

# Maximizing Stochastic Monotone Submodular Functions

Arash Asadpour

Stern School of Business, New York University, New York, NY, aasadpou@stern.nyu.edu

Hamid Nazerzadeh

Marshall School of Business, University of Southern California, Los Angeles, CA, hamidnz@marshall.usc.edu

We study the problem of maximizing a stochastic monotone submodular function with respect to a matroid constraint. Due to the presence of diminishing marginal values in real-world problems, our model can capture the effect of stochasticity in a wide range of applications. We show that the adaptivity gap — the ratio between the values of optimal adaptive and optimal non-adaptive policies — is bounded and is equal to  $\frac{e}{e-1}$ . We propose a polynomial-time non-adaptive policy that achieves this bound. We also present an adaptive myopic policy that obtains at least half of the optimal value. Furthermore, when the matroid is uniform, the myopic policy achieves the optimal approximation ratio of  $1 - \frac{1}{e}$ .

*Key words:* submodular maximization, stochastic optimization, adaptivity gap, influence spread in social networks

*Subject classifications:* Analysis of algorithms: performance guarantee; Nonlinear programming: submodular maximization; Programming: stochastic optimization.

*Area of review:* Optimization.

---

## 1. Introduction

The problem of maximizing submodular functions has been extensively studied in operations research and computer science. For a ground set  $\mathcal{A}$ , the set function  $f : 2^{\mathcal{A}} \rightarrow \mathbb{R}$  is called *submodular* if for any two subsets  $S, T \subseteq \mathcal{A}$ , we have  $f(S \cup T) + f(S \cap T) \leq f(S) + f(T)$ . An equivalent definition is that the marginal value of adding any element is diminishing. In other words, for any  $S \subseteq T \subseteq \mathcal{A}$  and  $j \in \mathcal{A}$ , we have  $f(T \cup \{j\}) - f(T) \leq f(S \cup \{j\}) - f(S)$ . Also, a set function  $f$  is called *monotone* if for any two subsets  $S \subseteq T \subseteq \mathcal{A}$ , we have  $f(S) \leq f(T)$ .

Due to the presence of diminishing marginal values, a wide range of optimization problems that arise in the real world can be modeled as maximizing a monotone submodular function with respect to some feasibility constraints. One instance is the welfare maximization problem (see, e.g., Dobzinski and Schapira 2006; Vondrák 2008; Feige 2009), which is to find an optimal allocation of resources to agents where the utilities of the agents are submodular. Here, submodularity corresponds to the law of diminishing return in the economy (see Samuelson and Nordhaus 2009).

Another application of this problem is capital budgeting in which a risk-averse investor with a limited budget is interested in finding the optimal investment portfolio over different projects (see, e.g., Weingartner 1963; Ahmed and Atamtürk 2011). Risk aversion can be modeled by concave utility functions where, informally speaking, the utility of a certain expectation is higher than the expectation of the utilities of the corresponding uncertain outcomes. As argued by Ahmed and Atamtürk (2011), the utility function of a risk-averse investor over a set of investment possibilities can be modeled by submodular functions. Such functions are also non-negative and monotone by nature.

Another example is the problem of viral marketing and maximizing influence through the network (see, e.g., Kempe et al. 2003; Mossel and Roch 2010), where the goal is to choose an initial “active” set of people and to provide them with free coupons or promotions so as to maximize the spread of a technology or behavior in a social network. It is well known that under a wide variety of models for influence propagation in networks (e.g., cascade model of Kempe et al. 2003), the expected size of the final cascade is a submodular function of the set of initially activated individuals. Also, due to budget limitations, the number of people that we can directly target is bounded. Hence, the problem of maximizing influence can be seen as a maximizing submodular function problem subject to cardinality constraints.

Yet another example is the problem of optimal placement of sensors for environmental monitoring (see, e.g., Krause and Guestrin 2005, 2007) where the objective is to place sensors in the environment in order to most effectively reduce uncertainty in observations. This problem can be modeled by entropy minimization and, due to the concavity of the entropy function, it is a special case of submodular optimization.

For the above problems and many others, the constraints can be modeled by a matroid. A finite *matroid*  $\mathcal{M}$  is defined by a pair  $(\mathcal{A}, \mathcal{I})$ , where  $\mathcal{A}$  is a ground set of elements and  $\mathcal{I}$  is a collection of subsets of  $\mathcal{A}$  (called the *independent sets*) with the following properties:

1. Every subset of an independent set is independent, i.e., if  $S \in \mathcal{I}$  and  $T \subseteq S$ , then  $T \in \mathcal{I}$ .
2. If  $S$  and  $T$  are two independent sets and  $T$  has more elements than  $S$ , then there exists an element in  $T$  that is not in  $S$  and when added to  $S$  still gives an independent set.

Two important special cases are *uniform matroid* and *partition matroid*. In a uniform matroid, all the subsets of  $\mathcal{A}$  of size at most  $k$ , for a given  $k$ , are independent. Uniform matroids represent cardinality constraints. A partition matroid is defined over a partition of set  $\mathcal{A}$ , where every independent set includes at most one element from each set in the partition.

The celebrated result of Nemhauser et al. (1978) shows that for maximizing nonnegative monotone submodular functions over uniform matroids, the greedy algorithm gives a  $(1 - \frac{1}{e} \approx 0.632)$ -approximation of the optimal solution. Later, Fisher et al. (1978) showed that for optimizing over

matroids, the approximation ratio of the greedy algorithm is  $\frac{1}{2}$ . Recently, Călinescu et al. (2011) proposed a better approximation algorithm with a ratio of  $1 - \frac{1}{e}$ . It also has been shown that this factor is optimal (in the value oracle model, where we only have access to the values of  $f(S)$  for all the subsets  $S$  of the ground set), if only a polynomial number of queries is allowed (see Nemhauser and Wolsey 1978; Feige 1998).

However, all these algorithms are designed for deterministic environments. In practice, one must deal with the stochasticity caused by the uncertain nature of the problem, the incomplete information about the environment, and so on. For instance, in welfare maximization, the quality of the resources may be unknown in advance, or in the capital budgeting problem some projects taken by an investor may fail due to unexpected events in the market. Yet another example is from viral marketing, where some people whom we target might not adopt the behavior (e.g., receive a coupon or a promotion but not purchase the product). Also, in the environmental monitoring example, it is expected that a non-negligible fraction of sensors might not be functioning properly due to various reasons.

All these possibilities motivate the study of submodular maximization in the stochastic setting. In such settings, the outcome of the elements in the selected set are not known in advance and they will only be discovered after they are chosen.

## Setting

Here, we provide a brief overview of our setting. Later in Section 2, we will discuss the model in details. We study the problem of maximizing a monotone submodular function  $f$  over a set of  $n$  random variables, namely,  $\mathcal{A} = \{X_1, X_2, \dots, X_n\}$ . A policy  $\pi$  picks the elements one by one (perhaps, based on the realized value of the previous elements) until it stops. Once  $\pi$  stops, the state of the world is a random vector  $\Theta^\pi = (\theta_1, \theta_2, \dots, \theta_n)$ , where  $\theta_j$  denotes the realization of  $X_i$ , if  $i$  is picked by the policy, and is equal to 0 otherwise. The objective of stochastic submodular maximization would be to optimize the expected value of a policy, i.e.,  $\text{Maximize}_{\pi} \mathbf{E}[f(\Theta^\pi)]$ , subject to feasibility. We model the feasibility constraint using a matroid. For a given matroid  $\mathcal{M}$  defined on the ground set of the aforementioned random variable set  $\mathcal{A}$ , a policy  $\pi$  is called feasible if the subset of random variables it picks is always an independent set of  $\mathcal{M}$ .

We consider both **adaptive** (general) and **non-adaptive** policies. In adaptive policies, at each point in time all the information regarding the previous actions of the policy is known. In other words, the policy has access to the actual realized value of all the elements it has picked so far. In contrast, non-adaptive policies do not have access to such information and should make their decisions (about which random variables to pick) before observing the outcome of any of them.

The main reason behind considering the notion of adaptivity is that they could be very hard, sometimes even impossible, to implement in practice due to various reasons. We will discuss this

in more detail later, but to name a couple, it could be very costly to wait long enough to observe the outcome of the previous actions, or sometimes it is not possible to measure such outcomes accurately. On the other hand, non-adaptive policies are highly restricted and hence may perform poorly in terms of value compared to adaptive ones. To study this tradeoff between simplicity and value, we study the notion of **adaptivity gap**, which is defined as the ratio of the expected value of the optimal adaptive policy versus the expected value of the optimal non-adaptive one.

### 1.1. Contributions

We show that the adaptivity gap of the stochastic monotone submodular maximization (SMSM) problem is equal to  $\frac{e}{e-1} \approx 1.59$ . In other words, we prove that there exists a non-adaptive policy that achieves at least an  $\frac{e-1}{e}$  fraction of the value of the optimal adaptive policy. We also provide an example to show that our analysis of the adaptivity gap is tight. For that, we use a simple instance of the stochastic max  $k$ -cover problem.

We will then present a generalization of the continuous greedy method of Călinescu et al. (2011) and *construct* a non-adaptive policy that approximates the optimal adaptive policy within a factor of  $\frac{e-1}{e} - \epsilon$  for any arbitrary  $\epsilon$ . Our algorithm runs in polynomial time in terms of the size of the problem and  $1/\epsilon$ . This policy does not necessarily coincide with the optimal non-adaptive policy; however, due to the tightness of our adaptivity gap result, this is essentially the best approximation ratio that one could hope for with respect to the optimal adaptive policy.

In Section 5, we focus on myopic policies. We study the natural extension of the myopic policy studied in Fisher et al. (1978) to a stochastic environment. This policy iteratively chooses an element with the maximum *expected* marginal value, *conditioned* on the outcome of the previous elements. We show that the approximation ratio of this policy with respect to the optimal adaptive policy is  $\frac{1}{2}$  for *general matroids*. In addition, we show that the approximation ratio of this adaptive myopic policy is  $\frac{1}{\kappa+1}$  if the feasible domain is given by the intersection of  $\kappa$  matroids. We also prove that over a uniform matroid (i.e., subject to a cardinality constraint), the approximation ratio of this policy is  $1 - \frac{1}{e}$ . We will discuss the results of Nemhauser and Wolsey (1978) and Feige (1998) to show that the approximation ratio of  $1 - \frac{1}{e}$  is optimal if only polynomial algorithms are allowed.

This paper settles the status of SMSM problem as presented in Table 1.1. The value of the optimal adaptive policy is denoted by  $\text{OPT}_{\text{NA}}$ . The bounds marked by † are tight.

### 1.2. Related Work

Goemans and Vondrák (2006) proposed adaptive and non-adaptive policies for the stochastic covering problem where the goal is to cover all elements of a target set using stochastic subsets with the minimum (expected) number of subsets. Guillory and Bilmes (2010) studied a similar set cover

**Table 1** Bounds for the Algorithms and the Adaptivity Gap

Constraint	NON-ADAPTIVE	MYOPIC ADAPTIVE	ADAPTIVITY GAP
UNIFORM MATROID	Myopic Policy $\frac{e-1}{2e} \text{OPT}_A$ [Proposition 2]	$\frac{e-1}{e} \text{OPT}_A$ † [Theorem 4]	$\frac{e}{e-1}$ † [Theorem 1]
GENERAL MATROID	Generalized Continuous Greedy $\frac{e-1}{e} \text{OPT}_A$ † [Theorem 2]	$\frac{1}{2} \text{OPT}_A$ † ( $\frac{1}{\kappa+1} \text{OPT}_A$ for the intersection of $\kappa$ matroids) † [Theorem 3]	

problem but they took a worst-case analysis approach. They showed that in the worst-case, the adaptivity gap can be very large and they provide optimal (up to a constant factor) algorithms for their problem.

Since the publication of our preliminary results in Asadpour et al. (2008), adaptive policies for other interesting notions of stochastic submodular maximization have been studied. Golovin and Krause (2011b) studied the notion of *adaptive submodularity*; a property that asks for the elements (or actions) chosen by a policy to have diminishing conditional expected marginal returns. They demonstrated the effectiveness and applicability of this notion over a wide range of important optimization problems by providing approximately optimal adaptive algorithms. In a follow-up work, Golovin and Krause (2011a) extended their results to settings with knapsack or matroids constraints. In addition, they provided examples that show that the adaptivity gap for certain covering problems (where the objective is to *minimize* the cost) may not be constant.

Very recently, Adamczyk et al. (2014) studied the problem of stochastic submodular probing with applications in online dating, kidney exchange, and Bayesian mechanism design. This problem investigates a generalized notion of adaptive policies. In the model, a ground set of elements is given where every element is *active* independently at random with some pre-known probability. A policy can probe elements sequentially, but if an element turns out to be active, it has to be included in the solution. The goal is, naturally, to find a feasible subset of active elements that maximizes the value of a submodular set function. The feasibility of the eventual subset of elements is determined by  $k^{in}$  matroid constraints. Also, the subset of elements being probed should be feasible and obey  $k^{out}$  matroid constraints. They provide  $(1 - 1/e)/(k^{in} + k^{out} + 1)$ -approximation policies for this problem for  $k^{in} \geq 0$  and  $k^{out} \geq 1$ . We note that as a complementary result to that paper, when  $k^{in} = 1$  and  $k^{out} = 0$ , the non-adaptive myopic policy provided by our Proposition 2 provides a  $(1 - 1/e)/(k^{in} + k^{out} + 1) = \frac{e-1}{2e}$ -approximation for the stochastic submodular probing problem.

Also, the non-adaptive policy resulting from Theorem 2 provides a  $(1 - 1/e - \epsilon)$ -approximation in polynomial time in terms of the size of the problem and  $1/\epsilon$ .

Another closely related work to ours is by Chan and Farias (2009) who studied a rather general stochastic depletion problem. Their optimization problem is defined over a time horizon where at each step a policy decides on an action. Each action stochastically generates a reward and depletes some of the resources. The goal is to maximize the expected reward subject to the resource constraints. They show that under certain “submodularity” constraints, the myopic policy obtains at least half the value of the optimal (offline) solution — a benchmark stronger than the optimal online policy. An important difference between the framework in Chan and Farias (2009) and ours is the ordering imposed in their model on the sequence of actions a policy can take, whereas in our model, in order to prove our results we need to exploit the matroid properties. Although our proposed adaptive policies are not applicable to their setting, our non-adaptive policy can be applied to their setting using a partition matroid to obtain a  $(1 - \frac{1}{e})$ -approximation with respect to the optimal adaptive (online) algorithms.

### 1.3. Organization

The rest of the paper is organized in the following way. In Section 2 we formally introduce our problem and define adaptive and non-adaptive policies and the concept of adaptivity gap. In Section 3 we study the adaptivity gap and show that it is equal to  $e/(e-1)$ . In Section 4 we provide a polynomial-time algorithm to find a non-adaptive policy that matches the adaptivity gap ratio. The simple myopic policies are investigated in Section 5, and worst-case performance guarantees have been provided for them. We discuss some natural variations of our model in Section 6. Finally, we conclude in Section 7 by mentioning some interesting directions for future research.

## 2. Model

In this section, we define our framework for stochastic submodular optimization. Consider a ground set  $\mathcal{A} = \{X_1, X_2, \dots, X_n\}$  of  $n$  independent random variables. Let  $\mathbb{R}_+$  denote the set of non-negative real numbers (including 0). Each random variable  $X_i$  is defined over a domain  $\Omega_i \subseteq \mathbb{R}_+$ , and its probability distribution function over  $\Omega_i$  has density  $g_i$ .<sup>1</sup> Let  $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$  be a *monotone submodular value function*. In other words,

$$\forall x, y \in \mathbb{R}_+^n : f(x \vee y) + f(x \wedge y) \leq f(x) + f(y),$$

where  $x \vee y$  denotes the componentwise maximum and  $x \wedge y$  the componentwise minimum of  $x$  and  $y$ . Note that submodular set functions are special cases of the above definition. This generalization

<sup>1</sup> Throughout this paper, we work with continuous probability distributions. All our theorems and their corresponding proofs directly apply to the probability distributions over discrete domains by using Dirac delta functions.

to the continuous domain allows us to capture wider aspects of the problem. For instance, we can model *partial* contributions from the elements. This could be useful in many of the applications that we have mentioned before. For example, in viral marketing, targeted people could have different levels of enthusiasm about the new product or technology. Or, in the sensor placement problem, some sensors in the network might not become fully functional and cover only a part of the area they were supposed to cover.

