Productivity Investment with Hidden Action

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October 2, 2019

Abstract

This paper considers a hidden action agency model in which an agent can be incentivized to simultaneously work and exert effort to increase productivity. We determine conditions under which the principal wants to incentivize concurrent working and productivity investment and conditions under which the agent would want to participate. Our results can help explain the skills gap.

JEL Code: L20, L21, L22, and L23

Key Words: Human Capital, Investment, Agency.

*I thank my colleagues Odilon Camara and Joao Ramos. I also thank the Marshall School of Business for generous research support.
1. Introduction

Investment in human capital has been and continues to be an important topic in the economics literature. Since the classic papers of Becker (1962) and Hashimoto (1981), the question of whether firm specific versus general human capital is advantageous for a firm has been extensively discussed. While general human capital in their environment is not profitable to the organization, firm specific investment is advantageous with the costs and the benefits being shared by the employee and the firm. Sharing is incentivized by the fact that it decreases the chance that either party will terminate the relationship and impose a loss on the other actor. This early literature studied the human capital investment problem in a perfectly competitive environment in the absence of an agency problem in the firm.

Theoretical work has extended this early study to consider the case of imperfectly competitive labor markets, and the results show that general training can be optimal in the imperfect setting (See Kessler and Lulfesmann (2006) or Acemoglu and Pischke (1999).). In a recent paper, Fudenberg and Rayo (2017) present a dynamic model of an apprentice who can work on skill enhancing and non-skill enhancing tasks while working for a principal. The principal optimally incentivizes the apprentice to pay for general training by working for low wages and by working inefficiently hard. This paper has interesting implications for the mix of the agent’s efforts, the length of the apprenticeship, and regulatory measures on apprenticeships, but it does not consider hidden action or information. Krakel (2016) considers human capital investment with moral hazard, where the agent elects whether or not to engage in training in a zero-one manner. He shows that firms will want to invest in general and specific human capital, depending on the assumptions. Cisternas (2018) studies a dynamic agency model of career concerns where work effort leads to skill acquisition through learning by doing during each period. Thus, he generalizes the traditional career concerns model of Holmstrom (1999) to the case where skills are endogenous over time.
This paper also extends the study of human capital investment to an agency setting, but with a focus on an agent choosing to develop additional skills while working. We assume that the agent multitasks by investing in human capital through the exertion of self improvement effort concurrently with work effort. This investment effort stochastically leads to possible higher productivity in the firm. We assume that both types of effort cannot be observed by the firm which must contract with the agent based on cash flow.

The literature on human resource development and education also studies skill acquisition as a task to be carried out simultaneously while working. These papers discuss the notion of self directed learning (SDL) in the workplace as an ever increasing trend, especially in knowledge-intensive jobs, for example, in areas such as computer programing, engineering, education, finance and some consumer services, where continuous learning is necessary for the development of one’s career. A recent survey conducted by Degreed (2016) makes this point. In the survey, employees across a broad spectrum of positions reported that they spent on average more than five times the amount of time on self directed learning as they did on corporate sponsored formal training. Specifically, they spent on average 3.3 hours per week studying on their own at work as opposed to 37 minutes per week learning formally from their employers. This interesting case study and the aforementioned description of SDL in the workplace nicely motivate our multitasking approach. In our modern economy, members of organizations self educate and engage in process innovation during normal work hours to stochastically improve their current productivity. Yet the contracting literature in economics has not fully examined this process. This paper helps fill that gap.

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1 Work effort and effort to develop skills are separate choice variables in contrast to the Cisternas model where skill acquisition is function of work effort through learning by doing. For example a surgeon becomes a better surgeon by doing surgeries.

2 See Ellinger (2004), Confessore et al. (1998), and Ardichvili et al. (2016) and their references for a discussion of SDL. See Lejeune et al. (2019) and Fontana et al. (2015a and 2015b) for examples of impact measurement of SDL. See Manuti et al. (2015) and their references for a discussion of SDL as informal learning in the workplace.

3 The sample consisted of 512 employees of which 13% were directors, vice presidents or C-level executives, 15% were managers or supervisors, 31% were salaried employees, 34% were hourly workers and 7% were contract workers.
In this setting, several interesting questions arise. Under what conditions would the principal want to incentivize such investment while working, and under what conditions would the agent want to participate? How does the principal’s contract change when investment is incentivized? How does the agent prioritize investment and work efforts? How does a second best equilibrium compare to the efficient equilibrium? We will address each of these questions.

We employ a simple quadratic cost model in which work and investment efforts are complements in cash flow production but substitutes in the agent’s cost of effort, the agent has limited liability, and all actors are risk neutral. This allows for tractability and closed form solutions which, otherwise, would not be possible. We show that there are two types of equilibria depending on the magnitude of a parameter, termed the C-S ratio, representing the ratio of the strength of complementarity of efforts in the production of cash flow to the strength of the substitutability of efforts in the agent’s effort cost. The principal must raise the agent’s incentive pay to induce the agent to see the efforts as complements, but this is costly due to limited liability. That is, the principal must give up a larger share of the surplus. If technological complementarities are strong relative to the agent’s view of these tasks as substitutes, that is, if C-S ratio is above a cutoff value, the principal optimally makes a great enough incentive payment to induce the agent to exert efforts to invest and to work. We call this the investment equilibrium and the agent multitasks. If the C-S ratio is below the cutoff and technological complementarities are weak relative to the substitutability of efforts in effort cost, the principal optimally makes a lower incentive payment which does not incentivize investment, even though a greater payment that induces effort in both tasks is more efficient. Here, we are back to a single task work effort model with limited liability. We call this a non-investment equilibrium. We show that an investment contract is always more high powered than a non-investment one. The result that inducing effort across both tasks requires higher incentive pay runs counter to the standard models, e.g., Holmstrom and Milgrom (1991), of
multitasking. The key feature of our model accounting for this novel difference is the presence of a limited liability constraint.

