbb{Z}_p$-flow if and only if $\overline{g}$ is not the zero polynomial.

Acknowledgements. The authors wish to express their deep gratitude to Professor Bojan Mohar for introducing the concept of nowhere-zero flows in bidirected graphs. The first author is indebted to the School of Mathematics, Institute for Research in Fundamental Sciences (IPM) for support. The research of the first author was in part supported by a grant (No. 88050212) from IPM.

References

[1] S. Akbari, N. Ghareghani, G.B. Khosrovshahi and A. Mahmoody, On zero-sum 6-flows of graphs, Linear Algebra Appl. 430 (2009), 3047-3052.

[2] N. Alon, Combinatorial Nullstellensatz, Recent trends in combinatorics, Combin. Probab. Comput. 8 (1999), No. 1-2, 7-29.

[3] A. Bouchet, Nowhere-zero integral flows on a bidirected graph, J. Combin. Theory, Ser. B 34 (1983), 279-292.

[4] R.L. Graham, M. Grötschel and L. Lovász, Handbook of Combinatorics, The MIT Press-North-Holland, Amsterdam, 1995.

[5] P. Hall, On representatives of subsets, J. London Math. Soc. 10 (1935), 26-30.

[6] F. Jaeger, Flows and generalized coloring theorems in graphs. J. Combin. Theory Ser. B 26 (1979), No. 2, 205-216.

[7] P.D. Seymour, Nowhere-zero 6-flows, J. Combin. Theory, Ser. B 30 (1981), No. 2, 130-135.

[8] W.T. Tutte, A contribution to the theory of chromatic polynomials, Canadian J. Math. 6, (1954). 80-91.

[9] D.B. West, Introduction to Graph Theory, Second Edition, Prentice Hall, 2001.

[10] R. Xu and C. Zhang, On flows in bidirected graphs, Disc. Math. (2005), No. 299, 335-343.