# Information-Theoretically Optimal Compressed Sensing via Spatial Coupling and Approximate Message Passing

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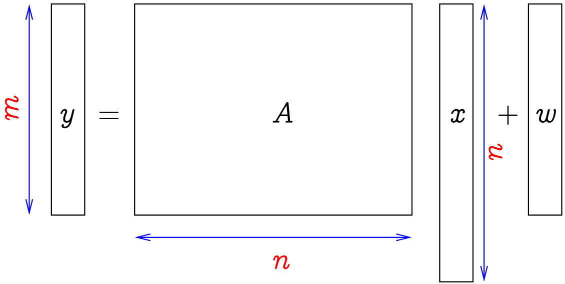
# General problem

$$y = Ax + \text{noise}\,,$$




- $x$ high-dimensional but highly structured
- How many linear measurements are needed?

# Normalization

$\rightarrow$ $w \sim \mathrm{N}(0, \sigma^2 I_{m\times m})$

$\rightarrow$ $m, n \to \infty,\ m/n = \delta$

$\rightarrow$ $A = [A_1 | \cdots | A_n] \qquad \|A_i\|_2 = \Theta(1)$

# Compressed sensing: Basic insights

Donoho, Candés, Romberg, Tao, Indyk, Gilbert, ... [2005-...]

| | | |
|---|---|---|
| **Structure** | $\rightarrow$ | $\|x\|_0 \leq k$ adversarial |
| **Rate** | $\rightarrow$ | $m = C\, k \log(n/k)$ |
| **Reconstruction** | $\rightarrow$ | Convex optimization |
| **Measurements** | $\rightarrow$ | Random isotropic vectors |
| **Robustness** | $\rightarrow$ | MSE $\leq C\sigma^2$ |

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# This paper

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| **Structure** $\rightarrow$ | $x = x_{\text{discr}} + x_{\text{other}};\ \|\|x_{\text{other}}\|\|_0 \le k$ oblivious |
| **Rate** $\rightarrow$ | $m = k + o(n)$ |
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# Outline

- A toy example (random signal).
- Results.
  - 'Spatially coupled' sensing matrices
  - How does spatial coupling work?
  - Bayes-optimal AMP
- Proof technique.
  - State evolution
- Supercooling.

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y &= Ax \,, \quad y \in \mathbb{R}^m \,,
\end{aligned}
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p_X = 0.2\,\delta_0 + 0.3\,\delta_1 + 0.2\,\delta_{-1} + 0.2\,\delta_3 + 0.1\,\text{Uniform}(-2,2).
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- Classical compressed sensing: $m = 0.97\,n + o(n)$
(Donoho 2006, universal, Donoho-Maleki-M. 2011 uniformly robust)

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# What is 0.1 here?

**Definition (Renyi's Information Dimension)**

For $X \sim p_X$, let $\langle X \rangle_m = \lfloor 2^m X \rfloor / 2^m$ be an $m$-digits rounding of $X$

$$\overline{d}(X) \equiv \lim\sup_{m \to \infty} \frac{H(\langle X \rangle_m)}{m}\,.$$

*Alternative characterization:*

- If

$$p_X = (1 - \varepsilon) \cdot \text{discrete} + \varepsilon \cdot \text{abs. continuous},$$

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then $\overline{d}(X) = \varepsilon$.