We rank investment and work efforts at an investment equilibrium and at the social optimum, and we compare all equilibria to the social optimum. We show that the agent would prefer that the principal’s threshold C-S ratio be lower for the investment equilibrium, but the principal incentivizes investment at the higher cutoff. In the interval of the C-S ratios between the agent’s lower cutoff and the principal’s higher one, the agent would gain more surplus than the principal would lose, if the principal were to implement an investment equilibrium as opposed to a non-investment equilibrium. However, due to the fact that the agent has limited liability, it is not possible to create a transfer from the agent to the principal in order to implement the more efficient investment equilibrium in this region. This interesting implication of the model can explain why many employers see a skills gap in hiring workers.4

While the one application of our model is skill development concurrent with working, the results of the model also apply to cases where the agent is using investment effort concurrent with working effort to improve the production process, in order to increase her or his productivity. Such multitasking effort might include updating or trying out new software, maintaining capital equipment, rearranging files and/or the office space for greater efficiency, coordinating with colleagues to create better communication, and creating different types of process innovation which might improve productivity. This type of investment effort is analogous to skill acquisition in that it is experimental and it may or may not actually lead to current productivity increases.

Section 2 presents the model, and Section 3 characterizes the principal’s optimal contracts. Section 4 concludes. All proofs are placed in the Appendix.

4See, for example, Bessen (2014) for a discussion of the skills gap.
2. The Basic Model

Consider a principal-agent situation where the agent has hidden effort. The agent’s effort can be directed towards the creation of cash flow for the firm, and the agent can exert effort to attempt to increase his or her productivity in the production of cash flow. The agent then has the classic multitasking problem of deciding whether to devote all effort to the production of cash flow or to devote some effort to cash flow production and some effort to the building of skill (or the enhancement of the production process) which then leads to a greater possibility of increased productivity in producing cash flow.

There are many examples of this setting. A researcher might read a related paper or review an analytical technique while researching some main topic in the hopes of improving productivity on that primary topic. The researcher cuts out competing time during normal work time to engage in this investment activity, and it may or may not payoff this period in terms of raising productivity on the main topic. The reading and review may be a bust for the researcher who does not precisely know the payoff while exerting this skill development effort. A programmer might review a new language in the hopes of improving his or her productivity on a current coding project. While examining the new language, the actual payoff on the current main project can clearly be uncertain to the programmer. A worker might try out a new trial software package in the hopes that it might improve productivity, but the end result of this effort for productivity increase could render that effort fruitless for this period’s output. A worker creating a physical commodity might engage in periodic technique development and practice intertwined with effort devoted to production, the result of which is an expectation of incremental productivity. Again, this effort may or may not pan out in terms of an actual productivity increase this period. We could go on and on providing examples. A key feature of each of these examples is that the process of concurrent exertion of investment and working efforts is experimental and it leads to a stochastic increase in productivity.
in the eyes of the investing agent during the work period. At the time of exertion of efforts, the agent does not know for sure that the investment effort will pay off in terms of boosting productivity this period. To this point, Richard Branson stated: "You don’t learn to walk by following rules. You learn by doing, and by falling over."

Let cash flow be denoted \( \hat{y} \), and assume that it can be high or low (zero), \( \hat{y} \in \{0, y\} \), where \( y > 0 \). The probability that cash flow will be high is given by

\[
\text{Prob}(y = \hat{y}) = \hat{a}L,
\]

where \( L \) is effort of the agent in attempting to create cash flow and \( \hat{a} \) is a random variable representing a productivity parameter for cash flow effort. The productivity parameter can take on two values given by \( \hat{a} \in \{a, a + \Delta\} \). The probability that the agent achieves the higher productivity parameter \( (a + \Delta) \) is

\[
\text{Prob}(\hat{a} = a + \Delta) = L_1.
\]

where \( L_1 \) represents the effort exerted by the agent to raise his/her productivity level above the guaranteed base level \( a \). We assume that \( a, \Delta > 0 \) and \( a + \Delta < 1 \). Given these assumptions, the firm’s expected cash flow is

\[
E(\hat{y}) = y[(a(1 - L_1)L + (a + \Delta)L_1L] = y(a + \Delta L_1)L.
\]

At a later point, we will present feasibility restrictions, denoted (Fi), on parameters which guarantee that, in equilibrium, the two probabilities are in the unit interval, \( L_1, \hat{a}L \in [0, 1) \). The two efforts are complements in production.

