Lecture 3: Calculus: Differentiation and Integration

3.1 First Order Derivatives

Consider functions of a single independent variable, $f : X \rightarrow R$, X an open interval of R. Write $y = f(x)$ and use the notation $f'(x)$ or dy/dx for the derivative of f with respect to x.

R1(Constant Function Rule) The derivative of the function $y = k$ is zero.

R2 (Power function rule) The derivative of the function $y = x^N$ is NX^{N-1}.

R3 (Multiplicative Constant Rule) The derivative of $y = kf(x)$ is $kf'(x)$.

Examples of R1-R3

#1. If $y = 4$, then $dy/dx = 0$.

#2. If $y = 3x^2$, then $dy/dx = 6x$.

R4 (Sum-difference Rule) $g(x) = \Sigma_i f_i(x) \Rightarrow g'(x) = \Sigma_i f_i'(x)$.

Example of R4

If $y = x^2 - 3x^3 + 7$, then $dy/dx = 2x - 9x^2$.

R5 (Product Rule) $h(x) = f(x)g(x) \Rightarrow h'(x) = f'(x)g(x) + f(x)g'(x)$

Example of R5

If $y = (3x+4)(x^2 - 4x^3)$, then $dy/dx = 3(x^2 - 4x^3) + (3x+4)(2x - 12x^2)$

R6 (Quotient Rule) $h(x) = f(x)/g(x) \Rightarrow h'(x) = [f'(x)g(x) - g'(x)f(x)]/[g(x)]^2$

Example of R6

If $y = (2x - 4)/(x^4 + 3x)$, then $dy/dx = [2(x^4 + 3x) - (4x^3 +3)(2x - 4)]/ (x^4 + 3x)^2$

R7 (Chain Rule) If $z = f(y)$ is a differentiable function of y and $y = g(x)$ is a differentiable function of x, then the composite function $f \circ g$ or $h(x) = f[g(x)]$ is a differentiable function of x and

$$h'(x) = f'[g(x)] \cdot g'(x).$$

Example of R7

If $h(x) = (g(x) + 3x)^2$, then $h'(x) = 2(g(x) + 3x)(g'(x) + 3)$

Functions which are one-to-one can be inverted. If $y = f(x)$, where f is one-to-one, then it is possible to solve for x as a function of y. We write this as $x = f^{-1}(y)$. For example if $y = a + bx$, then the inverse function is $x = -a/b + (1/b)y$. We have

R8 (Inverse Function Rule) Given $y = f(x)$ and $x = f^{-1}(y)$, we have $f^{-1'}(y) = 1/f'(x)$.

Examples of R8

If $y = a + bx$, then $dy/dx = b$ and $f^{-1'}(y) = 1/b$. If $y = x^2$, $x > 0$, then $dy/dx = 2x$ and $f^{-1}(y) = (y)^{1/2}$. In this case $f^{-1'}(y) = 1/2x = (1/2)(y)^{-1/2}$.

R9 (Exponential Function Rule) Let $y = e^{f(x)}$, then $dy/dx = f'(x)e^{f(x)}$.

Examples of R9

If $y = e^{3x}$, then $dy/dx = 3\, e^{3x}$. If $y = e^x$, then $dy/dx = e^x$.

R10 (Log function Rule) Let $y = \ln f(x)$, then $dy/dx = f'(x)/f(x)$.

Examples of R10

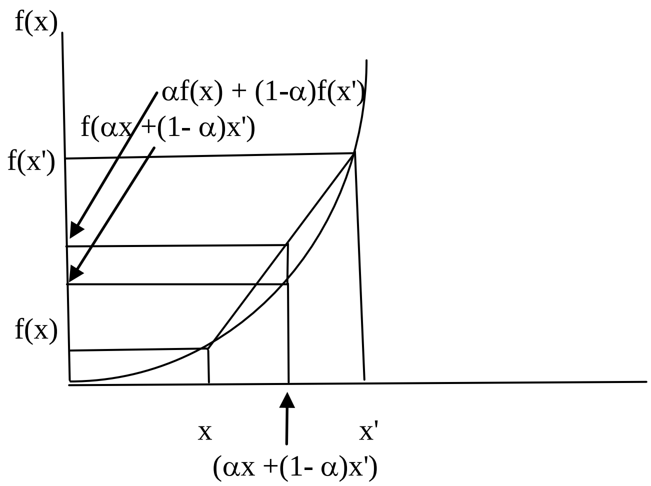
If $y = \ln(2x+3)$, then $dy/dx = 2/(2x+3)$. If $y = \ln x$, then $dy/dx = 1/x$.