A policy — formally defined below — consists of a sequence of actions  $(\pi_1, \pi_2, \dots)$ . Each *action* corresponds to the selection of an element from  $\mathcal{A}$  (or perhaps none, denoted by  $\emptyset$ , corresponding to the empty set). After an element is selected, its actual value is realized. The state of the problem  $\Theta$  is represented by an  $n$ -tuple  $(\theta_1, \theta_2, \dots, \theta_n)$ . Here,  $\theta_i = x_i$  if  $X_i$  has been selected by the policy in any of the previous steps and its actual value is realized to be  $x_i$ ; otherwise  $\theta_i = \circ$ . The symbol  $\circ$ , different from value zero, stands for null, representing that the corresponding element has not been selected yet.

Let  $\Omega := \prod_{i=1}^n (\Omega_i \cup \{\circ\})$  be the set of all possible states. Consider a policy  $\pi$  where the state of the problem before taking the  $j$ -th action is given by  $\Theta_{j-1}^\pi \in \Omega$ . The  $j$ -th action, denoted by  $\pi_j$ , is a mapping from the current state to an element from set  $\mathcal{A} \cup \{\emptyset\}$ . Before taking any action the state of the problem is  $\Theta_0^\pi = (\circ, \circ, \dots, \circ)$ . For instance, if we choose to pick  $X_3$  in our first action (i.e.,  $\pi_1 = X_3$ ) and its value is realized to be  $x_3$ , then the state would be  $\Theta_1^\pi = (\circ, \circ, x_3, \circ, \dots, \circ)$ . If at some step  $j$  the action  $\emptyset$  is selected, then the state of the problem will remain unchanged, i.e.,  $\Theta_j^\pi = \Theta_{j-1}^\pi$ .

We can now formally define the notion of policy.

**Definition 1 (Policy)** *A policy  $\pi : \Omega \rightarrow [0, 1]^n$  is a mapping from the state space  $\Omega$  to a distribution over the elements in  $\mathcal{A} \cup \{\emptyset\}$ . Let  $\Theta_{j-1}^\pi = (\theta_1, \theta_2, \dots, \theta_n)$  represent the state of the problem before taking the  $j$ -th action. Then,  $\pi(\Theta_{j-1}^\pi)$  is a distribution over  $\{X_k | \theta_k = \circ\} \cup \{\emptyset\}$ . If the policy is deterministic,  $\pi(\Theta_{j-1}^\pi)$  will correspond to exactly one element in  $\{X_k | \theta_k = \circ\} \cup \{\emptyset\}$ . The action of the policy, denoted by  $\pi_j$ , corresponds to choosing an element of  $\{X_k | \theta_k = \circ\} \cup \{\emptyset\}$  according to distribution  $\pi(\Theta_{j-1}^\pi)$ . Subsequently, if the policy chooses element  $X_i \neq \emptyset$ , the state of the problem after taking the  $j$ -th action will be  $\Theta_j^\pi = (\theta_1, \theta_2, \dots, \theta_{i-1}, x_i, \theta_{i+1}, \dots, \theta_n)$ , where  $x_i$  is the realized value of  $X_i$  (drawn from distribution  $g_i$ ). Finally, if at any point  $\pi_j(\Theta_{j-1}^\pi) = \emptyset$ , then the selection of the elements stops. We denote by  $\Theta^\pi$  the random  $n$ -tuple corresponding to the last state reached by the policy.*

We present the following simple policy as an example.

**Example** Suppose that the ground set of elements is  $\{X_1, X_2, X_3\}$ . Also, suppose that each element, once picked, will have a realized value of either 10 or 100 with probabilities 0.4 and 0.6, respectively. Consider a simple policy  $\pi$  that in the first step picks element  $X_3$ . In the next step,  $\pi$  will pick  $X_1$  if the realized value of  $X_3$  is realized to be 10. Otherwise, it picks either  $X_1$  or  $X_2$ , each of which with probability 0.5. After that,  $\pi$  does not pick anything. Note that as we have explained before, the initial step of the problem is by definition  $\Theta_0^\pi = (\circ, \circ, \circ)$ . The first action of this policy is  $\pi_1 = X_3$ . After that, the state of the problem, i.e.,  $\Theta_1^\pi$ , will be either  $(\circ, \circ, 10)$  or  $(\circ, \circ, 100)$  with probabilities 0.4 and 0.6, respectively.

Now, if  $\Theta_1^\pi = (\circ, \circ, 10)$ , then the second action  $\pi_2(\circ, \circ, 10)$  will be to choose  $X_1$ . Consequently, the state of the problem afterwards, i.e.,  $\Theta_2^\pi$ , will be either  $(10, \circ, 10)$  or  $(100, \circ, 10)$  with probabilities 0.4 and 0.6, respectively. On the other hand, if  $\Theta_1^\pi = (\circ, \circ, 100)$ , then the second action will be a random one that picks either  $X_1$  or  $X_2$ , each of which with probability 0.5. In the first case, the state of the problem, i.e.,  $\Theta_2^\pi$ , will be either  $(10, \circ, 10)$  or  $(100, \circ, 10)$  with probabilities 0.4 and 0.6, respectively. Similarly, in the second case,  $\Theta_2^\pi$  will be either  $(\circ, 10, 10)$  or  $(\circ, 100, 10)$  with probabilities 0.4 and 0.6, respectively.

From the discussion above, we conclude that for this specific policy  $\pi$  we have

$$\Theta_2^\pi = \begin{cases} (10, \circ, 10) & \text{w.p. } 0.4 \times 0.4 = 0.16, \\ (100, \circ, 10) & \text{w.p. } 0.4 \times 0.6 = 0.24, \\ (10, \circ, 100) & \text{w.p. } 0.6 \times 0.5 \times 0.4 = 0.12, \\ (100, \circ, 100) & \text{w.p. } 0.6 \times 0.5 \times 0.6 = 0.18, \\ (\circ, 10, 100) & \text{w.p. } 0.6 \times 0.5 \times 0.4 = 0.12, \text{ and} \\ (\circ, 100, 100) & \text{w.p. } 0.6 \times 0.5 \times 0.6 = 0.18. \end{cases} \quad (1)$$

After that, the policy stops. Hence,  $\Theta^\pi$  will be equal to  $\Theta_2^\pi$ .

Note that a policy sequentially picks new elements from the ground set  $\mathcal{A}$  (perhaps, based on the outcomes of the previous selections). Also, we emphasize that we only consider policies *without* replacements (i.e., only elements whose corresponding entries in the current state are null can be selected). As a consequence, once the value of an element is realized, it will be fixed and cannot be changed by picking that element again. As we discuss in Section 6.2, this assumption is only made to simplify the presentation, and all of our results hold without it.

The value of any state  $\Theta = (\theta_1, \theta_2, \dots, \theta_n)$ , denoted by  $f(\Theta)$ , is equal to the value of function  $f$  over those elements of  $\mathcal{A}$  that already have been chosen. Formally,

$$f(\Theta) := f(\zeta_1, \zeta_2, \dots, \zeta_n), \quad \text{where } \zeta_i = \begin{cases} \theta_i & \theta_i \neq \circ; \\ 0 & \theta_i = \circ. \end{cases} \quad (2)$$

For instance, for a state  $\Theta = (10, \circ, 100)$ , the value of  $f(\Theta)$  is equal to  $f(10, 0, 100)$ . Finally, *the expected value* of a policy  $\pi$  after the  $j$ -th action is equal to  $\mathbf{E}[f(\Theta_j^\pi)]$ . Here, the expectation is taken over all possible realizations of the state  $\Theta_j^\pi$  as an outcome of the first  $j$  actions taken by  $\pi$ .



Throughout this paper the parameter for the expectation will be clear from the context. We will make this parameter explicit wherever some confusion may arise. For instance, if needed, we can represent the aforementioned expectation by  $\mathbf{E}_\Theta[f(\Theta_j^\pi)]$ .

**Definition 2 (Value of a Policy)** *The value of a policy  $\pi$  is defined as the expected value of its final state, i.e.,  $\mathbf{E}[f(\Theta^\pi)]$ .*

Hence, for the example we discussed earlier, by following Eq. (1) we have

$$\begin{aligned} \mathbf{E}[f(\Theta^\pi)] = \mathbf{E}[f(\Theta_2^\pi)] &= 0.16f(10, 0, 10) + 0.24f(100, 0, 10) + 0.12f(10, 0, 100) \\ &+ 0.18f(100, 0, 100) + 0.12f(0, 10, 100) + 0.18f(0, 100, 0). \end{aligned}$$

Throughout this paper, we assume that all such expected values are finite. Due to the monotonicity of  $f$ , it is enough to assume that  $\mathbf{E}[f(\Theta^\pi)]$  is finite for a policy  $\pi$  that picks all the elements in  $\mathcal{A}$ .

As the final step, we take into account the constraints that might appear in our optimization problems.

**Definition 3 (A Feasible Policy)** *Consider a matroid  $\mathcal{M}(\mathcal{A}, \mathcal{I})$  defined on a ground set  $\mathcal{A} = \{X_1, X_2, \dots, X_n\}$  of  $n$  random variables. A policy  $\pi$  is feasible with respect to  $\mathcal{M}$  if and only if with probability 1 the set of elements chosen by the policy belongs to  $\mathcal{I}$ .*

Now, we are ready to formally define our problem.

**Definition 4 (SMSM) Maximizing a Stochastic Monotone Submodular Function with Matroid Constraint:** *An instance of the SMSM problem is defined with tuple  $(f, \mathcal{A}, \mathcal{I})$ , where  $\mathcal{A} = \{X_1, X_2, \dots, X_n\}$  is a set of  $n$  non-negative independent random variables,  $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$  is a monotone submodular value function, and matroid  $M(\mathcal{A}, \mathcal{I})$  represents the constraints. The goal is to find a feasible policy (with respect to  $\mathcal{M}$ ) that obtains the maximum (expected) value.*

In addition to “adaptive” policies, we also consider *non-adaptive* policies that do not rely on the outcomes of the previous actions in order to make a decision. When a policy progresses, more information about the actual realization of the state will be revealed. Non-adaptive policies do not use such information and choose all the elements in advance.

**Definition 5 (Non-Adaptive Policies)** *A policy  $\pi$  is called non-adaptive if, for each state  $\Theta$ ,  $\pi(\Theta)$  depends only on which elements were previously chosen but not their realizations. A non-adaptive policy is defined by the set (and not the order) of the chosen elements.*

Since non-adaptive policies choose their set of actions in advance (not sequentially), they are usually simpler to implement. Furthermore, sometimes it is not possible to implement an adaptive policy or it might be difficult to learn the realizations of the random variables. For instance, in the capital budgeting problem, it might be costly for the investor to wait for outcomes of the projects to be realized before making another decision. In the case of viral marketing, it is difficult to get feedback from each individual and find out about the outcome of the previous actions.

However, one drawback of non-adaptive policies is that they may not perform as well as the adaptive policies. The optimal policy for a given instance of the SMSM problem might be a complicated adaptive policy, far from being non-adaptive. Hence, there exists a trade-off between the value obtained by the policy on one side and its practicality and convenience (in terms of both computation and implementation) on the other side. However, contemplating between these two options is reasonable for strategists only if the performance of the non-adaptive policies is not too far from that of the adaptive ones. This observation motivates the comparison between the value of adaptive and non-adaptive policies in SMSM problems. This can be measured via studying the *adaptivity gap* as defined by Dean et al. (2005, 2008).

**Definition 6 (Adaptivity Gap)** *The adaptivity gap is defined as the upper bound on the ratio of the value of the optimal adaptive policy to the value of the optimal non-adaptive policy. More precisely,*

$$\text{ADAPTIVITY GAP} = \sup_{(f, \mathcal{A}, \mathcal{I})} \frac{\sup_{\pi \in \Pi(f, \mathcal{A}, \mathcal{I})} \mathbf{E}_{\Theta} [f(\Theta^{\pi})]}{\sup_{S \in \mathcal{I}} \mathbf{E}_{\Theta} [f(\Theta_S)]},$$

where the first supremum is taken over all instances  $(f, \mathcal{A}, \mathcal{I})$  of the SMSM problem,  $\Pi(f, \mathcal{A}, \mathcal{I})$  denotes the set of all feasible adaptive policies, and for  $S \in \mathcal{I}$  the vector  $\Theta_S$  corresponds to the outcome of the non-adaptive policy that picks set  $S$  of the elements.

### 3. The Adaptivity Gap of SMSM

It is easy to observe from Definition 6 that the adaptivity gap is at least equal to 1. This is due to the fact that every non-adaptive policy is a special case of adaptive (general) policies. However, a high adaptivity gap for a problem suggests that good adaptive policies are far superior than the non-adaptive ones. Similarly, a low (close to 1) adaptivity gap is a certificate that one can get the benefits of non-adaptive policies (such as the relative simplicity in their optimization and also implementation) without losing much in the obtained value compared to that of the adaptive policies. In this section, we show that for the SMSM problem, the adaptivity gap is bounded. The main result of this section is as follows:

**Theorem 1 (Adaptivity Gap)** *The adaptivity gap of SMSM is equal to  $\frac{e}{e-1}$ .*

In the rest of this section, we prove the above theorem by analyzing the optimal non-adaptive and adaptive policies and compare their performances. We start with an example that show that the above ratio is tight.

### 3.1. A Tight Example: Stochastic Maximum $k$ -Cover

Given a collection  $\mathcal{A}$  of the subsets of  $\{1, 2, \dots, m\}$ , the goal of the *maximum  $k$ -cover* problem is to find  $k$  subsets from  $\mathcal{A}$  such that their union has the maximum cardinality (see Feige (1998)). In the stochastic version, each random variable  $X_i$  with some probability covers a subset of  $\{1, 2, \dots, m\}$  and with the remaining probability covers no element (i.e., the realization of  $X_i$  would be the empty set). The subset that an element of  $\mathcal{A}$  would cover is revealed only after choosing the element according to a given probability distribution. It is easy to see that this problem is a special case of the SMSM with respect to a uniform matroid  $\mathcal{M} = (\mathcal{A}, \{S \subseteq \mathcal{A} : |S| \leq k\})$  constraint.

To give a lower bound on the adaptivity gap, consider the following instance: a ground set  $\{1, 2, \dots, m\}$  and a collection  $\mathcal{A} = \{X_j^{(i)} | 1 \leq i \leq m, 1 \leq j \leq m^2\}$  of its subsets are given. Note that here  $n$ , the total number of elements in the ground set, is equal to  $m^3$ . For every  $i, j$ , define  $X_j^{(i)}$  to be the one-element subset  $\{i\}$  with probability  $\frac{1}{m}$  and the empty set with probability  $1 - \frac{1}{m}$ . The goal is to cover the maximum number of the elements of the ground set by selecting at most  $k = m^2$  subsets from  $\mathcal{A}$ .

**Lemma 1** *The optimal non-adaptive policy is to pick  $m$  subsets from each of the collections  $\mathcal{A}^{(i)} = \{X_j^{(i)} | 1 \leq j \leq m^2\}$  for every  $i$ . For large enough values of  $m$ , the expected value of this policy is (arbitrarily close to)  $(1 - \frac{1}{e})m$ .*

The proof is given in the online appendix and is based on a convexity argument. We now consider the following myopic adaptive policy  $\mathcal{P}$ : start with  $i = 1$  and pick the elements of  $\mathcal{A}^{(i)}$  one by one until one of them is realized as  $\{i\}$  or all of elements in  $\mathcal{A}^{(i)}$  are chosen. Then increase  $i$  by one. Continue the iteration until either all the elements are covered or  $m^2$  subsets from  $\mathcal{A}$  are selected.