The firm cannot observe either type of effort. Both the firm and the agent know the default
productivity level $a$ and they know the probability distribution of $\hat{a}$, at the time of contracting. However, we assume that because of verifiability problems, the firm cannot contract on the agent’s productivity level. At time zero, the principal and the agent contract. After contracting, the agent exerts nonnegative investment effort and nonnegative working effort based on the expectation of productivity for that period. Possibly a greater productivity level, $a + \Delta$, is realized, but the base level, $a$, is guaranteed. At the end of the period, cash flow is realized.

For tractability, the cost of effort is assumed to be quadratic and given by

$$\left(\frac{c}{2}\right)(\delta L_t + L)^2.$$  

The parameter $\delta$ measures the extent to which the marginal cost of investment effort differs from that of normal cash flow effort, in that it is equal to the ratio of the marginal cost of investment effort to that of cash flow effort. If $\delta > 1$, then it is more costly on the margin for the agent to invest than it is to produce cash flow and conversely if $\delta < 1$. In the former case, investing is more of a chore than normal work effort, and in the latter it is a bit of a diversion from everyday work effort. Let us think of $\delta$ as a relative difficulty of investing parameter. The constant $c$ represents a cost of effort parameter. We will not consider situations where there is a capacity constraint on total effort, or, equivalently, we look at cases where such a constraint is non-binding.\(^5\) However, there is an opportunity cost of investment effort, because any positive amount raises the marginal and total cost of working effort in the cost of effort function. The two efforts are substitutes in cost.

Both the firm and the agent are assumed to be risk neutral, and the agent has an outside option utility of zero. The principal’s contract consists of a possible flat salary $S$ and an incentive

\(^5\)Effort is typically not measured in people hours, and if it were, few if any work 24-7.
percentage $b$ of cash flow. We will assume that the agent is subject to limited liability in the sense that no promised payment can be negative, $S,b \geq 0$.

The agent’s optimization problem is to

$$\max_{(L_t,L)} S + by(a + \Delta L_t)L - \left(\frac{c}{2}\right)(\delta L_t + L)^2. \quad (1)$$

The first order conditions for investment and cash flow efforts are

$$L_t : by\Delta L - \delta c(\delta L_t + L) = 0, \text{ and}$$

$$L : by(a + \Delta L_t) - c(\delta L_t + L) = 0. \quad (3)$$

It is interesting to note that condition (2) implies that $(by\Delta - c\delta)L - c\delta^2L_t = 0$. Thus, we will have a zero corner solution in investment effort, if $b(y\Delta/\delta) - c < 0$, for $b \in [0,1]$. This is the case where the ratio of the marginal benefit of investment to the relative difficulty of investment $b(y\Delta/\delta)$ is lower than the cost of effort parameter, given the optimal incentive share $b$. We will return to this interesting corner solution, when we consider the principal’s problem.

Assuming that a solution exists, we can solve (2) and (3) for the equilibrium values of $L_t$ and $L$ as functions of $b$. We have

$$L_t = \frac{a(by\Delta - c\delta)}{\Delta(2\delta c - by\Delta)} \equiv L_t(b), \text{ and}$$

$$L = \frac{ac\delta^2}{\Delta(2c\delta - by\Delta)} \equiv L(b). \quad (5)$$

At this point, we note that the principal will choose $b \in (0,1)$. Negative $b$ are ruled out by limited liability, and $b = 0$ would, by (2) and (3), imply that both efforts and profit are zero. A

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6 This form of limited liability does not allow payments from the agent to the principal.
\( b \geq 1 \), would imply that the firm’s profit is non-positive, under limited liability. Thus, we focus on \( b \in (0, 1) \).

From (4) and (5), for either effort level to be well defined and for \( L \) to be non-negative, it is necessary that \( 2 > y\Delta/\delta c \), for \( b \in (0, 1) \). Further, given \( 2 > y\Delta/\delta c \), we see that \( L_1 \geq 0 \) if and only if \( b(y\Delta/\delta c) \geq 1 \). Hence, it is necessary that \( y\Delta/\delta c > 1 \), if there is to be a positive solution \( L_1(b) > 0 \).

The ratio \( y\Delta/\delta c \), which we denote as \( \beta \), is a measure of the strength of the complementarity of the two tasks of working and investment in cash flow to the strength of these same two tasks as substitutes in the agent’s effort cost. That is, \( y\Delta = \partial^2[y(a + \Delta L_t)]/\partial L \partial L_t \) and \( \delta c = \partial^2[\delta L_t + L]/\partial L \partial L_t \). In what follows, we will refer to \( \beta \) as the C-S ratio and assume

\[ \text{A.1} \] The C-S ratio satisfies \( 2 > \beta > 1 \).

It turns out that the condition \( 2 > \beta \) in A.1 implies that the agent’s objective function is strictly concave in \((L_t, L)\) for \( b \in [0, 1] \).

