Lecture 4: Optimization

• Maximizing or Minimizing a Function of a Single Variable
• Maximizing or Minimizing a Function of Many Variables
• Constrained Optimization

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Maximizing a function of a single variable

- Given a real valued function, y = f(x) we will be concerned with the existence of extreme values of the dependent variable y and the values of x which generate these extrema. (maxima or minima)
- The function f(x) is called the *objective function* and the independent variable x is called the *choice variable*

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Max single variable

- The problem of finding the value or the set of values of the choice variable which yield extrema of the objective function is called optimization.
- In order to avoid boundary optima, we will assume that f : X $\rightarrow$ R, where X is an open interval of R. All of the optima characterized will be termed interior optima.

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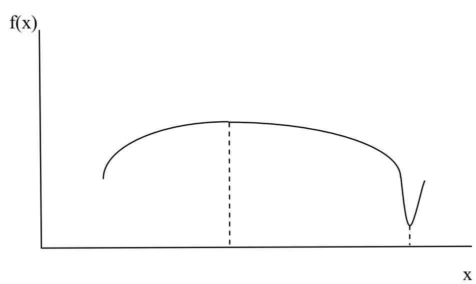
Definition of max or min

- Definition. f has a local maximum (minimum) at a point x', if for all x in an open interval (x' - μ, x' + μ), we have that f(x') > (<) f(x).



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Key result

Proposition 1. Let f be twice differentiable. Let there exists an $x^o \in X$ such that $f'(x^o) = 0$.

(i) If $f''(x^o) < 0$, then f has a local maximum at x^o. If, in addition, $f'' < 0$ for all x or if f is strictly concave, then the local maximum is a unique global maximum.

(ii) If $f''(x^o) > 0$, then f has a local minimum at x^o. If, in addition, $f'' > 0$ for all x or if f is strictly convex, then the local minimum is a unique global minimum.

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Terminology

- The zero derivative condition is called the *first order condition*
- The second derivative condition is called the *second order condition*.

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Examples

#1 Let $f = ax - bx^2$, $a,b > 0$ and $x > 0$. Find a maximum.

Here, $f' = 0$ implies $x' = a/2b$. Moreover, $f'' = -2b < 0$, for all x. Thus, we have a global maximum.

#2 Let $f = x + x^{-1}$, where $x > 0$. Find a minimum.

Here, $f' = 0$ implies that $x^{-2} = 1$, so that $x' = 1$. In this case, $f'' = 2x^{-3} > 0$. Thus, we have a global minimum.

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Maximizing or Minimizing a Function of Many Variables

- We consider a differentiable function of many variables $y = f(x_1,...,x_n)$.
- This function has a local maximum (minimum) at a point $x' = (x_1,...,x_n)$, if the values of the function are greater than (less than) image values of the function in a neighborhood of x'.
- The domain of f is thought of as a subset X of R^n, where each point of X has a neighborhood of points surrounding it which belongs to X. (Interior optima)

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Key Result

- *Proposition 2*. If a differentiable function f has a maximum or a minimum at $x^o \in X$, then $f_i(x^o) = 0$, for all i.
- This condition states that at a maximum or a minimum, all partial derivatives are zero. This depicts the top of a hill or a bottom of a valley.
- Operationally, the n partial derivative functions set equal to zero give us n equations in n unknowns to be solved for the extreme point x^o. These conditions are called the first order conditions (FOC).

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Example

Find the maximum of $(x_1x_2)^{1/4} - x_1 - x_2$. Computing, we have

$$.25(x_1)^{-.75}(x_2)^{.25} - 1 = 0$$

$$.25(x_2)^{-.75}(x_1)^{.25} - 1 = 0$$

These imply that $x_i = x$ and that x_i =1/16. At optimum, we have that y = .125.