The following lemma gives a lower bound on the number of elements in the ground set covered by the adaptive policy. The proof is given in the online appendix.

**Lemma 2** *The expected number of elements in  $\{1, 2, \dots, m\}$  covered by  $\mathcal{P}$  described above is  $(1 - o(1))m$ .*

By combining the results of Lemmas 1 and 2, we have that the adaptivity gap of stochastic maximum coverage is at least  $\frac{e}{e-1}$ .

### 3.2. Bounds on the Optimal Non-Adaptive Policy

In this section, we study the optimal non-adaptive policy in more detail. We start with some definitions. Let  $S \subseteq \mathcal{A}$  be a subset of variables. Also, let vector  $\Theta_S = (\theta_1, \dots, \theta_n)$  denote the realization of set  $S$ , where  $\theta_i = x_i$  for  $X_i \in S$  and  $\theta_i = \circ$  for  $X_i \notin S$ . The value obtained by choosing set  $S$  after the realization is equal to  $\mathbf{E}[f(\Theta_S)]$ . Let  $g_i$  and  $G_i$  be the probability density function (p.d.f.) and cumulative distribution function (c.d.f.) of random variable  $X_i$ , respectively. We introduce  $g_S : \mathbb{R}_+^n \rightarrow \mathbb{R}$  to represent the probability density function of observing  $\Theta = (\theta_1, \dots, \theta_n)$  while selecting  $S$ :

$$g_S(\Theta) := \Pr[\Theta_S = \Theta] = \prod_{i: X_i \in S} g_i(\theta_i),$$

where  $g_S(\theta)$  is defined to be zero if there exists  $j \notin S$  such that  $\theta_j \neq \circ$ . Note that the definition above relies on the assumption that the realizations of elements chosen by the policy are independent from each other. As we show in Section 6.3, the adaptivity gap can be arbitrarily large if this assumption does not hold.

Now, for the sake of simplicity in notation, we define the function  $F : 2^{\mathcal{A}} \rightarrow \mathbb{R}^+$  as the expected value obtained by choosing set  $S$ , i.e.,

$$F(S) = \mathbf{E}[f(\Theta_S)] = \int_{\Theta \in \Omega} f(\Theta) g_S(\Theta) d\Theta, \quad (3)$$

where, as defined before,  $\Omega = \prod_{i=1}^n (\Omega_i \cup \{\circ\})$  is the set of all possible states. Therefore, the optimal non-adaptive policy for SMSM is equivalent to choosing a set  $S \in \mathcal{I}$  that maximizes  $F(S)$  with respect to the desired matroid constraint. Hence, the following proposition holds:

**Proposition 1 (Optimal Non-Adaptive Policy)** *For any instance of SMSM, each set  $S^* \in \operatorname{argmax}_{S \in \mathcal{I}} F(S)$  corresponds to an optimal non-adaptive policy.*

In the online appendix we prove the lemma below that shows  $F$  is monotone and submodular.

**Lemma 3 (Properties of Function  $F$ )** *If  $f$  is monotone and submodular, then function  $F$ , defined in Eq. (3), is a monotone and submodular set function.*

We now define the notion of *fractional non-adaptive policies*. A fractional non-adaptive policy  $\mathcal{S}_y$  is defined with a vector  $y \in [0, 1]^n$ . Policy  $\mathcal{S}_y$  chooses elements in the (random) set  $Y$ ; a set that includes each  $X_i \in \mathcal{A}$  with probability  $y_i$ , independently for every  $i$ .

We call fractional non-adaptive policy  $\mathcal{S}_y$  *feasible in expectation* if vector  $y$  lies in the base polytope of  $\mathcal{M}$  denoted by  $\mathcal{B}(\mathcal{M})$ . The base polytope of  $\mathcal{M}$  is the convex hull of the characteristic vectors of all bases of  $\mathcal{M}$ , i.e.,

$$\mathcal{B}(\mathcal{M}) = \operatorname{conv}\{1_S | S \in \mathcal{I}, \text{ and } S \text{ is a basis}\}. \quad (4)$$

For instance, suppose  $\mathcal{M}$  is a uniform matroid of rank  $d$ , and for  $y = (y_1, y_2, \dots, y_n)$  we have  $\sum_{i=1}^n y_i \leq d$ . Observe that in this case, fractional non-adaptive policy  $\mathcal{S}_y$  is feasible in expectation, and the expected number of elements chosen by this policy is at most  $d$ .

With slight abuse of notation, we denote the expected value obtained by fractional non-adaptive policy  $\mathcal{S}_y$  by  $F(y)$ . Namely,

$$F(y) := \sum_{Y \subseteq \{0,1\}^n} \left[ \left( \prod_{i \in Y} y_i \prod_{i \notin Y} (1 - y_i) \right) F(Y) \right] = \sum_{Y \subseteq \{0,1\}^n} \left[ \left( \prod_{i \in Y} y_i \prod_{i \notin Y} (1 - y_i) \right) \mathbf{E}_{\Theta} [f(\Theta_Y)] \right], \quad (5)$$

where  $\Theta_Y$ , as previously defined, denotes the realization of the elements in set  $Y$ .

Fractional non-adaptive policies extend the space of (integral) non-adaptive policies. Therefore, it is easy to see that  $\max_{S \in \mathcal{I}} F(S) \leq \max_{y \in \mathcal{B}(\mathcal{M})} F(y)$ . We show that this inequality is in fact tight, and any vector  $y \in \mathcal{B}(\mathcal{M})$  can be rounded to an integral corner point  $Y \in \mathcal{B}(\mathcal{M})$  corresponding to a subset  $S \in \mathcal{I}$  such that  $F(S) \geq F(y)$ .

**Lemma 4 (Fractional vs. Integral Non-Adaptive Policies)** *For any  $y \in \mathcal{B}(\mathcal{M})$ , there exists a set  $S \in \mathcal{I}$  such that  $F(S) \geq F(y)$ . Moreover, such a set can be found in polynomial time.*

The above lemma implies that  $\max_{y \in \mathcal{B}(\mathcal{M})} F(y)$  is a *lower bound* on the performance of the optimal non-adaptive policy. The proof follows immediately from the pipage rounding procedure of Călinescu et al. (2011) who considered “multilinear extension,” the same as our function  $F$ , to study the problem of maximizing (deterministic) submodular functions.

### 3.3. Bounds on the Optimal Adaptive Policy

We start this section by making a few observations about adaptive policies. Consider an arbitrary adaptive policy ADAPT. Any such policy decides to choose a sequence of elements where the decision on which element to choose at any step might depend on the realized values of the previously chosen elements. Therefore, any specific state of the world will result in a distribution over the sequence of elements that will be chosen by ADAPT.<sup>2</sup> Any adaptive policy can be described by a (possibly randomized) decision tree in which at each step an element is being added to the current selection. Each path from the root to a leaf of this tree corresponds to a subset of elements and occurs with some certain probability. These probabilities, covering all the possible scenarios for ADAPT, sum to one. Let  $y_i$  be the probability that element  $X_i \in \mathcal{A}$  is eventually chosen by ADAPT. Also, let  $\beta_{\Theta}$  be the probability that the final state  $\Theta$  is reached by ADAPT. Then we have the following properties:

<sup>2</sup> We refer the reader to Definition 1 for policy and emphasize that we allow our policies to be *randomized*. This is the reason that we mention a distribution over different sequences of elements (and not just a specific sequence) here.

1.  $\int_{\Theta \in \Omega} \beta_{\Theta} d\Theta = 1.$
2.  $\forall \Theta \in \Omega : \beta_{\Theta} \geq 0.$
3.  $\forall i, x_i \in \Omega_i : \int_{\Theta: \theta_i = x_i} \beta_{\Theta} d\Theta^{-i} = y_i g_i(x_i),$

where  $d\Theta^{-i}$  represents  $\prod_{j \neq i} d\theta_j$ . The first two properties hold because  $\beta$  defines a probability measure on the space of all feasible outcomes. The third property implies that the probability that we observe an outcome  $x_i$  as a realized value of  $X_i$  among all possible states  $\Theta$  reached by the policy is equal to the probability that  $X_i$  is chosen (i.e.,  $y_i$ ) multiplied by the probability that the realization is equal to  $x_i$ . We emphasize that we use the independence among the random variables to ensure that this property holds. We also emphasize that, as we will see in Section 6.3, the adaptivity gap could be unbounded if there are dependencies among the variables.

Since every policy satisfies the above properties, we can establish an upper bound on the value of any adaptive policy. Hence, we define the function  $f^+ : [0, 1]^n \rightarrow \mathbb{R}$  as follows:

$$f^+(y) := \sup_{\alpha} \left\{ \int_{\Theta} \alpha_{\Theta} f(\Theta) d\Theta : \int_{\Theta} \alpha_{\Theta} d\Theta = 1, \alpha_{\Theta} \geq 0, \forall i, x_i \in \Omega_i : \int_{\Theta: \theta_i = x_i} \alpha_{\Theta} d\Theta^{-i} = y_i g_i(x_i) \right\}. \quad (6)$$

The supremum is taken over all probability measures  $\alpha$  that satisfy the three properties above.

Another observation is that for an optimal adaptive policy, vector  $y$  lies in the base polytope of  $\mathcal{M}$ ; see Eq. (4). Using these observations, we obtain the following lemma which is proved in the appendix:

**Lemma 5 (An Upper Bound on Adaptive Policies)** *The expected value of the optimal adaptive policy is at most  $\sup_{y \in \mathcal{B}(\mathcal{M})} \{f^+(y)\}$ .*

We are now ready to prove Theorem 1.

### 3.4. Proof of Theorem 1

Lemma 5 shows that  $\sup_{y \in \mathcal{B}(\mathcal{M})} f^+(y)$  is an upper bound on the performance of the optimal adaptive policy. Now consider any  $y \in \mathcal{B}(\mathcal{M})$ . Below, we present Lemma 6, which shows  $(1 - \frac{1}{e}) f^+(y) \leq F(y)$ . On the other hand, Lemma 4 implies that there exists a set  $S \in \mathcal{I}(\mathcal{M})$  such that  $F(y) \leq F(S)$ . Recall that  $F(S)$  is in fact the expected value gained by a non-adaptive policy that selects set  $S$ . Hence, for every  $y \in \mathcal{B}(\mathcal{M})$ , there exists a non-adaptive policy that achieves a value of at least  $(1 - \frac{1}{e}) f^+(y)$ . Therefore, the optimal non-adaptive policy corresponding to selecting set  $S^* \in \operatorname{argmax}_S \{F(S)\}$  has a value of at least  $(1 - \frac{1}{e}) \sup_{y \in \mathcal{B}(\mathcal{M})} f^+(y)$ , or equivalently, at least a  $(1 - 1/e)$  fraction of the value of the optimal adaptive policy. We remind the reader that the example in Section 3.1 shows that the factor is tight.

**Lemma 6 (Upper Bound on  $f^+$ )** *For any monotone submodular function  $f$  and any vector  $y \in \mathcal{B}(\mathcal{M})$ , we have  $f^+(y) \leq (\frac{e}{e-1})F(y)$ .*

The main technical ingredient in the proof of the lemma above is a generalization of Lemmas 3.7 and 3.8 in Vondrák (2007) to continuous submodular functions for which we provide a new stochastic dominance result.

*Proof of Lemma 6* We start with introducing the following notation. For a vector  $\Theta = (\theta_1, \theta_2, \dots, \theta_n)$ , define  $\Theta^{\uparrow j} = (\theta_1^{\uparrow j}, \theta_2^{\uparrow j}, \dots, \theta_n^{\uparrow j})$  as a random vector whose entries are defined as below.

$$\theta_i^{\uparrow j} := \begin{cases} \theta_i & i \neq j; \\ \max\{\theta_j, X\} & i = j, \text{ where } X \text{ is independently drawn from } g_j. \end{cases} \quad (7)$$

One can think of  $\Theta^{\uparrow j}$  as the same as  $\Theta$ , except for the  $j$ -th entry for which we draw a number  $X$  from the distribution  $g_j$  and override the entry with  $X$  if and only if it increases the value of the entry.

Now, for any  $j$  we define an independent Poisson clock  $\mathcal{C}_j$  that sends signals with rate  $y_j$  throughout the time. We start with  $\Theta = (0, 0, \dots, 0)$  at  $t = 0$ . Once a clock  $\mathcal{C}_j$  sends a signal, we replace  $\Theta$  by  $\Theta^{\uparrow j}$ . By abuse of notation, we denote the value of vector  $\Theta$  at time  $t$  by  $\Theta(t)$ . We first show that  $\mathbf{E}[f(\Theta(1))] \leq F(y)$ . We will do so by proving a stochastic dominance result.

Consider  $F(y)$ , defined in Eq. (5). Note that each entry  $j$  in  $\Theta_Y$  is zero (null) if  $j \notin Y$ . Otherwise, it is drawn from the probability distribution  $g_j$ , independently from other variables. Hence,  $F(y)$  can be written as  $E[f(\theta_1, \theta_2, \dots, \theta_n)]$ , where the entries are all independent and the c.d.f. of  $\theta_j$  is given by a function  $\eta_j : \mathbb{R}_+ \mapsto [0, 1]$  and  $\eta_j(x) = (1 - y_j) + y_j G_j(x)$ .

On the other hand, due to the independence of the Poisson clocks, the entries of  $\Theta(1)$  are independent random variables. In particular, the clock  $\mathcal{C}_j$  signals  $k$  times between  $t = 0$  and  $t = 1$ , where  $k$  is a Poisson random variable with rate  $y_j$ . If this clock signals  $k$  times, by the construction of  $\Theta(t)$  for  $0 \leq t \leq 1$ , the  $j$ -th entry of  $\Theta(1)$  will be the maximum of  $k$  random variables, each drawn independently from the probability distribution  $g_j$ . In this case, the c.d.f. of the  $j$ -th entry will be simply given by the function  $G_j(x)^k$ . Summing over all  $k$  and incorporating the Poisson distribution function, we can summarize that the c.d.f. of the  $j$ -th entry of  $\Theta(1)$  is given by a function  $\gamma_j : \mathbb{R}_+ \mapsto [0, 1]$ , where  $\gamma_j(x) = \sum_{k=0}^{\infty} \frac{y_j^k}{k!} e^{-y_j} G_j(x)^k$ .

By the properties of Poisson distribution, we have

$$\gamma_j(x) = \sum_{k=0}^{\infty} \frac{y_j^k}{k!} e^{-y_j} G_j(x)^k = \frac{e^{-y_j}}{e^{-y_j G_j(x)}} \sum_{k=0}^{\infty} \left( \frac{(y_j G_j(x))^k}{k!} e^{-y_j G_j(x)} \right) = \frac{e^{-y_j}}{e^{-y_j G_j(x)}} = e^{-y_j(1-G_j(x))}$$

However, comparing the two cumulative distribution functions discussed above, we have

$$\gamma_j(x) = e^{-y_j(1-G_j(x))} \geq 1 - y_j(1 - G_j(x)) = \eta_j(x).$$

This means that for every  $j$ , the random variable drawn from  $\gamma_j$  is stochastically dominated by that from  $\eta_j$ . Therefore,

$$\mathbf{E}[f(\Theta(1))] \leq F(y). \quad (8)$$

Now, we compare the value  $\mathbf{E}[f(\Theta(1))]$  with  $f^+(y)$ . Let  $t \in [0, 1)$  be fixed. For each  $j$ , the chance that the clock  $C_j$  sends a signal during the interval  $[t, t + dt)$  is simply  $y_j dt$  for sufficiently small  $dt$ . Hence,

$$\mathbf{E}[f(\Theta(t + dt)) - f(\Theta(t)) \mid \Theta(t) = \Theta] = \sum_{j=1}^n y_j dt (\mathbf{E}[f(\Theta^{\uparrow j})] - f(\Theta)). \quad (9)$$

Note that the expectation is taken over the random draw that appears in the definition of  $\Theta^{\uparrow j}$ , i.e., the random variable  $X$  in Definition (7).