Next consider the principal’s problem. We are interested in characterizing two types of solutions. The first is a solution in which both work effort and investment effort are positive. The second is a corner solution where work effort is positive, but investment effort is zero. We will begin by characterizing the conditional second best contract \((S, b)\) for the principal which would generate an interior solution \( L_1(b), L(b) > 0 \). By conditional, we mean conditional on generating an interior solution in both efforts. We will be interested in characterizing the set of parameters \( \beta \) for which this conditional "investment" solution exists. Keep in mind that this solution is profit maximizing, conditional on obtaining an interior solution in efforts (best in the set of interior solutions). It can be that a corner solution in investment effort may dominate the conditional investment solution for the principal, for some parameter values. In Section 3, we will show that in a subset of parameters
for which the conditional investment solution exists, the principal shuts down investment because
she or he prefers a contract which generates a corner solution in investment effort, $L_q = 0$.

Following our characterization of the conditional investment equilibrium for positive efforts, we
will characterize the set of parameters $\beta$ for which the agent sets investment effort at zero, given
the principal’s optimal contract for such $\beta$. Given such $\beta$, the agent’s new incentive compatibility
constraint for $L$ alone generates a new profit function for the principal. For this case, we will
characterize the principal’s optimal contract.

The principal will choose $S$ and $b$ so as to maximize the firm’s residual profit subject to the
agent’s limited liability constraints and the agent’s participation constraint.\footnote{Limited liability
here means that the principal does not stipulate any negative payments for the agent to pay as
part of the contract. All payments to the agent must be nonnegative.} The principal’s problem is to

$$
\max_{\{S, b\}} \{-S + (1 - b)y(a + \Delta L_L(b))L(b)\}
$$

subject to limited liability and participation:

$$
S, b \geq 0, \quad \text{and} \quad (\text{LL})
$$

$$
S + by(a + \Delta L_L(b))L(b) - \left(\frac{C}{2}\right)(\delta L_L(b) + L(b))^2 \geq 0. \quad (P)
$$

The following result allows us to simplify the principal’s problem.

**Lemma 1.** At a solution to the principal’s problem, $S = 0$ and the participation constraint is
non-binding for any $b \in (0, 1]$ and any solution with $L > 0$.

Using the result of Lemma 1, we can employ (4) and (5) to rewrite the principal’s problem as

$$
\max_{\{b\}} P(b) = \frac{ya^2c^2\delta^3}{\Delta} \frac{(1 - b)}{(2c\delta - by\Delta)^2}. \quad (7)
$$
The $b$ which solves (7) is

$$b = 2\left(1 - \frac{1}{\beta}\right).$$  \hspace{1cm} (8)

The optimal $b$ is a positive fraction under A.1. Note that the power of the contract, $b$, is increasing in the C-S ratio, $\beta$.

In the following result, we address the satisfaction of the second order condition to the principal’s problem and the question of whether the equilibrium $b$ is unique.

**Lemma 2.** There is a $b^* \in (0, 1)$ for which (8) holds, the second derivative of the objective function of problem (7) is negative at $b^*$, and $b^*$ is a unique interior maximizer of $P(b)$.

The condition that $2 > \beta$ is the condition which guarantees that the agent’s objective function is strictly concave, and it also implies that $b < 1$. The condition A.1, $2 > \beta > 1$, implies that there is an interior optimizer in $b$.

Next, we can substitute the optimal $b$ into the expressions for $L_1$ and $L$ given by (4) and (5). We have

$$L_1 = \frac{a(y\Delta - (3/2)c\delta)}{\Delta(2c\delta - y\Delta)}, \text{ and } L = \frac{ac\delta^2}{2\Delta(2c\delta - y\Delta)}. \hspace{1cm} (9)$$

Both $L_1$ and $L$ are well defined under A.1. Further, for an interior solution in $L_1$ we require that $\beta - (3/2) > 0$. Under this condition, we have that $\beta - 1 > 0$, so that, from (8), $b > 0$. We then have that an interior equilibrium $(L_1, L) > 0$ is guaranteed if

$$2 > \beta > \frac{3}{2}. \hspace{1cm} (10)$$

Condition (10) and the solution for $b$ imply that

$$b \in (2/3, 1).$$
For the investment equilibrium to be feasible, the two equilibrium probabilities, $L_l$ and $\hat{a}L$ must be in the unit interval. Given our assumptions, it suffices that $L_l, L < 1$. From (9), $L_l < 1$ if the feasibility condition

$$\Delta(\frac{2 - \Delta y}{\delta} - 3/2) > a$$

holds. Likewise, $L < 1$ if

$$\Delta(\frac{4}{\delta} - 2 \frac{\Delta y}{\delta a}) > a.$$ \(^9\)

If $\beta - \frac{3}{2} \leq 0$, then $L_l$ will be zero, and the agent will not invest. In this corner solution, the first order condition for the agent’s optimal choice of work effort, $L$, is $L = (bya)/c \equiv L(b)$. The principal’s optimal choice of $S$ is 0, and the profit function is

$$\hat{P}(b) = \frac{y^2a^2}{c}(1 - b)b,$$

where the optimal choice of $b$ can be shown to be $b = 1/2$. The equilibrium

$$L = ya/2c$$

is in the unit interval if

$$\frac{2c}{y} > a.$$ \(^{\text{F3}}\)

We have

**Proposition 1.** For both investment effort and cash flow effort to be positive and described by (9) and for the conditional second best to be characterized as $b = 2(1 - \frac{1}{y}) \in (2/3, 1)$, it suffices that $2 > \beta > 3/2$. At such an interior solution in efforts, the participation constraint is non-binding

\(^9\)Conditions F1 and F2 are independent, under our assumptions.
and \( S = 0 \). If \( 1 < \beta \leq 3/2 \), then investment effort will be zero, work effort is given by \( L = ya/2c \), and the optimal \( b = 1/2 \). At such a corner solution, the participation constraint is non-binding and \( S = 0 \).