# Why is this important?

**Theorem (Verdú, Wu, 2010)**

*Under mild regularity hypotheses, non-adaptive compressed sensing is possible if and only if*

$$m > \overline{d}(X)\, n + o(n)\,.$$

*(equivalently, $\delta > \overline{d}(X) + o(1)$).*

Shannon-theoretic argument. Exhaustive-search reconstruction :-(

# Results

# Two tricks

- ‘Spatially coupled’ sensing matrix.
  [Kudekar, Pfister, 2010]
  [cf. also Felstrom, Zigangirov, 1999; Kudekar, Richardson, Urbanke 2009-2011]

- AMP reconstruction, Posterior-expectation denoiser
  [Donoho, Maleki, Montanari 2009]

- Spatial coupling + MP reconstruction
  [Krzakala, Mézard, Sausset, Sun, Zdeborova, 2011]
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# Our contributions

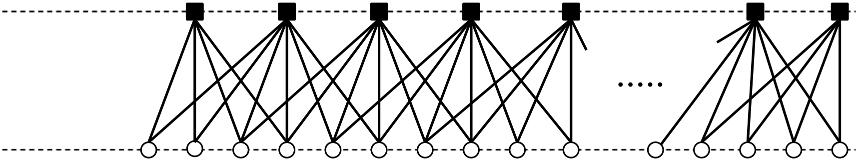
- Construction
- A rigorous proof
- Beyond random signals
- Robustness

# Spatially coupled sensing matrix

$$A = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & a_1 & a_2 & * & * & a_\ell & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_1 & b_2 & * & * & b_\ell & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1 & c_2 & * & * & c_\ell & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

- $\sim$ independent entries
- $\sim$ band diagonal
- $m, n, \ell \to \infty$, with $m/n \to \delta \in (0, 1)$, $\ell/n \to 0$

# How does spatial coupling work?



○ Coordinates of x

■ Coordinates of y

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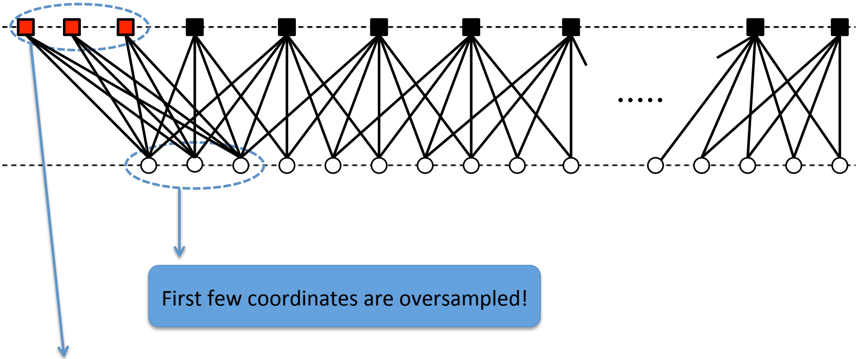

First few coordinates are oversampled!

Additional measurements associated to the first few coordinates

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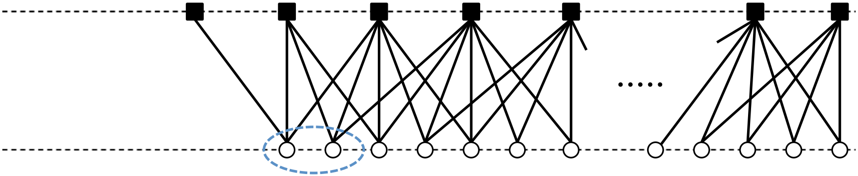

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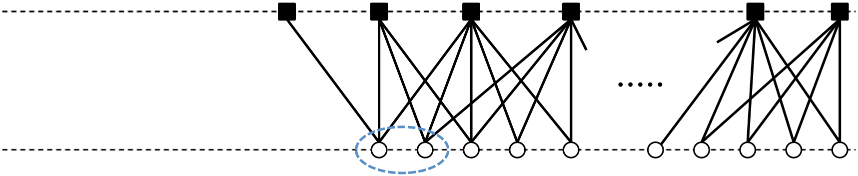

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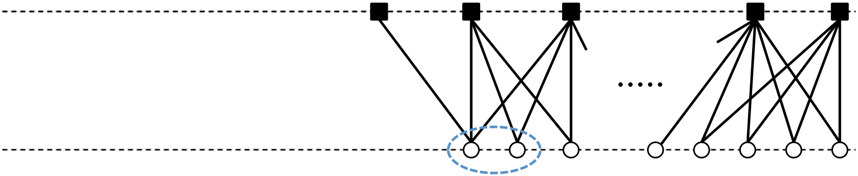