We define an auxiliary function  $f^* : [0, 1]^n \rightarrow \mathbb{R}$  as the following:

$$f^*(y) := \inf_{\Theta} \left\{ f(\Theta) + \sum_{j=1}^n y_j (\mathbf{E}[f(\Theta^{\uparrow j})] - f(\Theta)) \right\}, \quad (10)$$

where the infimum is taken over all possible states  $\Theta$ . Therefore, the right hand side of Eq. (9) is at least  $(f^*(y) - \mathbf{E}[f(\Theta)])dt$ . As a result, the following bound can be derived on the derivative of  $\mathbf{E}[f(\Theta(t))]$ :

$$\frac{d}{dt} \mathbf{E}[f(\Theta(t))] = \frac{1}{dt} \mathbf{E}[f(\Theta(t + dt)) - f(\Theta(t)) \mid \Theta(t) = \Theta] \geq f^*(y) - \mathbf{E}[f(\Theta(t))]. \quad (11)$$

Solving the differential equation above we obtain  $\mathbf{E}[f(\Theta(t))] \geq (1 - e^{-t})f^*(y)$ . In particular,

$$(1 - \frac{1}{e})f^*(y) \leq \mathbf{E}[f(\Theta(1))]. \quad (12)$$

In Appendix A.1, Lemma 12, we show that  $f^+(y) \leq f^*(y)$ ,  $y \in \mathcal{B}(\mathcal{M})$ . Combining with Inequalities (8) and (12), we have  $f^+(y) \leq \frac{e}{e-1}F(y)$ , which concludes the proof of the lemma.  $\square$

#### 4. A Polynomial-Time $(1 - 1/e - \epsilon)$ -Approximate Non-Adaptive Policy

In this section, we present polynomial-time non-adaptive policies for the SMSM problem. We assume that the values of function  $F$  are accessible via an oracle. We discuss the computation of function  $F$  in Section 6.1. We show that under standard assumptions (when  $g_j$ 's are bounded or constant Lipschitz continuous), the value of function  $F$  can be computed within any polynomially small error term using sampling.

Let us first consider the following myopic non-adaptive policy formally defined in Figure 1. This policy, at each step, adds a feasible element with the highest marginal contribution to the set of selected elements. Via Lemma 3, we showed that function  $F$  is monotone submodular.

The classic result of Fisher et al. (1978) implies that the greedy algorithm obtains an approximation ratio of  $\frac{1}{2}$  for maximizing monotone submodular functions, i.e., at the end of the algorithm  $F(S_i) \geq \frac{1}{2} \max_{S \in \mathcal{I}} \{F(S)\}$ . Therefore, using Theorem 1, we have



**Figure 1** The Non-Adaptive Myopic Policy

```

Initialize  $i = 0, S_0 = \emptyset, U_0 = \emptyset$ .
Repeat
   $i \leftarrow i + 1$ .
  Find  $X_i \in \operatorname{argmax}_{X \in \mathcal{A} \setminus (U_{i-1} \cup S_{i-1})} \mathbf{E}[F(S_{i-1} \cup \{X_i\})]$ .
  If  $S_{i-1} \cup \{X_i\} \in \mathcal{I}$ , then
     $S_i \leftarrow S_{i-1} \cup \{X_i\}; U_i \leftarrow U_{i-1}$ .
  else
     $U_i \leftarrow U_{i-1} \cup \{X_i\}; S_i \leftarrow S_{i-1}$ .
Until  $(U_i \cup S_i = \mathcal{A})$ .
The non-adaptive policy selects set  $S_i$ .
    
```

**Proposition 2 (Myopic Non-Adaptive Policy)** *The myopic non-adaptive policy for any instance of SMSM obtains an expected value of at least  $\frac{1}{2} \times (1 - \frac{1}{e}) \approx 0.316$  times the expected value of the optimal adaptive policy.*

The above result provides a performance guarantee for a simple policy; however, it does not match the adaptivity gap. Theorem 1 shows that there exists a non-adaptive policy corresponding to the selection of set  $S^* \in \operatorname{argmax}_{S \in \mathcal{I}(\mathcal{M})} F(S)$  that achieves at least a  $(1 - 1/e)$  fraction of the value of the optimal adaptive policy. However, this does not immediately provide an efficient algorithm to find such a non-adaptive policy. The reason is that finding  $\operatorname{argmax}_{S \in \mathcal{I}(\mathcal{M})} F(S)$ , i.e., maximizing a monotone submodular function with respect to a matroid constraint, is not computationally tractable. As mentioned before, the maximum  $k$ -cover, described in Section 3.1, is a special case of this problem. Feige (1998) shows that it is not possible to find an approximation ratio better than  $1 - \frac{1}{e}$  for the maximum  $k$ -cover problem unless  $NP \subset TIME(n^{O(\log \log n)})$ . Hence,  $1 - \frac{1}{e}$  is the best possible approximation ratio achievable for any policy that can be implemented within polynomial time.

In the proof of Theorem 1, we used a dynamic process  $\Theta(t) : t \in [0, 1]$  to find a vector  $y^*$  such that  $F(y^*) \geq (1 - 1/e) \sup_{y \in \mathcal{B}(\mathcal{M})} f^+(y)$ . This process takes an arbitrary  $y \in \mathcal{B}(\mathcal{M})$  and starts with  $\Theta(0) = (0, 0, \dots, 0)$ . It keeps track of  $n$  independent Poisson clocks  $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$ , where the rate of  $\mathcal{C}_j$  is  $y_j$ , and whenever a clock  $\mathcal{C}_j$  sends a signal throughout the time  $0 \leq t \leq 1$ , the process substitutes  $\Theta$  with  $\Theta^{\uparrow j}$  (see Eq. (7) for the definition of the latter). If we were given an appropriate vector  $y$  (for instance, the  $y$  that corresponds to the probability of each element being picked by the optimum adaptive policy), then we could simulate this process by **discretizing** the time (we will discuss discretizing later in detail). However, we do not know any such vector a priori. The idea of our algorithm is to use a **greedy** approach to select the entries that are going to be updated at any given point in time. See Figure 2 for the description of our algorithm.

**Figure 2** The Generalized Continuous Greedy Algorithm

- i. Let  $\delta = 1/(\lceil \frac{3d}{\epsilon} \rceil)$ , where  $d$  denotes the rank of the matroid  $\mathcal{M} = (\mathcal{A}, \mathcal{I})$ ;  $n = |\mathcal{A}|$ , and  $\epsilon$  is the discretization parameter. Start from  $t = 0$  and  $y(0) = 0$ .
- ii. Let  $R(t)$  be a set containing each element  $X_j \in \mathcal{A}$  independently with probability  $y_j(t)$ .  
Let  $\Theta_{R(t)}$  be the realization of set  $R(t)$ , i.e., a vector whose  $j$ -th entry is independently drawn from the probability distribution function  $g_j$  if  $X_j \in R(t)$ , and is 0 if  $X_j \notin R(t)$ .  
Let  $w_j(t)$  be an estimate of  $\mathbf{E}_{\Theta, R} \left[ f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)}) \right]$ , obtained by averaging over  $\frac{4}{\delta^2} (1 + \ln n - 0.5 \ln \delta)$  samples of  $\Theta_{R(t)}$ .
- iii. Let  $I(t)$  be a maximum-weight independent set in  $\mathcal{M}$  according to the weights  $w_j(t)$ . Let  $y(t + \delta) = y(t) + \delta \cdot \mathbf{1}_{I(t)}$ .
- iv. Increment  $t := t + \delta$ ; if  $t < 1$ , go back to (ii). Otherwise, return  $y(1)$ .

Our method is a generalization of the continuous greedy approach of Călinescu et al. (2011) performed on function  $F$  as defined in Eq. (5). In fact, for the deterministic problem (i.e., when each  $X_j$ , if picked, will always have a realized value of 1) our method becomes the equivalent of the continuous greedy method of Călinescu et al. (2011).

**The High Level Idea:** Our algorithm (whose description can be seen below) keeps track of a vector  $y(t)$ , starting from  $y(0) = (0, 0, \dots, 0)$  (see step (i)). It then updates  $y(t)$  by taking steps of size  $\delta$  (for some carefully chosen  $\delta$  as we will discuss later) throughout the time, until  $t = 1$ . In order to find out how to update  $y(t)$ , our algorithm first calculates the potential marginal contribution of each  $y_j$  to the current  $F(y(t))$ . In order to mimic the function  $F$  as defined in Eq. (5), our algorithm produces a randomly selected set  $R(t)$  that contains each element  $X_j$  with probability  $y_j(t)$  independently at random. Then it simulates a realization of  $R(t)$ , namely  $\Theta_{R(t)}$ , using the probability distributions  $g_j$ . The marginal contribution of the element  $y_j$  to  $F(y(t))$  for this specific realization can be simply measured by  $f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)})$ . In order to estimate the expected marginal contribution of  $y_j$ , our algorithm takes the average of the mentioned subtraction over a large enough number of samples (see step (ii)). Finally, our algorithm finds a maximum-weight independent set according to the weights defined as the expected marginal contribution of each element. This can be done via the classic greedy algorithm for finding the maximum-weight independent sets in matroids (see Rado 1957; Gale 1968; Edmonds 1971). Finally,  $y(t + \delta)$  is obtained by adding  $\delta$  to the entries of  $y(t)$  corresponding to the elements of the maximum-weight independent set (see step (iii)).

In order to analyze the generalized continuous greedy algorithm, we follow a similar path to that of the proof of Theorem 1 in the previous section. We show that our sampling in step (ii) provides a

good approximation of the weights, and in particular, the weight of the independent set selected at step (iii) in each iteration is close to the weight of the actual maximum-weight independent set. For the rest of this section, “with high probability” means with probability of at least  $(1 - 1/\text{poly}(n))$ . The proofs of the following lemmas can be found in the online appendix.

**Lemma 7** *If the set  $I(t)$  is chosen by step (iii) of the generalized continuous greedy algorithm at time  $t$ , then with high probability, for any  $t$  we have*

$$\sum_{j: X_j \in I(t)} \mathbf{E}_{\Theta, R} [f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)})] \geq \left( \max_{I \in \mathcal{I}} \sum_{j: X_j \in I} \mathbf{E}_{\Theta, R} [f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)})] \right) - 2d\delta \cdot OPT,$$

where  $OPT$  is the value of the optimum adaptive policy.

Now, we focus on the change in the value of  $F(y(t + \delta))$  compared to that of  $F(y(t))$ . We show that the actual difference between  $F(y(t + \delta))$  and  $F(y(t))$  is bounded from below by  $\delta(1 - d\delta)$  times the weight of the maximum independent set selected by step (iii). We note that this lemma could be seen as analogous to Eq. (9) in our continuous Poisson process.<sup>3</sup>

**Lemma 8** *At each  $t$  we have*

$$F(y(t + \delta)) - F(y(t)) \geq \delta(1 - d\delta) \sum_{j: X_j \in I(t)} \mathbf{E}_{\Theta, R} [f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)})],$$

where  $d$  is the rank of the matroid  $\mathcal{M}$ .

The final step in the analysis of our algorithm is to provide an analogous statement to the differential equation of Eq. (11). This is done through the following lemma whose proof is presented in the appendix.

**Lemma 9** *Throughout the algorithm, with high probability, for every  $t$  we have the following:*

$$F(y(t + \delta)) - F(y(t)) \geq \delta(1 - 3d\delta) \sup_{y \in B(\mathcal{M})} \{f^*(y)\} - F(y(t)).$$

Now we are ready to finish the analysis of the algorithm. Let us first denote the value of  $(1 - 3d\delta) \sup_{y \in B(\mathcal{M})} \{f^*(y)\}$  by  $\mathcal{X}$ . From Lemma 9 we have  $\mathcal{X} - F(y(t + \delta)) \leq (1 - \delta)(\mathcal{X} - F(y(t)))$  with high probability throughout the algorithm. By induction and considering the fact that  $F(y(0)) = 0$ , we get  $\mathcal{X} - F(y(k\delta)) \leq (1 - \delta)^k \mathcal{X}$  for every  $k$  such that  $k\delta \leq 1$ . In particular, for  $k = 1/\delta$  we will have

$$\mathcal{X} - F(y(1)) \leq (1 - \delta)^{1/\delta} \mathcal{X} \leq \frac{1}{e} \mathcal{X}.$$

<sup>3</sup> We remark that the proof of Lemma 8 does not depend on the accuracy of the estimation and only uses a stochastic dominance result.

Thus,  $F(y(1)) \geq (1 - 1/e)\mathcal{X} \geq (1 - 1/e - 3d\delta) \sup_{y \in B(\mathcal{M})} \{f^*(y)\}$ . Note that  $\delta = 1/(\lceil \frac{3d}{\epsilon} \rceil)$ . Hence, the point  $y(1) \in B(\mathcal{M})$  found by the generalized continuous greedy algorithm with high probability has the value of at least  $(1 - 1/e - \epsilon) \sup_{y \in B(\mathcal{M})} \{f^*(y)\}$ . Therefore, we obtain the following result:

**Lemma 10** *For any  $\epsilon > 0$ , the generalized continuous greedy algorithm finds a vector  $y^* \in \mathcal{B}(\mathcal{M})$  in polynomial time in  $|\mathcal{A}|$  and  $1/\epsilon$  such that  $F(y^*) \geq \sup_{y \in B(\mathcal{M})} (1 - 1/e - \epsilon)f^*(y)$ .*

By Lemma 4 and using vector  $y^*$  mentioned in the above lemma, we can find in polynomial time a subset  $S^* \in \mathcal{I}$  such that  $F(S^*) \geq F(y^*) \geq \sup_{y \in B(\mathcal{M})} (1 - 1/e - \epsilon)f^*(y)$ . Recall that by Lemma 12, we have  $f^*(y) \geq f^+(y)$  for every  $y \in \mathcal{B}(\mathcal{M})$ . Hence,  $F(S^*) \geq (1 - 1/e - \epsilon) \sup_{y \in B(\mathcal{M})} f^+(y)$ . However, by Lemma 5,  $\sup_{y \in B(\mathcal{M})} f^+(y)$  is an upper bound on the value of the optimal adaptive policy. Therefore, we can conclude the following:

**Theorem 2 (Approximately Optimal Polynomial-Time Adaptive Policy)** *For any  $\epsilon > 0$  and any instance of an SMSM problem, a non-adaptive policy that obtains a  $(1 - 1/e - \epsilon)$ -fraction of the value of the optimal adaptive policy can be found in polynomial time (with respect to  $|\mathcal{A}|$  and  $1/\epsilon$ ).*

## 5. Approximation Ratio of Simple Myopic Policies

In this section, we present an adaptive myopic policy for the SMSM problem with a general matroid constraint. The set of feasible solutions is the intersection of  $\kappa$  matroids. Let  $\mathcal{M}_1 = (\mathcal{A}, \mathcal{I}_1), \mathcal{M}_2 = (\mathcal{A}, \mathcal{I}_2), \dots, \mathcal{M}_\kappa = (\mathcal{A}, \mathcal{I}_\kappa)$  be  $\kappa$  matroids all defined on the ground set of elements  $\mathcal{A}$ . The feasible set for the SMSM problem that we study in this section is  $\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2 \cap \dots \cap \mathcal{I}_\kappa$ . We will present a myopic policy whose approximation ratio with respect to the optimal adaptive policy is  $\frac{1}{\kappa+1}$ . At the end of this section, we explain that the myopic policy achieves the approximation ratio of  $1 - \frac{1}{e}$ , if  $\kappa = 1$  and the matroid is uniform. The results of Fisher et al. (1978) and Feige (1998) for the (deterministic) submodular optimization imply that these bounds are tight.