In the conditional investment equilibrium, the principal sets the incentive share in the range where a majority of the cash flow goes to the agent. The investment equilibrium is, therefore, implemented with a high powered contract. If the C-S ratio \( \beta \) falls below a lower bound, the principal issues a lower incentive payment, and the agent ceases human capital investment and only exerts work effort. In this non-investment equilibrium, the principal’s optimal incentive share for the agent is lower or less high powered than it is in the investment equilibrium.

The comparative static results at the two solutions are straightforward and are as expected, with cost parameters decreasing effort and productivity parameters increasing efforts. The comparative static results on the incentive share at the interior solution point out that in firms where investment is less difficult and/or where investment is more productive, the sensitivity of the optimal contract will be greater in that agents will be promised greater contingent shares of the firm. The incentive share becomes less when efforts are more costly.

Whether the agent spends more time working as opposed to skill development depends on the difficulty of investment parameter. We have

**Proposition 2.** If \( \delta \geq 1 \), then \( L > L_d \). There exists a \( \hat{\delta} \in (0,1) \) such that if \( \delta < \hat{\delta} \), \( L_d > L \).

The threshold value \( \hat{\delta} \) is decreasing in \( \gamma \) and increasing in \( \Delta \).

The second best then admits solutions where there is more investment effort than work effort, if the difficulty of investment is sufficiently lower than that of working, \( \delta < \hat{\delta} < 1 \). Otherwise if \( \delta > \hat{\delta} \), work effort will exceed investment effort. A larger productivity increase, \( \Delta \), as the result of successful investment effort or a smaller marginal cost of effort, \( c \), will increase that threshold.

It is also of interest to examine how the agent’s choice of efforts differ from those that would
be chosen by a social planner at a first best solution. The social planner’s first best problem is to choose \( L_t \) and \( L \) so as to maximize

\[
W(L_t, L) \equiv ay(a + \Delta L_t)L - \frac{c}{2}(\delta L_t + L)^2.
\]

Under A.1’s restriction \( 2 > \beta \), it can be shown that \( W \) is strictly concave in \((L_t, L)\). If an interior solution exists, the first order conditions define a maximum of welfare. They are given by

\[
L_t : \quad y\Delta L - c\delta(\delta L_t + L) = 0, \quad \text{and} \quad (13)
\]

\[
L : \quad y(a + \Delta L_t) - c(\delta L_t + L) = 0. \quad (14)
\]

Solving (13) and (14), we obtain first best effort levels

\[
L^f = \frac{ac\delta^2}{\Delta(2c\delta - y\Delta)}, \quad \text{and} \quad L_t^f = \frac{a(y\Delta - c\delta)}{\Delta(2c\delta - y\Delta)}. \quad (15)
\]

We note that the planner would want to set human capital investment to zero if the condition \( \beta - 1 \leq 0 \) is met. However, this is ruled out by A.1. This is a weaker condition than the condition for zero human capital investment at the second best. That is, the upper threshold for the C-S ratio which causes the planner to cut off human capital investment \((\beta \leq 1)\) is lower than that of the firm at the second best \((\beta \leq 3/2)\). Let an \( f \) superscript denote the first best. If \( L_t^f = 0 \), then, at a non-investment equilibrium,

\[
L^f = ay/c. \quad (16)
\]

\(^{10}\)For \( L_t^f < 1 \), the feasibility condition \( \Delta \left( \frac{2c}{\Delta} - \frac{1}{2} \right) > a \) is necessary and sufficient. This condition implies F1. For \( L_t^f < 1 \), the condition \( \Delta \left( \frac{2}{\beta} - \frac{1}{2} \Delta \right) > a \) is necessary and sufficient. This condition implies F2.
Each of the first best efforts at the interior and corner solutions ((15) and (16)) are greater than their counterparts at the second best solutions ((9) and (11)). This is expected since with $b < 1$ the agent does not internalize the full marginal benefit of his or her actions.

At the first best interior solution, as in the second best above, the planner would want more effort to be devoted to investment than to work only if the marginal cost of investment effort is significantly less than that of working. We have

**Proposition 3.** If $\delta \geq 1$, then $L^I > L^I_1$. There exists a $\delta^f \in (0, 1)$ such that if $\delta < \delta^f$, $L_1 > L$.

The threshold value $\delta^f$ is decreasing in $c$ and increasing in $\Delta$.

### 3. The Equilibria

In this section, we will investigate globally optimal contracting by the principal and the welfare of the agent under different parametric configurations. From the basic model, we know that there are two possible equilibria. One equilibrium is such that the agent does not invest. At this equilibrium, we have $L_I = 0$, $L > 0$ and $b = 1/2$. This is the non-investment equilibrium and it occurs if $\beta \in (1, 3/2]$. The other equilibrium is the conditional investment equilibrium where $L_I, L > 0$ and $b = 2(1 - \frac{1}{\beta}) \in (2/3, 1)$, and it obtains when $\beta \in (3/2, 2)$. If $\beta \in (3/2, 2)$, we have not ruled out the possibility that the principal might prefer a corner solution, with $L_I = 0$, to the conditional investment equilibrium. The following proposition addresses this issue.

