Localization from Incomplete Noisy Distance Measurements

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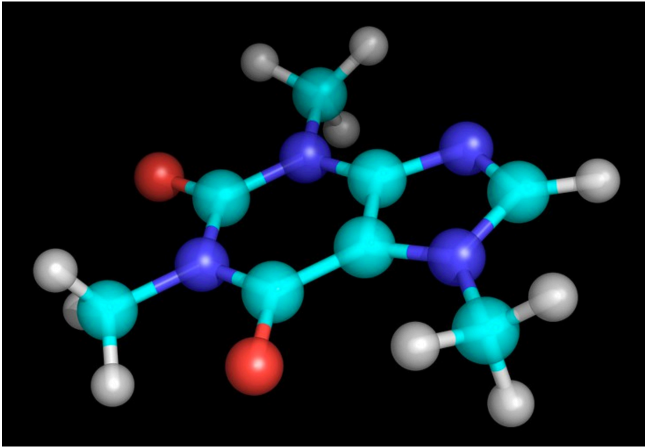
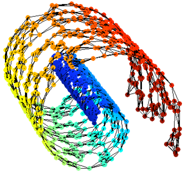
A chemistry question

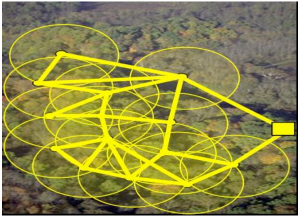


Which physical conformations are produced by given chemical bonds?

Other Motivations



(a) Manifold Learning



(b) Sensor Net. Localization

(c) Indoor Positioning

General 'geometric inference' problem

Given partial/noisy information about a cloud of points.

Reconstruct the points positions.

Notes

- Positions can be reconstructed up to rigid motions
- Well-posed problem only if G is connected
- In general, the problem (even uniqueness of reconstruction) is NP-hard [Saxe 1979]

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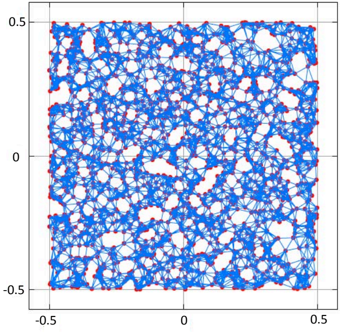
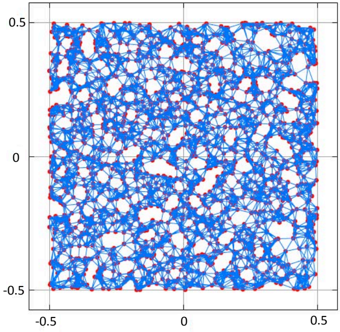


R.G.G. $G(n, r)$

$$x_1, \cdots, x_n \in [-0.5, 0.5]^d$$

$$r \geq \alpha(\log n/n)^{1/d}$$

This talk



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$r \geq \alpha(\log n/n)^{1/d}$

adversarial noise

$|\tilde{d}_{ij}^2 - d_{ij}^2| \leq \Delta$

Related work

- Triangulation
- Multidimensional scaling
- Divide and conquer (Singer 2008)

Few performance guarantees, especially in presence of noise

Outline

SDP relaxation and robust reconstruction

Optimization formulation

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} \|x_i\|_2^2 \\ \text{subject to} \quad & \left| \|x_i - x_j\|_2^2 - \widetilde{d}_{ij}^2 \right| \leq \Delta \end{aligned}$$

Nonconvex

Optimization formulation

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} Q_{ii} \\
\text{subject to} \quad & \left| Q_{ii} - 2\,Q_{ij} + Q_{jj} - \tilde{d}_{ij}^2 \right| \leq \Delta \\
& Q_{ij} = \langle x_i, x_j \rangle
\end{aligned}
$$