The policy is given in Figure 3. Remember that the feasible set of elements is given by  $\mathcal{I}$ . At each iteration, from the elements in  $\mathcal{A}$  that have not been picked yet, the policy chooses an element with the maximum expected marginal value. We denote by  $S_t$  the set of elements chosen by the adaptive policy up to iteration  $t$ . By abuse of notation, let  $s_t$  denote the realizations of all of the elements in  $S_t$ . Also,  $U_t$  is the set of elements probed but not chosen by the policy due to the matroid constraint.

Note the exit condition for the repeat loop. The repeat loop will continue until either a new element  $X_i$  is added to the set of selected elements (i.e.,  $S_t = S_{t-1} \cup \{X_i\} \neq S_{t-1}$ ) or all elements in the ground set  $\mathcal{A}$  are probed (i.e.,  $U_t \cup S_t = \mathcal{A}$ ). In the former, the policy will continue probing

**Figure 3** The Adaptive Myopic Policy

```

Initialize  $t = 0, S_0 = \emptyset, U_0 = \emptyset, s_0 = \emptyset$ .
While  $(U_t \cup S_t \neq \mathcal{A})$  do
     $t \leftarrow t + 1; S_t \leftarrow S_{t-1}; U_t \leftarrow U_{t-1}; s_t \leftarrow s_{t-1}$ .
    Repeat
        Find  $X_i \in \operatorname{argmax}_{X_i \in \mathcal{A} \setminus (U_t \cup S_t)} \mathbf{E}[F(S_{t-1} \cup \{X_i\}) | s_{t-1}]$ .
        If  $S_{t-1} \cup \{X_i\} \notin \mathcal{I}$ , then
             $U_t \leftarrow U_t \cup \{X_i\}$ ,
        else
             $S_t \leftarrow S_t \cup \{X_i\}$ .
        Observe  $x_i$  and update the state  $s_t$ , i.e.,  $s_t \leftarrow s_{t-1} \cup \{x_i\}$ .
    Until  $(S_t \neq S_{t-1})$  or  $(U_t \cup S_t = \mathcal{A})$ .
End (while)
    
```

more elements in order to possibly add new elements to  $S_t$ . In the latter, however, after exiting the repeat loop, the policy immediately jumps out of the while loop, too. Consequently, the policy ends.

**Theorem 3 (Adaptive Myopic Policy)** *If feasible domain  $\mathcal{I}$  is the intersection of  $\kappa$  matroids, then the approximation ratio of the adaptive myopic policy with respect to any optimal adaptive policy is  $\frac{1}{\kappa+1}$ .*

The proof is given in Appendix A.2. In order to prove the above bound, we will rely on the following lemma proved by Fisher et al. (1978). Note that in each iteration of the while loop, at least one new element of  $\mathcal{A}$  is probed. Hence,  $|\mathcal{A}|$  is an upper bound on the number of iterations of the while loop in the myopic policy. For the clarity of the presentation, if the policy ends at some iteration  $t^*$  where  $t^* < |\mathcal{A}|$ , we define  $U_i := U_{t^*}$  and  $S_i := S_{t^*}$  for all  $i > t^*$ . (In other words, we assume that the policy continues until the  $|\mathcal{A}|$ -th iteration but does not do anything after iteration  $t^*$ .)

**Lemma 11** (Fisher et al., 1978) *Let  $P \in \mathcal{I}_1 \cap \mathcal{I}_2 \cap \dots \cap \mathcal{I}_\kappa$  be any arbitrary subset in the intersection of  $\kappa$  matroids,  $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_\kappa$ . For  $0 \leq i \leq |\mathcal{A}|$ , let  $C_i = P \cap (U_{i+1} \setminus U_i)$ , where  $U_i$ 's are coming from The Adaptive Myopic Policy. We will have  $\sum_{i=0}^t |C_i| \leq \kappa t$ , for every  $1 \leq t \leq |\mathcal{A}|$ .*

Note that  $C_i$ 's,  $U_i$ 's, and  $S_i$ 's are random sets that depend on the decisions made by the policy and possibly based on the outcomes of the previous realizations of the chosen elements. However, the lemma holds for every realization path because the above property is a consequence of the matroid constraint, not the realizations of the element chosen by the policy.

Define  $\Delta_t := F(S_t) - F(S_{t-1})$ , for  $1 \leq t \leq |\mathcal{A}|$ . The main difficulty in proving the theorem compared to the non-stochastic case is that the realized values of  $\Delta_t$  are not always decreasing — note that  $\mathbf{E}[\Delta_t | s_t] \geq \mathbf{E}[\Delta_{t+1} | s_t]$  does not necessarily hold due to the stochastic nature of the problem. In addition, the sequence of elements chosen by the optimal adaptive policy is random. Please see the proof in the appendix for more details.

In addition, in Appendix A.2, we show that the myopic policy obtains a stronger approximation ratio if the constraint matroid is uniform.

**Theorem 4 (Uniform Matroid)** *Over a uniform matroid, the approximation ratio of the adaptive myopic policy with respect to the optimal adaptive policy is equal to  $1 - \frac{1}{e}$ .*

This approximation ratio is in fact tight. As mentioned earlier, even in deterministic settings, the problem of maximizing a submodular function with respect to cardinality constraints cannot be approximated with a ratio better than  $1 - \frac{1}{e}$  within polynomial time; see Feige (1998).

## 6. Discussions

In this section, we discuss some of our previous assumptions and explain how our results would change in their absence.

### 6.1. Computation of $F$

The problem of calculating  $F$  and functions similar to it has appeared in several related works and is usually solved via sampling. For instance, in the framework of Vondrák (2007, 2008), where the author deals with a similar notion of  $F$  but only with Bernoulli distribution for all distributions  $g_i$ -s, it is shown that with repeated sampling one can get an estimation of  $F$  within a multiplicative error of  $1 \pm 1/\text{poly}(n)$ . The following proposition generalizes their results to Lipschitz continuous functions.

**Proposition 3 (Additive Error)** *Suppose  $f$  is  $K$ -Lipschitz continuous and for all  $i$ ,  $\mathbf{Var}[X_i] \leq \mathcal{V}$ . Then, for any arbitrary values of  $0 < \epsilon < 1$  and  $\delta > 0$ , the average value of  $f(\Theta_S)$  for  $t = \lceil K^2 n^2 \mathcal{V} \epsilon^{-1} \delta^{-0.5} \rceil$  independent samples is within the interval  $(\mathbf{E}[f(\Theta_S)] - \epsilon, \mathbf{E}[f(\Theta_S)] + \epsilon)$ , with probability at least  $1 - \delta$ .*

The proof is given in the online appendix. The above lemma could be used to achieve any desirable polynomially small additive error. For instance,  $\lceil K^2 n^2 \mathcal{V} \rceil$  samples is enough to ensure that with probability  $1 - n^{-2}$ , the average sampled value of  $f(\Theta_S)$  is within the interval  $(\mathbf{E}[f(\Theta_S)] - n^{-2}, \mathbf{E}[f(\Theta_S)] + n^{-2})$ .

Note that if we have an upper bound on the values of  $f$  (for instance, where  $X$ 's can only take binary values), then we can use Chernoff inequality in the proof. That allows us to get the same

bounds by only  $O(\log K + \log n + \log \mathcal{V})$  samples. Also, since in this case the values  $F$  can take would be bounded, one can easily come up with multiplicative error terms. This is formalized via the following proposition (proved in the online appendix).

**Proposition 4 (Multiplicative Error)** *Suppose for any realization  $\Theta_S$  with  $\Pr[\Theta_S] > 0$ , we have  $1 \leq f(\Theta_S) \leq \mathcal{F}$ . Then, for any arbitrary values of  $0 < \epsilon < 1$  and  $0 < \delta$ , the average value of  $f(\Theta_S)$  for  $t = \lceil -2\mathcal{F}^2 \ln(\delta/2)\epsilon^{-1} \rceil$  independent samples is within the interval  $(\mathbf{E}[f(\Theta_S)](1 - \epsilon), \mathbf{E}[f(\Theta_S)](1 + \epsilon))$ , with probability at least  $1 - \delta$ .*

Throughout this paper we have assumed that the values of the form  $\mathbf{E}[f(\Theta_S)]$ , or equivalently  $F(S)$ , for every subset  $S$  are accessible via an oracle. However, if such an oracle is not available, then one can incorporate the above lemmas to achieve our results with the desirable degree of precision. To observe this, we note that our algorithms have at most  $n$  steps, and at each step there are at most  $n$  values of the form  $\mathbf{E}[f(\Theta_S)]$  that should be estimated. Hence, at most  $n^2$  estimations of function  $\mathbf{E}[f(\Theta_S)]$  will be needed. Due to Proposition 3, each of these estimations will have at most  $\epsilon$  additive error with probability at least  $1 - \delta$  if the average is taken over  $\lceil K^2 n^2 \mathcal{V} \epsilon^{-1} \delta^{-0.5} \rceil$  samples. Therefore, union bound implies that with a probability at least  $1 - n^2 \delta$ , all these samples will be within  $\epsilon$  error from their true value. Since all the inequalities in our proofs have at most  $n$  terms, an error of at most  $\epsilon n$  will appear in our final results. Hence, if we set  $\delta = n^{-3}$ , we need to average over  $\lceil K^2 n^2 \mathcal{V} (\frac{\epsilon}{n})^{-1} (n^{-3})^{-0.5} \rceil$  (or simply,  $\lceil K^2 n^{4.5} \mathcal{V} \epsilon^{-1} \rceil$ ) samples throughout our algorithms in order to ensure that with probability at most  $1 - n^2 \cdot n^{-3}$  (or simply,  $1 - 1/n$ ) our results will stay valid with an additional error of  $\epsilon$ .

## 6.2. Choosing Elements with Replacement

So far we have assumed that there is no benefit in selecting an element multiple times since once an element is picked by a policy, its realized value does not change. However, this assumption is made only to simplify the presentation and has no effect on our results. This follows from the observation that one can create identical and independent copies of each random variable in order to simulate multiplicity. Hence, all our approximation guarantees and the adaptivity gap, which are *independent* of the size of the problem, will hold when multiple selection of an element might change its value. Our tightness results for approximation guarantees come from deterministic examples. Hence, for the same reason, they hold as well when selection with replacement is allowed.

We note that the above observations are in sharp contrast with minimization problems (e.g., minimum set cover) and threshold problems in which the goal is to reach a value of function above a certain hard-constrained threshold. For instance, in the problem of stochastic set cover (where the goal is to cover *all* the elements of a ground set using the minimum number of a given collection of its subsets), allowing a policy to choose multiple copies of an element significantly reduces the adaptivity gap, as shown in Goemans and Vondrák (2006).

### 6.3. Large Adaptivity Gap without the Independence Assumption

In this section, we present an example that shows that the adaptivity gap without the independence assumption can be at least  $n/2$ . Consider an instance of SMSM with the ground set  $\mathcal{A} = \{X_0, X_1, \dots, X_n\}$ , where  $X_0$ , with equal probability of  $\frac{1}{n}$ , takes one of the values in  $\{1, 2, \dots, n\}$ . Also, for  $i$ ,  $1 \leq i \leq n$ , we have  $X_i = M \times \mathbf{I}\{X_0 = i\}$  for some  $M \gg n$ , i.e.,  $X_j = M$  if  $X_0 = j$ ; otherwise,  $X_j = 0$ . The goal is to maximize a *linear* function  $f$  that corresponds to the sum of the realized values of the chosen elements. The constraint is to select at most two elements, i.e., an independent set from the matroid  $\mathcal{M} = (\mathcal{A}, \{S \subseteq \mathcal{A}, |S| \leq 2\})$ .

It is easy to see that any non-adaptive policy would yield a value of at most  $2M/n$  because each element selected will contribute at most  $M/n$  in expectation. However, an adaptive policy can select  $X_0$  first and then select  $j = x_0$ , and therefore, obtain a value of at least  $M$  (exactly  $M + j$  to be more precise). Hence, the adaptivity gap is at least  $n/2$ .

## 7. Concluding Remarks

In this paper we studied the problem of maximizing monotone submodular functions with respect to matroid constraints in a stochastic setting. Our model can be applied to various problems that involve both diminishing marginal returns and a stochastic environment. In order to capture the effect of partial contributions, we considered real-valued submodular functions instead of submodular set functions. We showed that a myopic adaptive policy is guaranteed to achieve a  $(1 - \frac{1}{e} \approx 0.63)$ -approximation of the optimal (adaptive) policy. Also, we studied the concept of the adaptivity gap in order to compare the performance of non-adaptive policies (which are very easy to implement) to that of adaptive policies. We showed that a very straightforward myopic non-adaptive policy achieves a  $(\frac{1}{2} \times (1 - \frac{1}{e}) \approx 0.316)$ -approximation of the value of the optimal policy. Moreover, we showed that the adaptivity gap for these problems is at most  $\frac{e}{e-1}$ , which implies the existence of a non-adaptive policy that achieves a 0.63-approximation of the optimal policy. Finally, we provided a polynomial algorithm to find a non-adaptive policy that achieves a  $(0.63 - \epsilon)$ -approximation of the optimal policy for any positive  $\epsilon$ . In our proofs, we generalized the techniques from the previous works in the literature (especially those in Vondrák 2008 and Călinescu et al. 2011) to the continuous setting by proving new stochastic dominance results that could be of independent interest.

There are many interesting questions that we leave open for future research. One such question is whether our algorithms perform better for the specific submodular functions that appear in practice. For instance, the submodular functions that arise from the context of viral marketing have a very specific structure due to the properties of the underlying graph of relationships in social networks (such as being self-similar and scale-free, see Barabási and Réka 2009). Exploiting these



properties may lead to improved analysis of our policies or to new policies with better performance guarantees. Also, studying similar classes of functions for which the adaptivity gap is bounded would be an interesting extension. We believe that one promising class of such functions could be the almost-concave functions, which are the functions whose values can be approximated closely by a concave function.

*Acknowledgment* We would like to thank Jan Vondrák and Marek Adamczyk whose comments and suggestions significantly improved this work.