**Proposition 4.** The principal’s optimal contract and the possible equilibria are described as in the following.

(i) If $\beta \in (0, \frac{1}{2}(1 + 5^{1/2}))$, the principal sets $b = 1/2$ and $S = 0$. The agent sets investment effort at zero and work effort is $L = ya/2c$. The principal and the agent have positive expected payoffs. In the intermediate region $\beta \in (3/2, \frac{1}{2}(1 + 5^{1/2}))$, the agent prefers an investment contract, but the principal prefers and invokes a non-investment contract.
(ii) If $\beta \in \left[\frac{1}{2}(1 + 5^{1/2}), 2\right)$, the principal sets $b = 2(1 - 1/\beta) \in (2(1 - 1/(\frac{1}{2}(1 + 5^{1/2}))), 1)$ and $S = 0$. The agent sets positive investment effort and work effort as in (9). The principal and the agent have positive expected payoffs.

C-S ratios which are too small result in an equilibrium involving no investment on the part of the agent. At these small C-S ratios, the agent does not wish to exert positive investment effort, given optimal actions by the principal. It is not profitable for the principal to incentivize multitasking because the strength of complementarity of the two tasks in cash flow is small relative to the strength of the tasks as substitutes in the agent’s effort cost. There is an intermediate region of C-S ratios over which the agent would prefer the investment equilibrium, but the principal does not. In this region, although the agent would like to supply positive investment effort at the principal’s conditional investment incentive share, the principal would make greater profit if the non-investment equilibrium is invoked. If C-S ratios are above a threshold, then both the principal and the agent benefit from an investment type equilibrium, and it is invoked by the principal with a high powered contract. Here, the strength of complementarity of the two efforts in cash flow is strong relative to the strength of substitutability of the efforts in effort cost. This result is in striking contrast to the standard result of the multitasking models (Holmstrom and Milgrom 1991) where zero incentive payments can be optimal. Our model deviates from this approach due to the assumption of limited liability.

Let $\pi_{nl}$ and $\pi_l$ denote the principal’s expected profit in the investment and non-investment equilibria respectively, and let $u_{nl}$ and $u_l$ denote the agent’s expected utility at the same equilibria. In the intermediate region of C-S ratios, $\beta \in (3/2, \frac{1}{2}(1 + 5^{1/2}))$, the agent gains expected utility $\Delta u \equiv u_l - u_{nl} > 0$ by investment effort, and the principal loses profit with investing effort, $\Delta \pi \equiv \pi_{nl} - \pi_l > 0$. Define the difference $R = \Delta u - \Delta \pi$. If this were positive, then it would be more efficient or total surplus would be greater, if the investment equilibrium were implemented. Indeed
we have

**Lemma 3.** If $\beta \in (3/2, \frac{1}{2}(1 + 5^{1/2}))$, then $R > 0$.

While it is Pareto efficient for the firm to move to the investment equilibrium from the non-investment equilibrium in the intermediate region of C-S ratios, the firm, of course, would not do this because it does not maximize profit. Limited liability of the agent prevents the agent from offering the principal a lump sum transfer of $\Delta \pi$ as an inducement to implement the investment equilibrium. In the low state of the world with $\hat{y} = 0$, the agent could not pay the transfer. A transfer contingent on the good state would of course change the calculus of the agent’s optimization problem, affect effort and then unravel a transfer scheme from the agent to the principal.

Finally consider the case where skills increases are firm specific. If $\Delta$ is thought of as a skills gap between what skill level the principal desires and that which is available on the market, then the reticence of the principal to incentivize investment effort in the intermediate interval, where the employee is willing and the firm is unwilling, may help explain the persistence of a skills gap. In the above model, it is caused by the high standards on the C-S ratio of incentivizing concurrent investment effort optimally maintained by the firm. It is an "unwillingness to induce investment effort problem" in the face of low C-S ratios of skill development and working.

### 4. Conclusion

The main takeaways of this paper are succinctly described in the following. We find that concurrent work and productivity investment efforts are profitable for the firm only if the C-S ratio associated with productivity investment while working exceeds a threshold, and we show that the employee would like to see that threshold lowered. We show that it is not possible for the agent to issue a lump sum transfer to the principal in an intermediate region of C-S ratios so as to induce the principal to implement an investment contract below the principal’s threshold but above the agent’s lower
threshold. This inability for a Pareto efficient transfer is due to the agent’s limited liability. Thus, zero investment effort results, and this is inefficiently low. A skills gap may arise in cases where skills are firm specific due to the higher pre-conditions the firm optimally places on incentivizing concurrent skill development.

The investment multitasking contract is always higher powered than the non-investment contract, investment effort can be less or greater than work effort, and both of these efforts are less than their socially efficient levels. Work effort is always greater than investment effort if investment effort is at least as difficult as work effort. That multitasking requires a higher powered contract flies in the face of traditional multitasking results. The reason for this different implication is the presence of limited liability.