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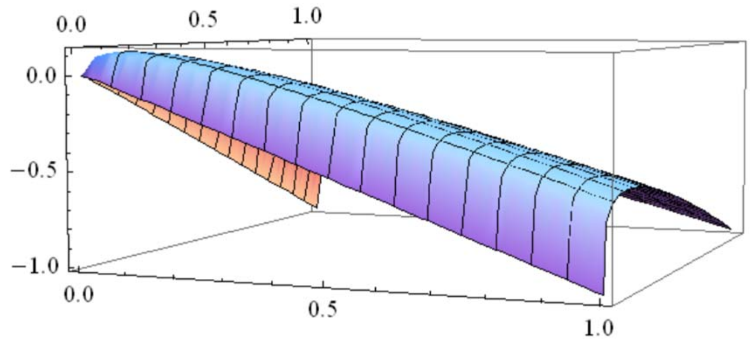
Illustration



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Example

Find the minimum of $z = x^2 + xy + 2y^2$. The FOC are

$2x + y = 0,$

$x + 4y = 0.$

Solving for the critical values x = 0 and y = 0.

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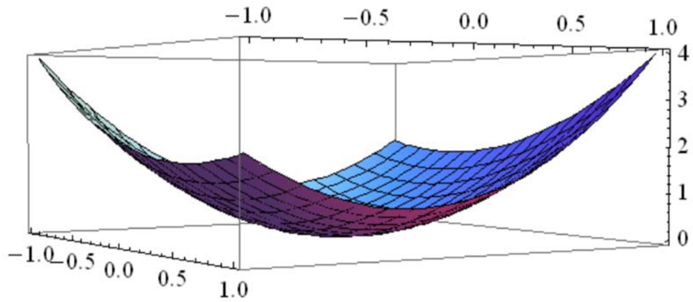
Illustration



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Constrained Optimization

- One of the most common problems in economics involves maximizing or minimizing a function subject to a constraint.
- We are again interested in characterizing interior or non-boundary constrained optima.
- The cost minimization problem subject to an output constraint is an example (Lecture 2).
- The basic problem is to maximize (minimize) a function of at least two independent variables subject to a constraint. We write the objective function as $f(x_1,...,x_n)$ and the constraint as $g(x_1,...,x_n) = 0$.

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Constrained Optimization

- The constraint set is written as C = { x | $g(x_1,...,x_n) = 0$}.
- We write the problem as

$$\underset{\{x_1,...,x_n\}}{\text{Max}}\ f(x_1,...,x_n) \text{ subject to } g(x_1,...,x_n) = 0.$$

- For minimization, replace Max with Min.

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Key Result

Proposition 3. Let f be a differentiable function whose n independent variables are restricted by the differentiable constraint g(x) = 0. Form the function $L(\lambda,x) \equiv f(x) + \lambda g(x)$, where λ is an undetermined multiplier. If x^o is an interior maximizer or minimizer of f subject to g(x) = 0, then there is a λ^o such that

(1) $\partial L(\lambda^o, x^o)/\partial x_i = 0$, for all i, and

(2) $\partial L(\lambda^o, x^o)/\partial \lambda = 0$.

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Discussion

- L is the Lagrangian function and λ is the Lagrangian multiplier.
- Conditions (1) and (2) are again called the FOC. They constitute n + 1 equations in n + 1 unknowns.

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Example

- Min $(p_1x_1 + p_2x_2)$ subject to $q^t = f(x_1,x_2)$.
  $\{x_1,x_2\}$
- Forming the Lagrangian, we have
  $L = (p_1x_1 + p_2x_2) + \lambda[q^t - f(x_1,x_2)]$.
- FOC

(1) $L_\lambda = q^t - f(x_1,x_2) = 0$,

(2) $L_1 = p_1 - \lambda f_1 = 0$,

(3) $L_2 = p_2 - \lambda f_2 = 0$.

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Example

- Condition (1) just says that the firm must obey its output constraint
- Conditions (2) - (3) say that

$$\frac{p_1}{p_2} = \frac{f_1}{f_2}.$$

- That is, the MRS should equal the price ratio.

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Numerical Example

- Let $f = x_1x_2$ and let $p_1 = 2$ and $p_2 = 2$. Solve the cost minimization problem with a target output of 16.
- The Lagrangian is

$L = 2x_1 + 2x_2 + \lambda(16 - x_1x_2)$.

- FOC
- (1) $x_1x_2 - 16 = 0$,
- (2) $2 - \lambda x_2 = 0$,
- (3) $2 - \lambda x_1 = 0$.

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Numerical Example

- Clearly $x_i = x$, so that using (1), $x^2 = 16$ and $x_i = 4$.

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