## Appendix A: Proofs

### A.1. Appendix to Section 3

*Proof of Lemma 5* An optimal adaptive policy only chooses independent sets of  $\mathcal{M}$ . Due to monotonicity, all of these are also the bases of the matroid. Hence, an optimal adaptive policy ends up selecting one of the basis of the matroid. Thus, vector  $y$ , whose  $i$ -th entry represents the probability that element  $X_i \in \mathcal{A}$  is ultimately chosen by this policy, is a convex combination of the characteristic vectors of the basis of  $\mathcal{M}$ . Moreover, the expected value of the adaptive policy is bounded by  $f^+(y)$  because the policy has to satisfy the three properties mentioned earlier.  $\square$

**Lemma 12** For  $y \in \mathcal{B}(\mathcal{M})$ , we have  $f^+(y) \leq f^*(y)$ .

*Proof* Consider the function  $f^+$ . Following the notation of  $\Theta^{\uparrow j}$ , for every  $x \in \mathbb{R}$  we define  $\Theta^{\uparrow j;x}$  as a vector that has all entries except the  $j$ -th one as the same as  $\Theta$ , and its  $j$ -th entry is defined as  $\max\{\theta_j, x\}$ . Fix any  $y \in \mathcal{B}(\mathcal{M})$ , any feasible probability measure  $\alpha$  with respect to  $y$  (see definition (6) for  $f^+$ ), and any given vector  $\underline{\Theta} \in \mathbb{R}_+^n$ . We have

$$\begin{aligned} \int_{\Theta} \alpha_{\Theta} f(\Theta) d\Theta &\leq \int_{\Theta} \alpha_{\Theta} \left[ f(\underline{\Theta}) + \sum_{j=1}^n \left( f(\underline{\Theta}^{\uparrow j, \theta_j}) - f(\underline{\Theta}) \right) \right] d\Theta \\ &= f(\underline{\Theta}) + \sum_{j=1}^n y_j \left[ \int_{\theta_j > \underline{\theta}_j} \left( f(\underline{\Theta}^{\uparrow j, \theta_j}) - f(\underline{\Theta}) \right) g_j(\theta_j) d\theta_j \right]. \end{aligned}$$

The inequality above holds due to the submodularity of  $f$ , and the equality is a consequence of the properties of the feasible probability measure  $\alpha$ . However,  $\int_{\theta_j > \underline{\theta}_j} f(\underline{\Theta}^{\uparrow j, \theta_j}) g_j(\theta_j) d\theta_j$  is simply  $\mathbf{E}[f(\underline{\Theta}^{\uparrow j})]$ . Hence,

$$\int_{\Theta} \alpha_{\Theta} f(\Theta) \leq f(\underline{\Theta}) + \sum_{j=1}^n y_j \left( \mathbf{E}[f(\underline{\Theta}^{\uparrow j})] - f(\underline{\Theta}) \right).$$

Note that the inequality above holds for any arbitrary  $y \in \mathcal{B}(\mathcal{M})$ , any feasible probability measure  $\alpha$ , and any vector  $\underline{\Theta} \in \mathbb{R}_+^n$ . Thus, we have

$$f^+(y) = \sup_{\alpha} \left\{ \int_{\Theta} \alpha_{\Theta} f(\Theta) d\Theta \right\} \leq \inf_{\underline{\Theta}} \left\{ f(\underline{\Theta}) + \sum_{j=1}^n y_j \left( \mathbf{E}[f(\underline{\Theta}^{\uparrow j})] - f(\underline{\Theta}) \right) \right\} = f^*(y). \quad (13)$$

$\square$

## A.2. Appendix to Section 5

*Proof of Theorem 3:* Let  $P$  be the (random) set of elements chosen by the optimal adaptive policy. Let  $S$  denote the final selection of the elements by our myopic policy, i.e.,  $S = S_{|\mathcal{A}|}$ . Moreover, let  $t$  be any arbitrary number between 1 and  $|\mathcal{A}|$ . Consider a realization  $s_t$  of  $S_t$ . By submodularity and monotonicity of  $F$  we have

$$\mathbf{E}[F(P) - F(S)|s_t] \leq \mathbf{E}[F(P \cup S) - F(S)|s_t] \leq \mathbf{E}\left[\sum_{l \in P \setminus S} (F(S+l) - F(S))|s_t\right].$$

The expectations in the above inequality are taken over the probability distribution of all possible realizations conditioned on the realized values of elements in  $S_t$  are equal to  $s_t$ .

Since the above inequality holds for all  $s_t$ , we have

$$\mathbf{E}[F(P)] - \mathbf{E}[F(S)] \leq \mathbf{E}\left[\sum_{l \in P \setminus S} (F(S+l) - F(S))\right]. \quad (14)$$

Now, define  $C_t = P \cap (U_{t+1} \setminus U_t)$ , for all  $0 \leq t \leq |\mathcal{A}|$ . In other words,  $C_t$  represents the elements of the optimum solution  $P$  that our myopic policy has probed (but not picked) after picking its  $t$ -th and before picking its  $(t+1)$ -st elements. By the construction of the policy, we have  $U_0 \subseteq U_1 \subseteq \dots \subseteq U_{|\mathcal{A}|}$ . Hence,  $(U_{t+1} \setminus U_t)$  and  $(U_{t'+1} \setminus U_{t'})$  are disjoint for any  $t \neq t'$ . Therefore,  $C_t$ 's are all disjoint. On the other hand, Lemma 11 implies that  $|C_0| = 0$  and hence  $C_0 = \emptyset$ . Therefore,  $\bigcup_{t=1}^{|\mathcal{A}|} C_t$  represents all the elements in  $P$  that were not selected by our policy, i.e.,  $\bigcup_{t=1}^{|\mathcal{A}|} C_t = P \setminus S$ . As a result, we can rewrite Ineq. (14) as follows:

$$\mathbf{E}[F(P)] - \mathbf{E}[F(S)] \leq \sum_{t=1}^{|\mathcal{A}|} \mathbf{E}\left[\sum_{l \in C_t} (F(S+l) - F(S))\right].$$

By expanding the expectation we have

$$\mathbf{E}[F(P)] - \mathbf{E}[F(S)] \leq \sum_{t=1}^{|\mathcal{A}|} \int_{s_{t-1}: S_{t-1} \in \mathcal{I}} \mathbf{E}\left[\sum_{l \in C_t} F(S+l) - F(S)|s_{t-1}\right] \mathbf{Pr}[s_{t-1}] ds_{t-1} \quad (15)$$

Fix any  $s_{t-1}$  and  $l \in C_t$ . Note that the myopic policy probes the elements in decreasing order of their expected marginal value and it has picked the  $t$ -th element of  $S$  (i.e., the only element in  $S_t \setminus S_{t-1}$ ) before  $l \in C_t$ . Hence, we have  $\mathbf{E}[F(S_{t-1} + l) - F(S_{t-1})|s_{t-1}] \leq \mathbf{E}[F(S_t) - F(S_{t-1})|s_{t-1}] = \mathbf{E}[\Delta_t|s_{t-1}]$ . Now, since  $F$  is submodular and  $S_{t-1} \subseteq S$ , for any  $l \in C_t$  we have  $\mathbf{E}[\sum_{l \in C_t} F(S+l) - F(S)|s_{t-1}] \leq \mathbf{E}[\sum_{l \in C_t} \Delta_t|s_{t-1}]$ . By plugging the above inequality into (15), we get

$$\mathbf{E}[F(P)] - \mathbf{E}[F(S)] \leq \sum_{t=1}^{|\mathcal{A}|} \int_{s_{t-1}: S_{t-1} \in \mathcal{I}} \mathbf{E}\left[\sum_{l \in C_t} \Delta_t|s_{t-1}\right] \mathbf{Pr}[s_{t-1}] ds_{t-1}.$$

Using telescopic sums and the linearity of expectation, we derive the following; here,  $\Delta_{|\mathcal{A}|+1}$  is defined as 0:

$$\begin{aligned} \mathbf{E}[F(P)] - \mathbf{E}[F(S)] &\leq \sum_{t=1}^{|\mathcal{A}|} \int_{s_{t-1}: S_{t-1} \in \mathcal{I}} \mathbf{E}\left[\sum_{l \in C_t} \sum_{j=t}^{|\mathcal{A}|} (\Delta_j - \Delta_{j+1})|s_{t-1}\right] \mathbf{Pr}[s_{t-1}] ds_{t-1} \\ &= \sum_{j=1}^{|\mathcal{A}|} \sum_{t=1}^j \int_{s_{t-1}: S_{t-1} \in \mathcal{I}} \mathbf{E}\left[\sum_{l \in C_t} (\Delta_j - \Delta_{j+1})|s_{t-1}\right] \mathbf{Pr}[s_{t-1}] ds_{t-1}. \end{aligned}$$

Note that by applying the law of total probability, for every  $t$  and  $j$  the integral term in the above is in fact equal to  $\mathbf{E}[\sum_{l \in C_t} (\Delta_j - \Delta_{j+1})]$ . Again, we use the law of total probability, but this time by conditioning on  $s_{j-1}$  instead of  $s_{t-1}$ . We will have

$$\mathbf{E}[F(P)] - \mathbf{E}[F(S)] \leq \sum_{j=1}^{|\mathcal{A}|} \sum_{t=1}^j \int_{s_{j-1}: S_{j-1} \in \mathcal{I}} \mathbf{E}[\sum_{l \in C_t} (\Delta_j - \Delta_{j+1}) | s_{j-1}] \Pr[s_{j-1}] ds_{j-1}.$$

Note that the term  $(\Delta_j - \Delta_{j+1})$  in the innermost summation does not depend on the index of the sum, i.e.,  $l \in C_t$ . Hence, the r.h.s. can be written as follows:

$$= \sum_{j=1}^{|\mathcal{A}|} \sum_{t=1}^j \int_{s_{j-1}: S_{j-1} \in \mathcal{I}} (E[|C_t| \mathbf{E}[\Delta_j - \Delta_{j+1} | s_{j-1}] | s_{j-1}]) \Pr[s_{j-1}] ds_{j-1}.$$

Note that conditioned on  $s_{j-1}$ , the term  $\mathbf{E}[\Delta_j - \Delta_{j+1} | s_{j-1}]$  is by definition a constant, and we can take it out from the outer expectation. Hence,

$$\mathbf{E}[F(P)] - \mathbf{E}[F(S)] \leq \sum_{j=1}^{|\mathcal{A}|} \int_{s_{j-1}: S_{j-1} \in \mathcal{I}} \left( \mathbf{E}[\sum_{t=1}^j |C_t| | s_{j-1}] \mathbf{E}[\Delta_j - \Delta_{j+1} | s_{j-1}] \right) \Pr[s_{j-1}] ds_{j-1}. \quad (16)$$

We now use Lemma 11, which implies that in every realization  $\sum_{t=1}^j |C_t| \leq \kappa j$ . We also use the fact that due to the submodularity and the rule of the policy, we have  $\mathbf{E}[(\Delta_j - \Delta_{j+1}) | s_{j-1}] \geq 0$ . We conclude that

$$\begin{aligned} \mathbf{E}[F(P)] - \mathbf{E}[F(S)] &\leq \sum_{j=1}^{|\mathcal{A}|} \int_{s_{j-1}: S_{j-1} \in \mathcal{I}} \left( \mathbf{E}[\sum_{t=1}^j |C_t| | s_{j-1}] \mathbf{E}[\Delta_j - \Delta_{j+1} | s_{j-1}] \right) \Pr[s_{j-1}] ds_{j-1} \\ &\leq \sum_{j=1}^{|\mathcal{A}|} \int_{s_{j-1}: S_{j-1} \in \mathcal{I}} \kappa j \mathbf{E}[(\Delta_j - \Delta_{j+1}) | s_{j-1}] \Pr[s_{j-1}] ds_{j-1} \\ &= \sum_{j=1}^{|\mathcal{A}|} \int_{s_{j-1}: S_{j-1} \in \mathcal{I}} \kappa \mathbf{E}[\Delta_j | s_{j-1}] \Pr[s_{j-1}] ds_{j-1} \\ &= \kappa \sum_{j=1}^{|\mathcal{A}|} \mathbf{E}[\Delta_j] = \kappa \mathbf{E}[F(S)]. \end{aligned}$$

Therefore,  $\mathbf{E}[F(P)] \leq (\kappa + 1) \mathbf{E}[F(S)]$ , as desired. Finally, Fisher et al. (1978) have shown that even in the non-stochastic setting, in the worst case, the approximation ratio of the greedy algorithm (hence the myopic policy) is equal to  $\frac{1}{\kappa+1}$ . Therefore, this bound is tight.  $\square$

*Proof of Theorem 4:* This proof is similar to the proof of Kleinberg et al. (2004) for submodular set functions. The main technical difficulty in proving our claim is that the optimal adaptive policy here is a random set whose distribution depends on the realized values of the elements of  $\mathcal{A}$ .

Let  $P$  denote the (random) set chosen by an optimal adaptive policy. Also, denote the marginal value of the  $t$ -th element chosen by the myopic policy by  $\Delta_t$ , i.e.,  $\Delta_t = F(S_t) - F(S_{t-1})$ . Now consider a realization  $s_t$  of  $S_t$ . Because  $F$  is stochastic monotone submodular, we have

$$\mathbf{E}[F(P) | s_t] \leq \mathbf{E}[F(P \cup S_t) | s_t] \leq \mathbf{E}[F(S_t) + \sum_{l \in P} (F(S_t + l) - F(S_t)) | s_t]. \quad (17)$$

The above expectations are taken over all realization of  $P$  such that the realized values of elements in  $S_t$  are according to  $s_t$ . Because the myopic policy chooses the element with maximum marginal value, for every

$l \in P$ , we have  $\mathbf{E}[\Delta_{t+1}|s_t] \geq \mathbf{E}[F(S_t + l) - F(S_t)|s_t]$ . Therefore, we get  $\mathbf{E}[F(P)|s_t] \leq \mathbf{E}[F(S_t) + k\Delta_{t+1}|s_t]$ . Since this inequality holds for every possible path in the history, by adding up all such inequalities for all  $t$ ,  $0 \leq t \leq k-1$ , we have

$$\mathbf{E}[F(P)] \leq \mathbf{E}[F(S_t)] + k\mathbf{E}[\Delta_{t+1}] = \mathbf{E}[\Delta_1 + \dots + \Delta_t] + k\mathbf{E}[\Delta_{t+1}].$$

We multiply the  $t$ -th inequality,  $0 \leq t \leq k-1$ , by  $(1 - \frac{1}{k})^{k-1-t}$ , and add them all. The sum of the coefficients of  $\mathbf{E}[F(P)]$  is equal to

$$\sum_{t=0}^{k-1} (1 - \frac{1}{k})^{k-1-t} = \sum_{t=0}^{k-1} (1 - \frac{1}{k})^t = \frac{1 - (1 - \frac{1}{k})^k}{1 - (1 - \frac{1}{k})} = k(1 - (1 - \frac{1}{k})^k). \quad (18)$$

On the right hand side, the sum of the coefficients corresponding to the term  $\mathbf{E}[\Delta_t]$ ,  $1 \leq t \leq k$ , is equal to

$$k(1 - \frac{1}{k})^{k-t} + \sum_{j=t}^{k-1} (1 - \frac{1}{k})^{k-1-j} = (1 - \frac{1}{k})^{k-t} + k(1 - (1 - \frac{1}{k})^{k-t}) = k. \quad (19)$$

Thus, by inequalities (18) and (19), we conclude  $(1 - (1 - \frac{1}{k})^k)\mathbf{E}[F(P)] \leq \sum_{t=1}^k \mathbf{E}[\Delta_t] = \mathbf{E}[F(S_k)]$ . Hence, the approximation ratio of the myopic policy is at least  $1 - \frac{1}{e}$ .  $\square$

## References

- Adamczyk, M., M. Sviridenko, J. Ward. 2014. Submodular stochastic probing on matroids. *STACS*. 29–40.
- Ahmed, S., A. Atamtürk. 2011. Maximizing a class of submodular utility functions. *Math. Programming* **128**(1-2) 149–169.
- Asadpour, A., H. Nazerzadeh, A. Saberi. 2008. Stochastic submodular maximization. *WINE*. 477–489.
- Barabási, A. L., A. Réka. 2009. Emergence of scaling in random networks. *Science* **286**(5439) 509–512.
- Călinescu, G., C. Chekuri, M. Pál, J. Vondrák. 2011. Maximizing a monotone submodular function subject to a matroid constraint. *SIAM J. Comput.* **40**(6) 1740–1766.
- Chan, C. W., V. F. Farias. 2009. Stochastic depletion problems: Effective myopic policies for a class of dynamic optimization problems. *Math. Oper. Res.* **34**(2) 333–350.
- Chung, F. R. K., L. Lu. 2006. Concentration inequalities and martingale inequalities: A survey. *Internet Math.* **3**(1) 79–127.
- Dean, B. C., M. X. Goemans, J. Vondrák. 2005. Adaptivity and approximation for stochastic packing problems. *SODA*. 395–404.
- Dean, B. C., M. X. Goemans, J. Vondrák. 2008. Approximating the stochastic knapsack problem: The benefit of adaptivity. *Math. Oper. Res.* **33**(4) 945–964.
- Dobzinski, S., M. Schapira. 2006. An improved approximation algorithm for combinatorial auctions with submodular bidders. *SODA*. 1064–1073.