Finally, the results of the basic model hold for applications where the agent invests in enhancement of the production process as opposed to skill development effort. Inducing the multitasking of such effort with work effort also requires a higher powered contract. Moreover, if the C-S ratio is below the principal’s threshold, enhancement of the production process by the agent would fall below an efficient level.

Appendix

Proof of Lemma 1: The relevant Lagrangian is

\[
\mathcal{L} = -S + (1 - b)g(a + \Delta L_t(b))L(b) + \lambda[S + b_{\text{y}}(a + \Delta L_t(b))L(b) - \left(\frac{c}{2}\right)(\delta L_t(b) + L(b))^2] + \mu_S S + \mu_b b.
\]

The first order condition for \( S \) is

\[
-1 - \lambda + \mu_S = 0.
\]

By \( \lambda \geq 0 \), we have that \( \mu_S > 0 \). Thus, \( S = 0 \).
With $S = 0$, the agent’s objective function, $A(L_l, L) = by(a + \Delta L_l)L - (\frac{\xi}{2})(\delta L_l + L)^2$, is strictly concave in $(L_l, L)$, under $2 > \beta$. Further, $A(0,0) = 0$. The gradient condition for a strictly concave function then tells us that $A(L_l, L) > A_{L_l} \cdot (L_l) + A_L \cdot (L) = 0$, in equilibrium, with $L > 0$ and $L_l \geq 0$. It follows that (P) is non-binding. ■

**Proof of Lemma 2:** Taking the second derivative of the objective function of (7), we obtain

$$\frac{\partial^2 P}{\partial b^2} = \frac{2a^2c^2\delta^3y^2[-4c\delta - (3 + b)y\Delta]}{(-2c\delta + by\Delta)^2}. $$

The sign of this expression is that of $-4c\delta + (3 - b)y\Delta$. If we substitute for the optimal $b^*$ from (8), we obtain $-2c\delta + y\Delta$, which is assumed to be negative.

At any critical point, $P(b^*)$ is locally strictly concave. Given that the function is twice differentiable and, thus, has continuous first derivative, the existence of more than one interior critical point would imply that the negative second derivative property at an interior critical point would be violated at some point on the convex combination of the two alleged maxima. Thus, uniqueness holds. Finally, existence of $b^*$ is guaranteed by the conditions $P'(0) > 0$ and $P'(1) < 0$. By continuity there is a critical point, $b^*$. First note that $sign P'(0) = sign (4c\delta - c\delta + y\Delta) > 0$, by $\beta > 1$. Next, $P'(1) = \frac{-y\alpha^2c^2\delta^3}{\Delta(2c\delta - \Delta y)^2} < 0$. ■

**Proof of Proposition 1:** The text proves all but the statements regarding the solution with $L_l = 0$. For this case, the principal’s Lagrangian is

$$\mathcal{L} = -S + (1 - b)yaL(b) + \lambda[-S + byaL(b) - \frac{c}{2}L(b)^2] + \mu_S S + \mu_y b.$$

The first order condition for $S$ is again

$$-1 - \lambda + \mu_S = 0.$$

By $\lambda \geq 0$, we have that $\mu_S > 0$. Thus, $S = 0$. From (13), $L = bya/c$, so that participation constraint
is now given by  

\[
bya(bya/c) - \frac{c}{2}(bya/c)^2 > 0.
\]

The principal solves  

\[
\max_{b} \frac{y^2a^2}{c}(1 - b)b.
\]

It follows that \( b = 1/2 \). \( \blacksquare \)

**Proof of Proposition 2:** We have  

\[
L - L_I = \frac{a(c\delta^2 + 3c\delta - 2y\Delta)}{2(2c - y\Delta)}
\]

and the sign of this expression is that of \( \delta + 3 - 2\beta \). However, \( \beta \) takes on a supremum value of 2, so that \( \delta + 3 - 2\beta > 0 \) and work effort is greater than investment effort if \( \delta \geq 1 \).

If \( \delta < 1 \), then we need more information to determine the sign of (15). The threshold level of \( \delta, \hat{\delta}^s \), which sets \( \delta + 3 - 2\beta = 0 \) is defined by  

\[
\hat{\delta}^s = \left(\frac{9c^2 + 8cy\Delta}{2c}\right)^{1/2} - 3c.
\]

It is clear that \( \hat{\delta}^s > 0 \), from the fact that the numerator of \( \hat{\delta}^s \) is positive. For \( \hat{\delta}^s < 1 \), we would require \( 2c > y\Delta \). Under A.1, we have that \( 2c\delta > y\Delta \), so that indeed \( 2c > y\Delta \), if \( \delta < 1 \). Thus, under A.1, there would be a \( \delta \in (\frac{y\Delta}{2c}, \hat{\delta}^s) \) such that \( L - L_I < 0 \). Note that \( \frac{y\Delta}{2c} < \hat{\delta}^s \), by A.1, so that the latter interval is non-empty. We have that \( \hat{\delta}^s \) is clearly increasing in \( \Delta \), and  

\[
\frac{d\hat{\delta}^s}{dc} = -\frac{2y\Delta}{c\sqrt{c(3c + 8y\Delta)}} < 0.
\]