3.2 Higher Order Derivatives: The Second Derivative

1. If a function is differentiable, then its derivative function may itself be differentiable. If so then the derivative of the derivative is called the second derivative of the function. It measures the rate of change of the rate of change of the function. The sign of the second derivative tells us about the curvature (concavity versus convexity) of the function. The second derivative is written as d^2y/dx^2 or $f''(x)$.

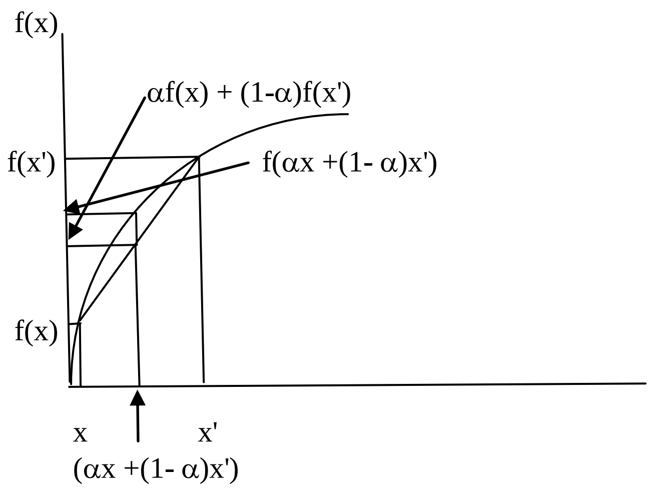
2. In terms of computation, we merely differentiate the derivative function. For example, if $y = ax^2 + bx$, then $f'(x) = 2ax + b$ and $f''(x) = 2a$.

3. A function f(x) is strictly convex (concave) if for all x, x', $f(\alpha x + (1-\alpha)x') < (>)\ \alpha f(x) + (1-\alpha)f(x')$, for $\alpha \in (0,1)$. The following figure illustrates a strictly convex function.



Visual inspection of this function tells us that the first derivative is increasing. Thus, it is true that if the second derivative is positive, then the first derivative is increasing and the function is strictly convex.

A strictly concave function is illustrated in the following figure.



Again, visual inspection shows us that the first derivative is decreasing for a strictly concave function. Thus, a negative second derivative is sufficient for strict concavity.

4. Examples. The function $y = x^2$ $(x > 0)$ is strictly convex and we have that $d^2y/dx^2 = 2 > 0$.

The function $y = x^{1/2}$ $(x > 0)$ is strictly concave. We have that $d^2y/dx^2 = (-1/4)x^{-3/2} < 0$.

3.3 Partial Derivatives of Functions of Many Variables

1. Consider a function of n independent variables. It would be of the form

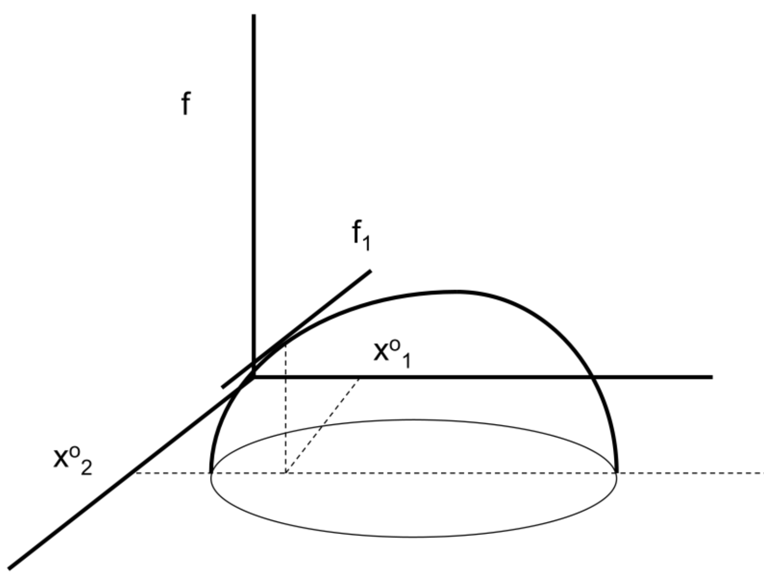
$$y = f(x_1, \ldots, x_n),\ f: R^n \to R.$$

2. *Def.* The *partial derivative* of the function $f(x_1,x_2,\ldots, x_n)$, $f: R^n \to R$, at a point $(x_1^o, x_2^o, \ldots, x_n^o)$ *with respect to x_i* is given by

$$\lim_{\Delta x_i \to 0} \Delta y / \Delta x_i = \frac{f(x_1^o, \ldots, x_i^0 + \Delta x_i, \ldots, x_N^0) - f(x_1^0, \ldots, x_N^0)}{\Delta x_i}$$

3. The notation for a partial derivative is $f_i(x_1, \ldots, x_n)$ or $\partial f/\partial x_i$.

4. Illustration



5. The mechanics of differentiation are very simple:

- When differentiating with respect to x_i, regard all other independent variables as constants

- Use the simple rules of differentiation for x_i.