- Edmonds, J. 1971. Matroids and the greedy algorithm. *Math. Programming* **1** 127–136.
- Feige, U. 1998. A threshold of  $\ln n$  for approximating set cover. *J. ACM* **45**(4) 634–652.
- Feige, U. 2009. On maximizing welfare when utility functions are subadditive. *SIAM J. Comput.* **39**(1) 122–142.
- Fisher, M., G. Nemhauser, L. Wolsey. 1978. An analysis of approximations for maximizing submodular set functions-ii. *Math. Programming Stud.* **8** 73–87.
- Gale, D. 1968. Optimal assignments in an ordered set: an application of matroid theory. *J. Combin. Theory* **4** 176–180.
- Goemans, M. X., J. Vondrák. 2006. Stochastic covering and adaptivity. *LATIN*. 532–543.
- Golovin, D., A. Krause. 2011a. Adaptive submodular optimization under matroid constraints. *CoRR* **abs/1101.4450**.
- Golovin, D., A. Krause. 2011b. Adaptive submodularity: Theory and applications in active learning and stochastic optimization. *J. Artif. Intell. Res. (JAIR)* **42** 427–486.
- Guillory, A., J. Bilmes. 2010. Interactive submodular set cover. *ICML*. 415–422.
- Kempe, D., J. M. Kleinberg, É. Tardos. 2003. Maximizing the spread of influence through a social network. *KDD*. 137–146.
- Kleinberg, J. M., C. H. Papadimitriou, P. Raghavan. 2004. Segmentation problems. *J. ACM* **51**(2) 263–280.
- Krause, A., C. Guestrin. 2005. Near-optimal nonmyopic value of information in graphical models. *UAI*. 324–331.
- Krause, A., C. Guestrin. 2007. Near-optimal observation selection using submodular functions. *AAAI*. 1650–1654.
- Mossel, E., S. Roch. 2010. Submodularity of influence in social networks: From local to global. *SIAM J. Comput.* **39**(6) 2176–2188.
- Nemhauser, G., L. Wolsey. 1978. Best algorithms for approximating the maximum of a submodular set function. *Math. Oper. Res.* **3**(3) 177–188.
- Nemhauser, G., L. Wolsey, M. Fisher. 1978. An analysis of the approximations for maximizing submodular set functions. *Math. Programming* **14** 265–294.
- Rado, R. 1957. A note on independence functions. *Proc. London Math. Soc.* **7** 300–320.
- Samuelson, P. A., W. D. Nordhaus. 2009. *Microeconomics (19th ed.)*. McGraw-Hill.
- Vondrák, J. 2007. Submodularity in combinatorial optimization. *PhD thesis, Charles University, Prague* URL [http://theory.stanford.edu/~jvondrak/data/KAM\\_thesis.pdf](http://theory.stanford.edu/~jvondrak/data/KAM_thesis.pdf).
- Vondrák, J. 2008. Optimal approximation for the submodular welfare problem in the value oracle model. *STOC*. 67–74.
- Weingartner, H. 1963. *Mathematical programming and the analysis of capital budgeting problems*. Prentice-Hall.

## Online Appendix

### Appendix A: Proofs

#### A.1. Proofs From Section 3.1

*Proof of Lemma 1* Consider an arbitrary non-adaptive policy which picks set  $S \subset \mathcal{A}$ , containing  $m^2$  sets from  $\mathcal{A}$ . For each  $i$ , define  $k_i = |S \cap \mathcal{A}^{(i)}|$ .

Moreover, each element  $i$  in the ground set is covered if and only if at least one of its corresponding chosen subsets are realized as a non-empty subset. Hence, it will be covered with probability  $1 - (1 - \frac{1}{m})^{k_i}$ . Therefore, the expected value of this policy is  $\sum_i 1 - (1 - \frac{1}{m})^{k_i}$ . Note that  $1 - (1 - \frac{1}{m})^x$  is a concave function with respect to  $x$  and also  $\sum_i k_i = m^2$ . Hence, the expected value of the policy is maximized when  $k_1 = k_2 = \dots = k_m = m$ . In this case, the expected value is  $(1 - (1 - \frac{1}{m})^m)m \approx (1 - \frac{1}{e})m$  for large  $m$ .  $\square$

*Proof of Lemma 2* Let  $X_k$  be the indicator random variable corresponding to the event that the subset chosen at the  $k$ -th step is realized as a non-empty subset for any  $1 \leq k \leq m^2$ . Note that the number of elements covered by  $\mathcal{P}$  is  $\sum_{k=1}^{m^2} X_k$ . Moreover, all  $X_k$ 's are independent random variables.

By the description of  $\mathcal{P}$ , as long as  $\sum_{i=1}^k X_k < m^2$ ,  $X_k$  will be one with probability  $\frac{1}{m}$  and will be zero with probability  $1 - \frac{1}{m}$ . Also, when  $\sum_{i=1}^t X_k = m$ , we have already covered all the elements in the ground set. Therefore,  $X_{t+1}, \dots, X_{m^2}$  will all be equal to zero. With this observation, we define i.i.d random variables  $Y_1, Y_2, \dots, Y_{m^2}$ , where each  $Y_i$  is set to be one with probability  $\frac{1}{m}$  and zero with probability  $\frac{1}{m}$ . Observe that  $\min\{m, Y = \sum_k Y_k\}$  has the same probability distribution as  $\sum_k X_k$ . Note that  $\mathbf{E}[Y] = m$ . Using Chernoff bound, we have  $\Pr[Y \leq m - m^{2/3}] \leq e^{-\frac{m^{4/3}}{2m}} = e^{-m^{1/3}}$ . Thus, with probability at least  $1 - e^{-m^{1/3}}$ , we have  $Y > m - m^{2/3}$ . Hence,  $\mathbf{E}[\sum_{k=1}^{m^2} X_k] = \mathbf{E}[\min\{m, Y\}] \geq (1 - e^{-m^{1/3}})(m - m^{2/3}) = m - o(m)$ , which completes the proof of the lemma.  $\square$

#### A.2. Proofs From Section 3.2

*Proof of Lemma 3* The intuition behind the proof is that  $F$  can be thought of as a linear combination of some monotone submodular functions. For monotonicity, let  $S \subsetneq \mathcal{A}$  and  $i$  be so that  $X_i \notin S$ . Consider any arbitrary realization  $\bar{\Theta}_S = (\theta_1, \theta_2, \dots, \theta_n)$ , and let  $Z = (\zeta_1, \zeta_2, \dots, \zeta_n)$  be their corresponding values in Eq. (2) such that  $f(\bar{\Theta}_S) = f(Z)$ . Because  $X_i \notin S$ , we have  $\theta_i = 0$  and consequently,  $\zeta_i = 0$ . Now, by adding  $X_i$  to  $S$  we will have the following.

$$\begin{aligned} E[f(\Theta_{S \cup \{X_i\}}) | \bar{\Theta}_S] &= \int_{x_i \in \Omega_i} f(\zeta_1, \dots, \zeta_{i-1}, x_i, \zeta_{i+1}, \dots, \zeta_n) g(x_i) dx_i \\ &\geq \int_{x_i \in \Omega_i} f(\zeta_1, \dots, \zeta_{i-1}, 0, \zeta_{i+1}, \dots, \zeta_n) g(x_i) dx_i \\ &= f(\zeta_1, \dots, \zeta_{i-1}, 0, \zeta_{i+1}, \dots, \zeta_n) = f(\bar{\Theta}_S), \end{aligned}$$

where the inequality holds due to the monotonicity of the function  $f$  and the fact that  $\Omega_i \subseteq \mathbb{R}_+$ . Now, we apply the derived inequality to bound the value of  $F$  over the subset  $S \cup \{X_i\}$ .

$$\begin{aligned} F(S \cup \{X_i\}) &= \mathbf{E}[f(\Theta_{S \cup \{X_i\}})] = \int_{\bar{\Theta}_S \in \Omega} E[f(\Theta_{S \cup \{X_i\}}) | \bar{\Theta}_S] g_S(\bar{\Theta}_S) d\bar{\Theta}_S \\ &\geq \int_{\bar{\Theta}_S \in \Omega} f(\bar{\Theta}_S) g_S(\bar{\Theta}_S) d\bar{\Theta}_S = E[f(\Theta_S)] = F(S). \end{aligned}$$

This completes the proof of monotonicity of function  $F$ .

The proof of submodularity also follows from a similar path. Let  $S$  and  $T$  be any arbitrary subsets of the ground set  $\mathcal{A}$ . We have

$$F(S) + F(T) = E[f(\Theta_S)] + E[f(\Theta_T)],$$

where  $\Theta_S$  (resp.  $\Theta_T$ ) is the realization of the elements if we choose the set of elements in  $S$  (resp.  $T$ ). Hence,

$$F(S) + F(T) = \int_{\Theta: \theta_i = 0, i \notin S} f(\Theta) \prod_{i: X_i \in S} g_i(\theta_i) d\theta_i + \int_{\Theta: \theta_i = 0, i \notin T} f(\Theta) \prod_{i: X_i \in T} g_i(\theta_i) d\theta_i. \quad (20)$$

Note that  $g_i$ 's are probability distributions corresponding to independent random variables. Hence, for every set  $A \in \mathcal{A}$  we have

$$\int_{\theta_i \in \Omega_i: X_i \in A} \prod_{i: X_i \in A} g_i(\theta_i) d\theta_i = 1.$$

Combining the above equality for  $A = \mathcal{A} \setminus S$  and  $A = \mathcal{A} \setminus T$  with Eq. (20) results in the following.

$$\begin{aligned} F(S) + F(T) &= \int_{\Theta: \theta_i = 0, i \notin S} f(\Theta) \left( \int_{\theta_i \in \Omega_i: X_i \notin S} \prod_{i: X_i \notin S} g_i(\theta_i) d\theta_i \right) \prod_{i: X_i \in S} g_i(\theta_i) d\theta_i \\ &\quad + \int_{\Theta: \theta_i = 0, i \notin T} f(\Theta) \left( \int_{\theta_i \in \Omega_i: X_i \notin T} \prod_{i: X_i \notin T} g_i(\theta_i) d\theta_i \right) \prod_{i: X_i \in T} g_i(\theta_i) d\theta_i \\ &= \int_{\Theta} f(\Theta(S)) \prod_i g_i(\theta_i) d\theta_i + \int_{\Theta} f(\Theta(T)) \prod_i g_i(\theta_i) d\theta_i \\ &= \int_{\Theta} \left( f(\Theta(S)) + f(\Theta(T)) \right) \prod_i g_i(\theta_i) d\theta_i, \end{aligned}$$

where for every  $A \in \mathcal{A}$ ,  $\Theta(A)$  is defined as  $(\theta_1 \times \mathbb{1}_{X_1 \in A}, \theta_2 \times \mathbb{1}_{X_2 \in A}, \dots, \theta_n \times \mathbb{1}_{X_n \in A})$ .

The submodularity of means that

$$f(\Theta(S)) + f(\Theta(T)) \geq f(\Theta(S) \vee \Theta(T)) + f(\Theta(S) \wedge \Theta(T)).$$

However,  $\Theta(S) \vee \Theta(T)$  is nothing but the component-wise maximum of  $\Theta(S)$  and  $\Theta(T)$ . Hence, its  $i$ -th entry is the pairwise maximum of  $\theta_i \times \mathbb{1}_{X_i \in S}$  and  $\theta_i \times \mathbb{1}_{X_i \in T}$ , or equivalently,  $\theta_i \times \mathbb{1}_{X_i \in S \cup T}$ . Similarly, the  $i$ -th entry of  $\Theta(S) \wedge \Theta(T)$  is  $\theta_i \times \mathbb{1}_{X_i \in S \cap T}$ .

In summary,

$$\begin{aligned} F(S) + F(T) &= \int_{\Theta} \left( f(\Theta(S)) + f(\Theta(T)) \right) \prod_i g_i(\theta_i) d\theta_i \\ &\geq \int_{\Theta} \left( f(\Theta(S) \vee \Theta(T)) + f(\Theta(S) \wedge \Theta(T)) \right) \prod_i g_i(\theta_i) d\theta_i \\ &\geq \int_{\Theta} \left( f(\Theta(S \cup T)) + f(\Theta(S \cap T)) \right) \prod_i g_i(\theta_i) d\theta_i \\ &= \mathbf{E}[f(\Theta_{S \cup T})] + \mathbf{E}[f(\Theta_{S \cap T})] \\ &= F(S \cup T) + F(S \cap T). \end{aligned}$$

This completes the proof of the lemma. □

### A.3. Proof from Section 4

*Proof of Lemma 7* By definition (7), the  $j$ -th entry of  $\Theta_{R(t)}^{\uparrow j}$ , for any fixed  $R(t)$  is as follows:

- If  $X_j \in R(t)$ , then it is the maximum of two independently sampled random variables from the c.d.f.  $G_j$ .

Hence, its own c.d.f. will be  $G_j^2$ .

- If  $X_j \notin R(t)$ , then it is simply a random variable sampled from c.d.f.  $G_j$ .

Similar to the rest of the paper, we assume that we have oracle access to such values, i.e. we know the values  $\mathbf{E}_{\Theta}[f(\Theta_{R(t)})]$  and  $\mathbf{E}_{\Theta}[f(\Theta_{R(t)}^{\uparrow j})]$  (and consequently, the value of  $\mathbf{E}_{\Theta}[f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)})]$ ) when  $R(t)$  is fixed and the expectation is taken over all realizations of  $\Theta$ . (For computing these values using sampling refer to the appendix.) For the simplicity of notation, for a fixed  $R(t)$  we define  $f(R(t), j) = \mathbf{E}_{\Theta}[f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)})]$ . The interpretation of  $f(R(t), j)$  is the expected marginal contribution of adding element  $j$  to the set  $R(t)$ , while allowing to improve the value of  $\theta_j$  if  $j$  is already in  $R(t)$ .

Note that similar to Lemma 3.2 in Călinescu et al. (2011)  $R(t)$  is a random set containing each element  $j$  independently at random with probability  $y_j$ . Now, consider an estimate  $w_j(t)$  of  $\mathbf{E}_R[f(R(t), j)]$  obtained by  $\text{blah}$  independent samples  $R_i$  of  $R(t)$ .

Let us call an estimate  $w_j$  *bad* if  $|w_j(t) - \mathbf{E}_R[f(R(t), j)]| > \delta \text{OPT}$ . Following the proof of that lemma, we define  $X_i = (f(R_i, j) - \mathbf{E}_R[f(R(t), j)])/\text{OPT}$ ,  $k = \frac{4}{\delta^2}(1 + \ln n - 0.5 \ln \delta)$ , and  $a = \delta k$ . We have  $|X_i| \leq 1$  since  $\text{OPT} \geq \max_j f(\{j\}, j) \geq \max_{R \subseteq \mathcal{A}} f(R, j)$ , where the first inequality holds because of monotonicity of  $f$  and the second inequality is due to the submodularity of  $f$  and hence, diminishing marginal return of any element  $j$ . The estimate is bad if and only if  $|\sum_i X_i| > a$ . But the Chernoff bound (see Theorem 2.2 in Călinescu et al. 2011) implies that  $\Pr[|\sum_i X_i| > a] \leq 2e^{-\delta^2 k/2} = 2e^{-2-2 \ln n + \ln \delta} \leq \delta/(3n^2)$ .