Nonconvex

Optimization formulation (better notation)

$$\begin{aligned} \text{minimize} \quad & \mathrm{Tr}(Q) \\ \text{subj.to} \quad & \left| \langle M_{ij}, Q \rangle - \widetilde{d}_{ij}^2 \right| \le \Delta \\ & Q_{ij} = \langle x_i, x_j \rangle \end{aligned}$$

$$\begin{aligned} M_{ij} &= e_{ij} \, e_{ij}^T \,, \\ e_{ij} &= (0, \dots, 0, \underbrace{+1}_{i}, 0, \dots, 0, \underbrace{-1}_{j}, 0, \dots, 0) \end{aligned}$$

Semidefinite programing relaxation

$$\begin{array}{ll} \text{minimize} & \text{Tr}(Q) \\ \text{subj.to} & \left| \langle M_{ij}, Q \rangle - \tilde{d}_{ij}^2 \right| \le \Delta \\ & \cancel{Q_{ij} = \langle x_i, x_j \rangle} \quad Q \succeq 0 \end{array}$$

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Semidefinite programing relaxation

SDP-BASED LOCALIZATION

Input : Distance measurements $\widetilde{d}_{ij}$, $(i, j) \in G$
Output : Low-dimensional coordinates $x_1, \ldots, x_n \in \mathbb{R}^d$

1: Solve the following SDP problem:

$$\begin{aligned} \text{minimize} \quad & \mathrm{Tr}(Q), \\ \text{s.t.} \quad & \big|\langle M_{ij}, Q\rangle - \widetilde{d}_{ij}^2\big| \le \Delta, \quad (i,j) \in G, \\ & Q \succeq 0. \end{aligned}$$

2: Eigendecomposition $Q = U \Sigma U^T$;

3: Top d e-vectors $X = U_d \Sigma_d^{1/2}$;

Robustness?

Theorem (Javanmard, Montanari '11)

Assume $r \geq 10\sqrt{d}(\log n/n)^{1/d}$. *Then, w.h.p.,*

$$d(X, \hat{X}) \leq C_1 (nr^d)^5 \frac{\Delta}{r^4},$$

Further, there exists a set of 'adversarial' measurements such that

$$d(X, \hat{X}) \geq C_2 \frac{\Delta}{r^4}.$$

$$d(X, \hat{X}) \approx \frac{1}{n} \sum_{i=1}^{n} \|x_i - \widehat{x}_i\|$$

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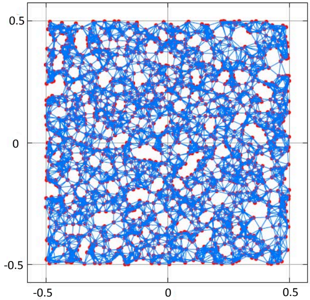
Lower bound

Proof: Lower bound

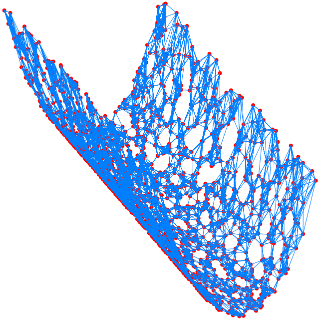
Proof: Lower bound (first attempt)



Scale the coordinates by $a = \sqrt{\frac{\Delta}{r^2} + 1}$

$$d(X, \hat{X}) \geq \frac{\Delta}{r^2}$$

Proof: Lower bound

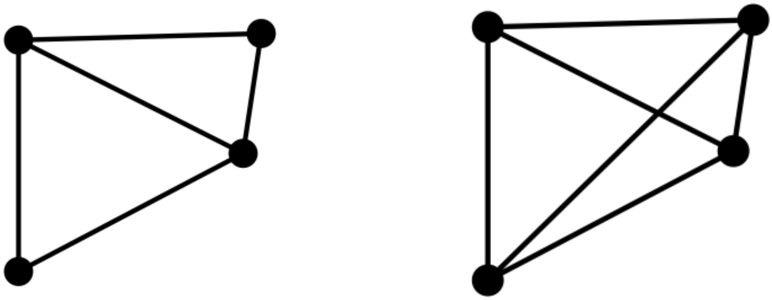


$$\mathcal{T} : [-0.5, 0.5]^d \to \mathbb{R}^{d+1}$$

$$\mathcal{T}(t_1, t_2, \cdots, t_d) = (R \sin \frac{t_1}{R}, t_2, \cdots, t_d, R(1 - \cos \frac{t_1}{R})), \quad R = \frac{r^2}{\sqrt{\Delta}}$$