6. Examples

a. If $y = f(x_1,x_2) = x_1^3 x_2^2$, then $f_1 = 3x_1^2 x_2^2$, $f_2 = x_1^3 2x_2^1$.

b. If $y = f(x_1,x_2) = 2x_1 + x_1x_2$, then $f_1 = 2 + x_2$ and $f_2 = x_1$.

c. If $y = f(x_1,x_2) = x_1g(x_2)$, then $f_1 = g(x_2)$ and $f_2 = x_1g'(x_2)$.

d. If $y = f(x_1,x_2) = \ln (x_1 + 4x_2)$, then $f_1 = 1/(x_1 + 4x_2)$ and $f_2 = 4/(x_1 + 4x_2)$.

e. If $y = f(x_1,x_2) = x_1 e^{x_2}$, then $f_1 = e^{x_2}$ and $f_2 = x_1 e^{x_2}$.

3.4 Antiderivatives and Indefinite Integrals

1. In differential calculus, we were interested in the derivative of a given real-valued function, whether it was algebraic, exponential or logarithmic. Here we are concerned with the inverse of the operation of differentiation. That is, the operation of searching for functions whose derivatives are a given function.

2. Consider any arbitrary real-valued function

$$f: X \to R$$

defined on a subset x of the real line, i.e. $X \subset R$. By the antiderivative (primitive) of f; we mean any differentiable function

$$F: X \to R$$

whose derivative is the given function f. Hence,

$$\frac{dF}{dx} = F'(x) = f(x).$$

Clearly if F is an antiderivative of f then so is F + c, where c is a constant. F + c then represents the set of all antiderivative functions of f and this is called the indefinite integral of f. The indefinite integral is denoted as

$$\int f(x)dx = F(x) + c.$$

3. The basic rules of integration

R1 (Power Function Rule) $\int x^N \, dx = \frac{1}{N+1} x^{N+1} + c.$

R2 (Multiplicative Constant Rule) $\int cf(x) \, dx = c\int f(x) \, dx.$

R3 (Sum Rule) $\int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx.$

R4 (Exponential Function Rule) $\int e^x \, dx = e^x + c.$

R5 (Logarithmic Function Rule) $\int \frac{1}{x} \, dx = \ln|x| + c.$

(Note that ln is defined only for positive x. Thus, the use of absolute value.)

3. Examples of the basic rules.

#1 Let $f(x) = x^4$, then

$$\int x^4 dx = \frac{x^5}{5} + c.$$

check:

$$\frac{d\left(x^5/5\right)}{dx} = x^4.$$

#2 Let $f(x) = x^3 + 5x^4$, then

$$\int [x^3 + 5x^4] \, dx = \int x^3 \, dx + 5\int x^4 \, dx$$

$$= x^4/4 + 5(x^5/5) + c$$

$$= x^4/4 + x^5 + c$$

check.

$$\frac{d}{dx}(x^4/4 + x^5) = x^3 + 5x^4.$$

#3 Let $f(x) = x^2 - 2x$

$$\int [x^2 - 2x] \, dx = \int x^2 \, dx + \int -2x \, dx$$

$$= x^3/3 - 2\int x \, dx + c'$$

$$= x^3/3 - 2(x^2/2) + c$$

$= x^3/3 - x^2 + c.$

#4 Let f(x) = e^{2x^1}, then $\int e^{2x^1}dx = \frac{1}{2}e^{2x^1} + c.$

#5 Let f(x) = $\frac{2}{x}$, then $\int \frac{2}{x}dx = 2lnx + c.$

3.5 Antiderivatives and Definite Integrals

1. Let f(x) be continuous on an interval $X \subset R$, where $f: X \rightarrow R$. Let F(x) be an antiderivative of f, then $\int$ f(x) dx = F(x) + c.

2. Now choose a, b $\in$ X such that a < b. Form the difference

$$[F(b) + c] - [F(a) + c] = F(b) - F(a).$$

(Note that this is independent of the constant c)

This difference F(b) - F(a) is called the *definite integral of f from a to b*. The point a is termed the *lower limit of integration* and the point b, the *upper limit of integration.*

3. Notation: We would write

$$\int_a^b f(x)dx \equiv F(x)\Big]_a^b \equiv F(x)\Big|_a^b = F(b) - F(a)$$

4. Examples

#1 Let f(x) = x^3, find

$$\int_0^1 x^3 dx = \frac{1}{4}x^4\Big|_0^1 = \frac{1}{4}(1)^4 - \frac{1}{4}(0)^4$$