# Bayes-optimal AMP

$$
\begin{aligned}
x^{t+1} &= \eta_t(x^t + (Q_t \odot A)^* r^t)\,, \\
r^t &= y - Ax^t + \mathsf{b}_t \odot r^{t-1}\,.
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$Q_t, \mathsf{b}_t$ explicitly given normalizations

$\eta_t(y) \equiv \mathbb{E}\{X | X + \tau_t Z = y\}$

(reduces to simple expression in most cases)

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# A theorem

**Theorem (Donoho, Javanmard, Montanari, 2011)**

*Let $\{(x(n), y(n))\}_{n\geq 0}$ be a sequence of instances and assume the empirical distributions converge $p_{x(n)} \to p_X$.*

*Using Gaussian spatially-coupled matrices, Bayes-optimal AMP recovers $x(n)$ with high probability from*

$$m > \overline{d}(X)\, n + o(n)\,,$$

*noiseless measurements.*

*Further, if*[a] *$m > \overline{D}(X)n + o(n)$, and measurements are noisy*

$$\mathrm{MSE} \leq C(p_X)\sigma^2\,.$$

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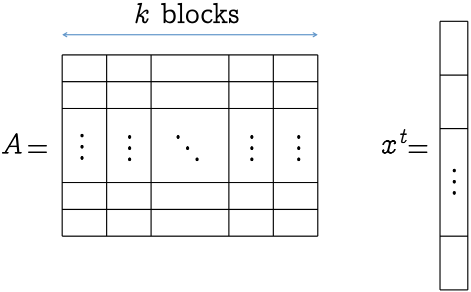
## Proof technique

# State evolution

A block Gaussian sensing matrix




$$\mathsf{MSE}^{(t)} \in \mathbb{R}^k, \qquad \mathsf{MSE}^{(t)}(i) = \lim_{n\to\infty} \frac{n}{k} \|x^t_{B_i} - x_{B_i}\|^2.$$

We show a state evolution recursion:

$$\mathsf{MSE}^{(t+1)} = \mathcal{F}(\mathsf{MSE}^{(t)}; p_X)$$

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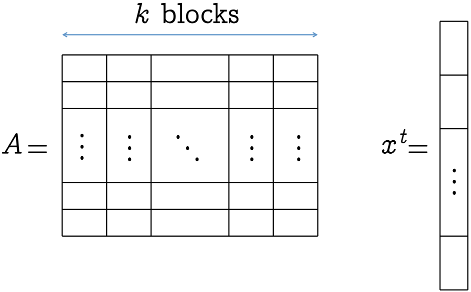


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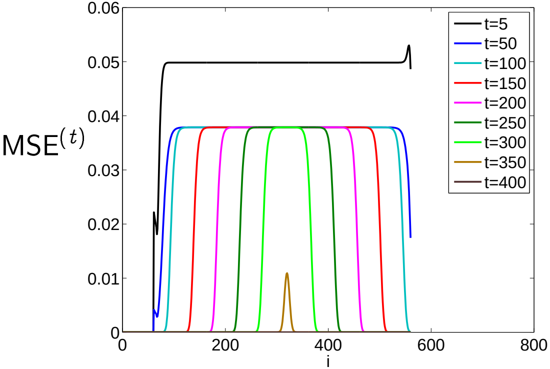
# An illustration



# Steps of the proof

- Analysis of the state evolution
- Continuum state evolution
- An energy functional $\mathcal{E}(\cdot)$
  - Fixed point of the state evolution $\Phi_\infty \to \nabla\mathcal{E}(\Phi_\infty) = 0$

# Supercooling

# Question

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I will discuss it in my talk on Thursday!

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