Rigidity theory and upper bound

Uniqueness $\Leftrightarrow$ Global rigidity



Global rigidity

Assume noiseless measurements.
Is the reconstruction unique? (up to rigid motions)

Depends both on G and on $(x_1, \dots, x_n)$

Global rigidity: Characterization

Theorem (Connelly 1995; Gortler, Healy, Thurston, 2007)

$(G, \{x_i\})$ *is globally rigid in* $\mathbb{R}^d$

$\Leftrightarrow$

$(G, \{x_i\})$ *admits a stress matrix* Ω, *with* $\mathrm{rank}(\Omega) = n - d - 1$.

Stress matrix

Definition

$\Omega \in \mathbb{R}^{n\times n}$ is a *stress matrix* if $\text{supp}(\Omega) \subseteq E$ and

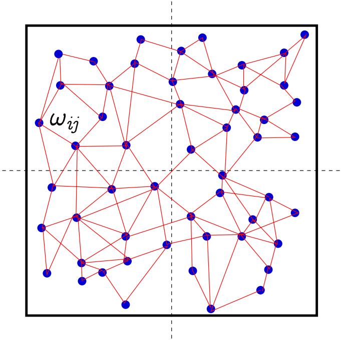
$$\Omega u = \Omega x^{(1)} = \dots \Omega x^{(d)} = 0\,.$$

$u = (1, \dots, 1) \in \mathbb{R}^n$
$x^{(\ell)} \in \mathbb{R}^n$ vector of positions' ℓ-th coordinate

$$\text{rank}(\Omega) \le n - d - 1$$

Stress matrix: Some intuition

... imagine putting springs on the edges ...



Equilibrium $x_1, \dots, x_n$:

$$[\text{force on } i] = \sum_{j \in \partial i} \omega_{ij}(x_j - x_i) = 0$$

$\Omega_{ij} = \omega_{ij}$, $\Omega_{ii} = -\sum_{j \in \partial i} \omega_{ij}$

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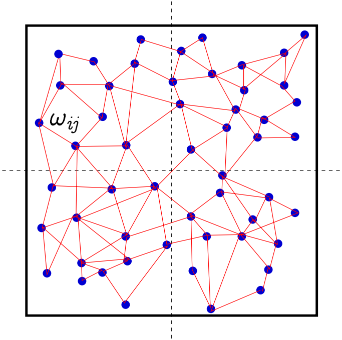

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Infinitesimal rigidity

Consider a continuos motion preserving distances instantaneously

$$(x_i - x_j)^T(\dot{x}_i - \dot{x}_j) = 0, \forall (i,j) \in E$$

Trivial motions

$$\dot{x}_i = Ax_i + b, \quad A = -A^T \in \mathbb{R}^{d \times d}$$

rotation translation

Definition

$(G, \{x_i\})$ is *infinitesimally rigid* if rotations and translations are the only infinitesimal motions.

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$$(x_i - x_j)^T(\dot{x}_i - \dot{x}_j) = 0, \forall (i,j) \in E$$

$$R_{G,X} \cdot \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = 0$$

Definition

$R_{G,X} \in \mathbb{R}^{|E| \times nd}$ is the *rigidity matrix* of framework $(G, \{x_i\})$.

$$dim(\text{null}(R_{G,X})) \geq \underbrace{\frac{d(d-1)}{2}}_{A} + \underbrace{d}_{b} = \binom{d+1}{2}$$

$(G, \{x_i\})$ is infinitesimally rigid if $\text{rank}(R_{G,X}) = nd - \binom{d+1}{2}$.