**Proof of Proposition 3:** From (15),  

\[
\text{sign} (L^f - L_I^f) = \text{sign}(c\delta^2 + c\delta - y\Delta) = \text{sign}(1 + \delta - \beta),
\]

with \( \beta \in (1,2) \). For \( \delta \geq 1 \), the maintained condition \( 2 > \beta \) implies that \( 1 + \delta - \beta > 0 \), so that it is socially optimal for the agent to devote more time to working than to learning. For \( \delta < 1 \), the critical \( \hat{\delta}^f \) which sets \( (c\delta^2 + c\delta - y\Delta) = 0 \) is  

\[
\hat{\delta}^f = \left(\frac{c^2 + 4cy\Delta}{2c}\right)^{1/2} - c.
\]

Note that \( \hat{\delta}^f > 0 \). Also \( \hat{\delta}^f < 1 \), if and only if \( \frac{y\Delta}{2c} < 1 \). By A.1, \( \frac{y\Delta}{2c} < \delta < 1 \). Moreover, \( \hat{\delta}^f > \frac{y\Delta}{2c} \) if \( \frac{y\Delta}{2c} < 1 \), which is again implied by A.1. For \( \delta \in (\frac{y\Delta}{2c}, (\frac{c^2 + 4cy\Delta}{2c})^{1/2} - c) \), it is socially optimal for the agent to devote more time to investment than to working. Clearly, \( \hat{\delta}^f \) is increasing in \( \Delta \), and  

\[
\frac{d\hat{\delta}^f}{dc} = -\frac{y\Delta}{c\sqrt{c(3c + 4y\Delta)}} < 0.
\]

**Proof of Proposition 4:** In the non-investment equilibrium, the principal’s expected profit and the agent’s expected utility are  

\[
\pi_{nl} = \frac{a^2\mu^2}{4c}, \quad u_{nl} = \frac{a^2\mu^2}{8c},
\]

respectively. In an investment equilib-
rium, the principal’s expected profit and the agent’s expected utility become \( \pi_l = \frac{a^2c^2\delta^3}{4\Delta^2(2\delta - \Delta y)} \), and

\[ u_t = \frac{a^2c\delta^2(\Delta y - \delta)}{2\Delta^2(2\delta - \Delta y)} \]  

The principal would prefer that the agent invest if \( \pi_l > \pi_{nl} \), assuming that the sufficiency conditions for the investment equilibrium are met. We can write this condition as

\[ \pi_l > \pi_{nl} \text{ if and only if } 1 - 2\beta^2 + \beta^3 > 1, \text{ for } \beta \in (3/2, 2). \]

The function \( f(\beta) = 1 - 2\beta^2 + \beta^3 \) is strictly increasing in the C-S ratio \( \beta \) with \( f(3/2) = -0.125 \) and \( f(1/2) = 0 \). Clearly, the principal prefers the investment equilibrium only in the interval \( \beta \in (1/2, 1 + 5^{1/2}) \).

Next, consider the agent’s welfare. If an investment contract is offered and the sufficiency conditions for \( L_t > 0 \) are met, \( \beta \in (3/2, 2) \), then the agent would prefer the investment equilibrium if \( u_t > u_{nl} \). We can write this condition as

\[ u_t > u_{nl} \text{ if and only if } \beta^3 + 4\beta - 2\beta^2 - 4 > 0, \text{ for } \beta \in (3/2, 2). \]

The function \( g(\beta) = \beta^3 + 4\beta - 2\beta^2 - 4 \) is strictly increasing in \( \beta^3 + 4\beta - 2\beta^2 - 4 \) with \( g(3/2) = 0.875 > 0 \). Thus, the agent prefers investment in the entire feasible region for the C-S ratio \( \beta \). That is, unlike the principal, the agent prefers the investment equilibrium in the intermediate region of C-S ratios given by \( \beta \in (3/2, 1/2) \).

Finally consider the case where \( \beta \in (0, 1/2) \). For \( \beta \) in this region, the principal prefers the non-investment equilibrium. If we can show that the agent selects this equilibrium if \( b = 1/2 \), then the result obtains. If the principal sets \( b = 1/2 \), the agent optimizes over a selection of \( L_t \) according to the FOC (2) with \( b = 1/2 \). This condition is \( (y\Delta/2 - \delta c)L - \delta^2 cL_t = 0 \). Thus, for a positive solution in \( L_t \), it is necessary that \( (y\Delta/2 - \delta c) > 0 \) or \( \beta > 2 \). The latter is impossible since \( \beta < 2 \).
Proof of Lemma 3: We have that
\[ R = \frac{-a^2(2c^3 \delta^3 - 4c^2 \delta^2 \Delta y + 6y^2 \Delta^2 \delta - 3y^3 \delta^3)}{8c \Delta^2(2c \delta - \Delta y)}, \]
so that, given our assumptions, the sign of \( R \) is given by the sign of
\[-2c^3 \delta^3 + 4c^2 \delta^2 \Delta y - 6y^2 \Delta^2 \delta + 3y^3 \delta^3.\]
Dividing this expression by \( c^3 \delta^3 > 0 \), we obtain
\[-2 + 4\beta - 6\beta^2 + 3\beta^3.\]
This expression is positive for \( \beta \in (3/2, \frac{1}{2}(1 + 5^{1/2})) \). ■

References