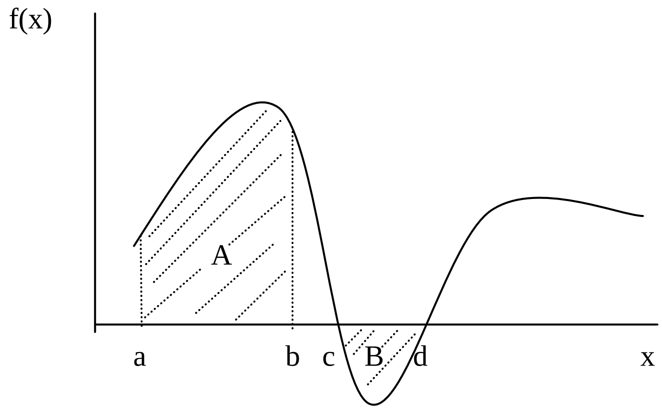
$$= \frac{1}{4}.$$

#2 f(x) = $2x\,e^{x^2}$, find $\int_3^5 f(x)dx$,

$$\int_3^5 2xe^{x^2} = e^{x^2}\Big|_3^5 = e^{25} - e^9$$

#3 f(x) = $2x + x^3$, find $\int_0^1 f(x)dx = (x^2 + \frac{1}{4}x^4)|_0^1 = \frac{5}{4}$

5. The absolute value of the definite integral represents the area between f(x) and the x-axis between the points a and b.



The area A = $\int_a^b f(x)dx$ and the area B = $(-1)\int_c^d f(x)dx$.

3.6 Differentiation of an Integral

1. The following rule applies to the differentiation of an integral.

$$\frac{\partial}{\partial y}\int_{p(y)}^{q(y)} f(x,y)dx = \int_p^q f_y(x,y)dx + f(q,y)q'(y) - f(p,y)p'(y).$$

2. Example: In Economics, we study a consumer's demand function in inverse form

$$p = p(Q),$$

where Q is quantitiy demanded and p denotes the maximum uniform price that the consumer is willing to pay for a given quantity level Q. We assume that p' is negative. The definite integral

$$\int_0^Q p(z)dz = TV(Q)$$

is called total value at Q. It gives us the maximum revenue that could be extracted from the consumer for Q units of the product. The dollar amount

$$TV(Q) - p(Q)Q = CS(Q)$$

is called consumer's surplus. Let C(Q) be the cost of supplying Q units. If a firm could extract maximum revenue from the consumer, its profit function would be

$$\int_0^Q p(z)dz - C(Q).$$

The output level maximizing the firm's profit sets the derivative of the previous definite integral equal to zero. This implies

$$p(Q) = C'(Q).$$

The firm should set the last quantity unit sold, so that price is equal to marginal cost. Each unit should have a price equal to the maximum price the consumer is willing to pay. This is called a perfectly discriminating monopolist.

3.7 Some Notes on Multiple Integrals

1. In this section, we will consider the integration of functions of more than one independent variable. The technique is analogous to that of partial differentiation. When performing integration with respect to one variable, other variables are treated as constants. Consider the following example:

$$\int_c^d \int_a^b f(x,y)dxdy \ .$$

We read the integral operators from the inside out. The bounds a,b refer to those on x, while the bounds c,d refer to y. Likewise, dx appears first and dy appears second. The integral is computed in two steps:

#1. Compute $\int_a^b f(x,y)dx = g(y)$.

#2. Compute $\int_c^d g(y)dy = \int_c^d \int_a^b f(x,y)dxdy$.

If there were n variables, you would follow the same recursive steps n times. Each successive integration eliminates a single independent variable.

2. Some Examples.

Example 1: Suppose that z = f(x,y). We wish to compute integrals of the form

$$\int_c^d \int_a^b f(x, y)dxdy.$$

Consider the example f = x^2y, where c = a = 0 and d = 2, b = 1. We have

$$\int_0^2 \int_0^1 x^2 ydxdy.$$

Begin by integrating with respect to x, treating y as a constant

$$\int_0^1 yx^2 dx = \frac{1}{3} yx^3 |_0^1 = \frac{1}{3} y.$$

Next, we integrate the latter expression with respect to y.

$$\int_0^2 \frac{1}{3} ydy = \frac{1}{3}\frac{1}{2} y^2 |_0^2 = \frac{1}{6} 4 = \frac{2}{3}.$$

Example 2: Compute

$$\iiint (3xy + 2z)dxdydz,$$

where the limits of integration are 0 and 1, in each case. Begin with x.

$$(\frac{3x^2y}{2} + 2zx) |_0^1 = \frac{3y}{2} + 2z.$$

Next, y

$$(\frac{3y^2}{4} + 2zy) |_0^1 = \frac{3}{4} + 2z.$$

Finally, z

$$(\frac{3z}{4} + z^2) |_0^1 = \frac{7}{4}.$$