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(Idea of the) proof of the upper bound

- Noise is analogous to stretching/compressing the edges
- Need to measure the stability of the framework
- Global/infinitesimal rigidity is checked by rank of Stress/Rigidity matrix.
- Needed: Quantitative rigidity theory
  (Rank $\Rightarrow$ bounds on singular values)

An experiment

(d) Graph I

(e) Graph II

Quantitative rigidity theory

$$U(X) = \sum_{(i,j)\in E} \frac{1}{2}(\|x_i - x_j\|^2 - d_{ij}^2)^2 - \sum_i f_i^T x_i$$

$$\dot{X} = -\nabla_X U$$

$x_i + \delta x_i$ equilibrium positions in the presence of force

$$(\Omega \otimes I_d + R_{G,X} R_{G,X}^T) \cdot \delta x \approx f$$

$$\delta x = \mathcal{P}^{\perp}_{\langle X,u\rangle}(\delta x) + \mathcal{P}_{\langle X,u\rangle}(\delta x)$$

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(A piece of the) proof of the upper bound

Solution of SDP $\rightarrow Q$
Gram matrix $\rightarrow Q_0$ ($Q_0 = XX^T$, $X = [x_1|x_2|\cdots|x_n]^T$)

$$\begin{aligned} Q &= Q_0 + R \\ &= Q_0 + XY^T + YX^T + R^\perp \end{aligned}$$

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Steps of the proof

Lemma

For a stress matrix $\Omega \succeq 0$,

$$\|R^{\perp}\|_* \leq \frac{\lambda_{\max}(\Omega)}{\lambda_{\min}(\Omega|_{\langle u,x\rangle^{\perp}})}|E|\Delta.$$

Lemma

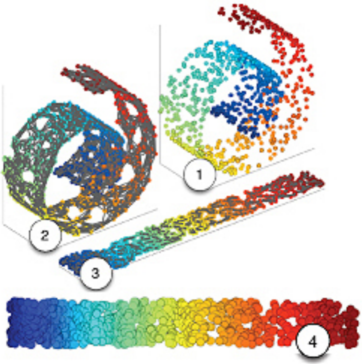
$$\lambda_{\min}(\Omega|_{\langle u,x\rangle^{\perp}}) \geq C_1(nr^d)^{-2}r^4, \quad \lambda_{\max}(\Omega) \leq C_2(nr^d)^2.$$

Lemma

$$\|XY^T + YX^T\|_1 \leq C(nr^d)^5\frac{n^2}{r^4}.$$

Discussion

Manifold learning



Discussion: Manifold learning

Data: $x_1, \dots x_n \in \mathbb{R}^m$

Assumption: $\{x_i\}_{i \in [n]}$ are close to d-dimensional submanifold of $\mathbb{R}^m$

Objective: d-dimensional embedding $z_1, \dots, z_n \in \mathbb{R}^d$ that preserves local geometry.

Where does the assumption come from?

- You need some assumption!
- PCA assumes linear approximately linear manifold:

$$\Big[x_1 \Big| \cdots \Big| x_n\Big] \approx UV^*, \qquad U \in \mathbb{R}^{m\times d}, V \in \mathbb{R}^{n\times d},$$

$$x_i \approx \sum_{a=1}^{d} v_{a,i}\, u_a$$

- Vision.

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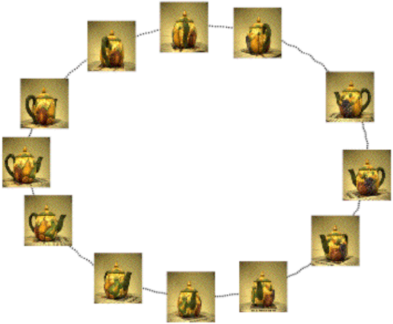
Vision 1



Vision 2

(a) (b)

Human brain effectively projects on low dimension (?).

Vision 2

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Manifold learning

MANIFOLD LEARNING

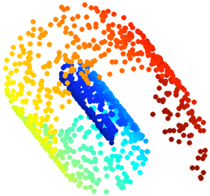
Input : Data points $x_1, \ldots, x_n \in \mathbb{R}^m$
Output : Low-dimensional coordinates $z_1, \ldots, z_n \in \mathbb{R}^d$, $d \ll m$

1: Proximity graph G on $\{1, \ldots, n\}$;

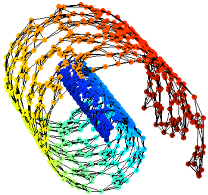
2: For $(i, j) \in E$ set $\widetilde{d}_{ij} = \|x_i - x_j\|_2$;

3: Find positions $z_i \in \mathbb{R}^d$, such that $\|z_i - z_j\|_2 \approx \widetilde{d}_{ij}$

Discussion : Manifold learning



(f) Data set

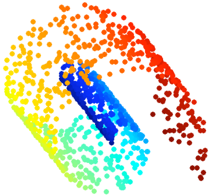


(g) Proximity graph

$$\tilde{d}_{ij} = \|x_i - x_j\|_{\mathbb{R}^m}, \qquad d_{ij} = d_{\mathcal{M}}(x_i, x_j)$$

$$\Delta = ?$$

Discussion : Manifold learning



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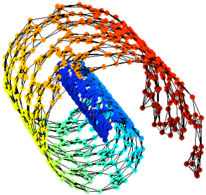

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Manifold learning: Reconstruction error

Lemma (M. Bernstein et. al., 2000)

Let $r_0 = r_0(\mathcal{M})$ *be the radius curvature defined by*

$$\frac{1}{r_0} = \max_{\gamma,t}\{\|\ddot{\gamma}(t)\|\}.$$

Then

$$(1 - \frac{d_{ij}^2}{24 r_0^2}) d_{ij} \leq \tilde{d}_{ij} \leq d_{ij}.$$

$$\Delta \propto \frac{r^4}{r_0^2}$$

$$d(X, \hat{X}) \leq C \frac{(nr^d)^5}{r_0^2}$$

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References

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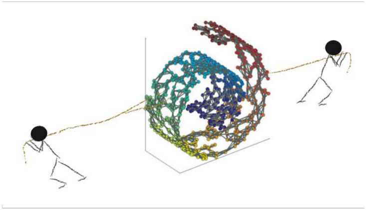
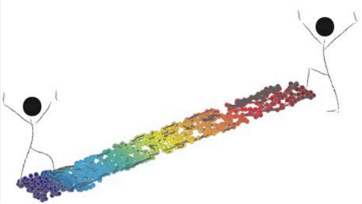
Thanks!

Maximum Variance Unfolding





[cf. Weinberger and Saul, 2006]

Optimization formulation

$$\begin{aligned} \text{maximize} \quad & \sum_{1 \le i,j \le n}^{n} \|x_i - x_j\|^2 \\ \text{subj.to} \quad & \|x_i - x_j\|^2 = d_{ij}^2 \quad \forall (i,j) \in E \\ & \sum_{i=1}^{n} x_i = 0 \end{aligned}$$

Nonconvex

Optimization formulation (better notation)

$$\begin{aligned} \text{minimize} \quad & \mathrm{Tr}(Q) \\ \text{subj.to} \quad & \langle M_{ij}, Q \rangle = d_{ij}^2 \\ & u^T Q u = 0 \\ & Q_{ij} = \langle x_i, x_j \rangle \end{aligned}$$

$$\begin{aligned} M_{ij} &= e_{ij} e_{ij}^T \,, \\ e_{ij} &= (0, \dots, 0, \underbrace{+1}_{i}, 0, \dots, 0, \underbrace{-1}_{j}, 0, \dots, 0) \end{aligned}$$

Semidefinite programing relaxation

$$\begin{aligned} \text{minimize} \quad & \mathrm{Tr}(Q) \\ \text{subj.to} \quad & \langle M_{ij}, Q \rangle = d_{ij}^2 \\ & u^T Q u = 0 \\ & \cancel{Q_{ij} = \langle x_i, x_j \rangle} \qquad Q \succeq 0 \end{aligned}$$

$$\begin{aligned} M_{ij} &= e_{ij} e_{ij}^T , \\ e_{ij} &= (0, \dots, 0, \underbrace{+1}_{i}, 0, \dots, 0, \underbrace{-1}_{j}, 0, \dots, 0) \end{aligned}$$