Note that in each step we compute  $n$  estimates (one for each  $X_j \in \mathcal{A}$ ) and the total number of steps is  $1/\delta$ . By the union bound, the probability of having any bad estimate is at most  $\frac{n}{\delta} \times \frac{\delta}{3n^2}$ . Hence, with probability at least  $(1 - \frac{1}{3n})$  all estimates throughout the algorithm are good, i.e.,

$$|w_j(t) - \mathbf{E}_R[f(R(t), j)]| \leq \delta \text{OPT}, \quad (21)$$

for all  $j$  and  $t$ . Now, let  $I \in \mathcal{I}$  be any independent set. Note that  $|I| = d$ , and with high probability there is no bad estimates. Thus, with high probability

$$\left( \sum_{j: X_j \in I} \mathbf{E}_R[f(R(t), j)] \right) + d\delta \cdot \text{OPT} \geq \sum_{j: X_j \in I} w_j(t) \geq \left( \sum_{j: X_j \in I} \mathbf{E}_R[f(R(t), j)] \right) - d\delta \cdot \text{OPT}. \quad (22)$$

Now, let  $I^*$  denote an independent set achieving the  $\max_{I \in \mathcal{I}} \sum_{j: X_j \in I} \mathbf{E}_{\Theta, R}[f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)})]$ . Our algorithm finds a set  $I(t) \in \mathcal{I}$  that maximizes  $\sum_{j: X_j \in I} w_j$ . Hence, with high probability

$$\begin{aligned} \sum_{j: X_j \in I(t)} \mathbf{E}_{\Theta, R}[f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)})] &= \sum_{j: X_j \in I(t)} \mathbf{E}_R \left[ \mathbf{E}_{\Theta} [f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)})] \right] \\ &\stackrel{\text{by definition of } f(R(t), j)}{=} \sum_{j: X_j \in I(t)} \mathbf{E}_R[f(R(t), j)] \\ &\stackrel{\text{by the first inequality in (22)}}{\geq} \sum_{j: X_j \in I(t)} w_j(t) - d\delta \cdot \text{OPT} \end{aligned}$$



$$\begin{aligned}
 & \boxed{\text{by the choice of } I(t)} && \geq \sum_{j: X_j \in I^*} w_j(t) - d\delta \cdot \text{OPT} \\
 & \boxed{\text{by the second inequality in (22)}} && \geq \sum_{j: X_j \in I^*} \mathbf{E}_R[f(R(t), j)] - 2d\delta \cdot \text{OPT} \\
 & \boxed{\text{by definition of } f(R(t), j)} && = \sum_{j: X_j \in I^*} \mathbf{E}_{\Theta, R} \left[ f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)}) \right] - 2d\delta \cdot \text{OPT},
 \end{aligned}$$

which completes the proof of the lemma.  $\square$

*Proof of Lemma 8:* As before, let  $R(t)$  denote the random set that contains  $X_j$  with probability  $y_j(t)$  independently at random. Also, let  $D(t)$  be a random set that contains  $X_j$  with probability  $\delta_j(t) = y_j(t + \delta) - y_j(t)$  independently at random. Note that according to our algorithm  $\delta_j(t) = \delta \cdot \mathbf{1}_{j \in I(t)}$ . We show that  $F(y(t + \delta)) \geq \mathbf{E}_{\Theta, R, D}[f(\Theta_{R(t)} \vee \Theta_{D(t)})]$ , where the expectation is taken over all the random sets  $R(t)$  and  $D(t)$  and their corresponding realization of  $\Theta$ . We emphasize that due to the independence of each entry and also the monotonicity of  $f$ , we only need to show the entry-by-entry stochastic dominance of the left hand side over the right hand side. Let  $\alpha(x)$  be the c.d.f. of the  $j$ -th entry of the left hand side, i.e.  $F(y(t + \delta))$ . Also, let  $\beta(x)$  denote the c.d.f. of the  $j$ -th entry of  $\mathbf{E}_{\Theta, R, D}[f(\Theta_{R(t)} \vee \Theta_{D(t)})]$ . By the definition of  $F$  we know that  $F(y(t + \delta)) = \mathbf{E}_{\Theta, R}[f(\Theta_{R(t+\delta)})]$ , where each  $X_j$  is contained in  $R(t + \delta)$  with probability  $y_j(t + \delta) = y_j(t) + \delta_j(t)$ . Hence, for the c.d.f. of its  $j$ -th entry we have

$$\alpha(x) = 1 - \left( y_j(t) + \delta_j(t) \right) + G_j(x) \left( y_j(t) + \delta_j(t) \right),$$

for every  $x \in \mathbb{R}_+$ .

On the other hand, the  $j$ -th entry of  $\Theta_{R(t)} \vee \Theta_{D(t)}$  can be understood as follows. The value of the entry is 0 iff  $X_j \notin R(t)$  and  $X_j \notin D(t)$ . This happens with probability  $(1 - y_j(t))(1 - \delta_j(t))$ . Similarly, with probability  $y_j(t)(1 - \delta_j(t)) + (1 - y_j(t))\delta_j(t)$  it will be a variable drawn from the c.d.f.  $G_j$ . Finally, with probability  $y_j(t)\delta_j(t)$  it will be the maximum of two independent random variables each drawn from c.d.f.  $G_j$ , where consequently, its c.d.f. will be  $G_j^2$ . In conclusion, the c.d.f. will be

$$\beta(x) = \left( (1 - y_j(t))(1 - \delta_j(t)) \right) + \left( y_j(t)(1 - \delta_j(t)) + (1 - y_j(t))\delta_j(t) \right) G_j(x) + \left( y_j(t)\delta_j(t) \right) G_j(x)^2,$$

for every  $x \in \mathbb{R}_+$ . But we have,

$$\begin{aligned}
 \beta(x) - \alpha(x) &= \left( y_j(t)\delta_j(t) \right) - \left( 2y_j(t)\delta_j(t) \right) G_j(x) + \left( y_j(t)\delta_j(t) \right) G_j(x)^2 \\
 &= \left( y_j(t)\delta_j(t) \right) \left( 1 - G_j(x) \right)^2 \\
 &\geq 0.
 \end{aligned}$$

This establishes the entry-by-entry stochastic dominance of  $F(y(t + \delta)) = \mathbf{E}_{\Theta, R}[f(\Theta_{R(t+\delta)})]$  over  $\mathbf{E}_{\Theta, R, D}[f(\Theta_{R(t)} \vee \Theta_{D(t)})]$ . Thus,

$$\begin{aligned}
 F(y(t + \delta)) - F(y) &= \mathbf{E}_{\Theta, R}[f(\Theta_{R(t+\delta)})] - \mathbf{E}_{\Theta, R}[f(\Theta_{R(t)})] \\
 &\geq \mathbf{E}_{\Theta, R, D}[f(\Theta_{R(t)} \vee \Theta_{D(t)})] - \mathbf{E}_{\Theta, R}[f(\Theta_{R(t)})] \\
 &\geq \sum_j \Pr[D(t) = \{X_j\}] \left( \mathbf{E}_{\Theta, R}[f(\Theta_{R(t)} \vee \Theta_{\{X_j\}})] - \mathbf{E}_{\Theta, R}[f(\Theta_{R(t)})] \right)
 \end{aligned}$$

$$\begin{aligned}
\boxed{\text{by the definition of } \Theta_{R(t)}^{\uparrow j}} &= \sum_{j: X_j \in I(t)} \delta(1-\delta)^{|I(t)|-1} \mathbf{E}_{\Theta, R} \left[ f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)}) \right] \\
\boxed{\text{by } |I(t)| = d} &\geq \delta(1-d\delta) \sum_{j: X_j \in I(t)} \mathbf{E}_{\Theta, R} \left[ f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)}) \right].
\end{aligned}$$

□

*Proof of Lemma 9:* Let  $\bar{y} \in B(\mathcal{M})$  be an arbitrary point in the base polytope of  $\mathcal{M}$ . This point can be written as the convex combination of some independent sets of  $\mathcal{M}$ . By taking this convex combination on the corresponding inequalities proved in Lemma 7 with high probability we have

$$\begin{aligned}
\sum_{j: X_j \in I(t)} \mathbf{E}_{\Theta, R} \left[ f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)}) \right] &\geq \max_{I \in \mathcal{I}} \sum_{j: X_j \in I} \mathbf{E}_{\Theta, R} \left[ f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)}) \right] - 2d\delta \text{OPT} \\
&\geq \sum_{j \in \mathcal{A}} \bar{y}_j \mathbf{E}_{\Theta, R} \left[ f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)}) \right] - 2d\delta \text{OPT} \\
&= -F(y(t)) + \left[ \mathbf{E}_{\Theta, R} [f(\Theta_{R(t)})] + \sum_{j \in \mathcal{A}} \bar{y}_j \mathbf{E}_{\Theta, R} \left[ f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)}) \right] \right] - 2d\delta \text{OPT} \\
&\geq -F(y(t)) + \inf_{\Theta} \left[ f(\Theta) + \sum_{j \in \mathcal{A}} \bar{y}_j [f(\Theta^{\uparrow j}) - f(\Theta)] \right] - 2d\delta \text{OPT},
\end{aligned}$$

where the first equality is due to the fact that by definition  $F(y(t)) = \mathbf{E}_{\Theta, R} [f(\Theta_{R(t)})]$ .

However, the infimum part is nothing but the definition of  $f^*(\bar{y})$  in (10). Therefore, with high probability

$$\sum_{j: X_j \in I(t)} \mathbf{E}_{\Theta, R} \left[ f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)}) \right] \geq f^*(\bar{y}) - F(y(t)) - 2d\delta \cdot \text{OPT},$$

for any arbitrary  $\bar{y} \in B(\mathcal{M})$ . Hence, with high probability

$$\sum_{j: X_j \in I(t)} \mathbf{E}_{\Theta, R} \left[ f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)}) \right] \geq \sup_{y \in B(\mathcal{M})} \{f^*(y)\} - F(y(t)) - 2d\delta \cdot \text{OPT}.$$

Also, note that OPT is the optimum value of the adaptive policy which is bounded by  $\sup_{y \in B(\mathcal{M})} f^+(y)$  (see Lemma 5). Also, we know from inequality (13) that  $f^+(y) \leq f^*(y)$  for every  $y \in B(\mathcal{M})$ . Hence,  $\text{OPT} \leq \sup_{y \in B(\mathcal{M})} f^*(y)$ . Consequently, with high probability

$$\sum_{j: X_j \in I(t)} \mathbf{E}_{\Theta, R} \left[ f(\Theta_{R(t)}^{\uparrow j}) - f(\Theta_{R(t)}) \right] \geq (1-2d\delta) \sup_{y \in B(\mathcal{M})} \{f^*(y)\} - F(y(t)).$$

By applying the result of Lemma 8 on the above inequality, with high probability we get the following.

$$\begin{aligned}
F(y(t+\delta)) - F(y(t)) &\geq \delta(1-d\delta) \left[ (1-2d\delta) \sup_{y \in B(\mathcal{M})} \{f^*(y)\} - F(y(t)) \right] \\
&\geq \delta \left[ (1-3d\delta) \sup_{y \in B(\mathcal{M})} \{f^*(y)\} - F(y(t)) \right],
\end{aligned}$$

which completes the proof of the lemma. □

#### A.4. Proof from Section 6.1

*Proof of Proposition 3* Note that  $F(S) = \mathbf{E}[f(\Theta_S)]$ . By Lemma 14 below we have  $\mathbf{Var}[f(\Theta_S)] \leq K^2|S|^2\mathcal{V}$ . Suppose that we take  $t$  independent samples of the value of  $f(\Theta_S)$  and take their average as an estimation for the actual value of  $f(\Theta_S)$ . The derived sample will have a variance lower than  $K^2|S|^2\mathcal{V}/t$ . The proof is completed by taking  $t = \lceil K^2n^2\mathcal{V}\epsilon^{-1}\delta^{-0.5} \rceil$  and applying the Chebyshev inequality.  $\square$

*Proof of Proposition 4* We use Chernoff bound. In particular, we use Theorems 6 and 7 in Chung and Lu (2006). Clearly, since  $0 \leq f(\Theta_S) \leq \mathcal{F}$ , we have  $\mathbf{Var}[f(\Theta_S)] \leq \mathcal{F}^2/2$ . Suppose we take  $t$  samples and let  $\Upsilon_t$  be their total summation. Theorem 6 in Chung and Lu (2006) ensures that

$$\Pr[\Upsilon_t \geq t \cdot f(\Theta_S) + \lambda] \leq e^{-\frac{\lambda^2}{2(\mathbf{Var}(\Upsilon_t) + \mathcal{F}\lambda/3)}}.$$

Note that  $\Upsilon_t$  consists of  $t$  independent samples of  $f(\Theta_S)$ . It means that  $\mathbf{Var}(\Upsilon_t) = t \cdot \mathbf{Var}(f(\Theta_S))$ , and hence, it is bounded from above by  $t\mathcal{F}^2/2$ . Now, let  $\lambda = t\epsilon$ . We will have

$$\Pr\left[\frac{\Upsilon_t}{t} \geq f(\Theta_S)(1 + \epsilon)\right] \leq \exp\left(-\frac{t^2\epsilon^2}{2(t\mathcal{F}^2/2 + \mathcal{F}t\epsilon f(\Theta_S)/3)}\right) \leq \exp\left(-\frac{t^2\epsilon^2}{2t\mathcal{F}^2}\right) = \exp\left(-\frac{t\epsilon}{2\mathcal{F}^2}\right).$$

We emphasize that our bound in the second inequality is loose. This is only for the sake of achieving the same upper-tail and the lower-tail bounds and subsequently, having a clearer representation.

Now, Theorem 7 in Chung and Lu (2006) implies that

$$\Pr[\Upsilon_t \leq t \cdot f(\Theta_S) - \lambda] \leq e^{-\frac{\lambda^2}{2t\mathcal{F}^2}}.$$

Again, by letting  $\lambda = t\epsilon$  we will have

$$\Pr\left[\frac{\Upsilon_t}{t} \geq f(\Theta_S)(1 + \epsilon)\right] \leq \exp\left(-\frac{t^2\epsilon^2}{2t\mathcal{F}^2}\right) = \exp\left(-\frac{t\epsilon}{2\mathcal{F}^2}\right).$$

Now, we can conclude the following for the chance that the average of  $t$  samples (i.e.  $\Upsilon_t/t$ ) deviates from its expected value (i.e.  $f(\Theta_S)$ ) by a factor more than  $(1 \pm \epsilon)$ .

$$\Pr\left[f(\Theta_S)(1 - \epsilon) \leq \frac{\Upsilon_t}{t} \leq f(\Theta_S)(1 + \epsilon)\right] \geq 1 - 2\exp\left(-\frac{t\epsilon}{2\mathcal{F}^2}\right).$$

The above inequality for  $t = \lceil -2\mathcal{F}^2 \ln(\delta/2)\epsilon^{-1} \rceil$  completes the proof.  $\square$

**A.4.1. Bounding the Variance of  $f(\Theta_S)$**  Suppose  $S = \{X_{i_1}, X_{i_2}, \dots, X_{i_m}\}$  and let  $S_j = \{X_{i_1}, X_{i_2}, \dots, X_{i_j}\}$ . We write  $F(S)$  as the telescopic sum of its marginal values.

$$f(\Theta_S) = \sum_{j=1}^m [f(\Theta_{S_j}) - f(\Theta_{S_{j-1}})].$$

First, we bound the variance of each term of the summation.

**Lemma 13** *For every  $1 \leq j \leq m$ , if  $f$  is  $K$ -Lipschitz continuous, then*

$$\mathbf{Var}[f(\Theta_{S_j}) - f(\Theta_{S_{j-1}})] \leq K^2 \mathbf{Var}[X_{i_j}].$$

*Proof* We will prove a stronger result here. We show that the claim holds for any possible realizations of  $\Theta_{S_{j-1}}$ . Fix  $\Theta_{S_{j-1}} = (\theta_1, \theta_2, \dots, \theta_n)$ . Note that  $\Theta_{S_j}$  will be different from  $\Theta_{S_{j-1}}$  in only one dimension, namely the one associated with  $X_{i_j}$ . Define  $h(x) := f(\Theta_{S_j}) - f(\Theta_{S_{j-1}})$  for the specific realization in which the value of  $X_{i_j}$  is  $x$ . Equivalently,

$$h(x) = f(\zeta_1, \dots, \zeta_{i_j-1}, x, \zeta_{i_j+1}, \dots, \zeta_n) - f(\zeta_1, \dots, \zeta_n), \quad \forall x \in \mathbb{R}_+$$

$$\text{where } \zeta_i = \begin{cases} \theta_i & \theta_i = x_i, \\ 0 & \theta_i = \circ. \end{cases}$$

Note that  $X_{i_j} \notin S_{i_j-1}$ , hence,  $\theta_{i_j} = \circ$  and consequently,  $\zeta_{i_j} = 0$ . Therefore, because  $f$  is monotone, the function  $h$  defined above will be non-negative and increasing. Moreover,  $h$  is also  $K$ -Lipschitz continuous. Let  $\mu = \mathbf{E}[X_{i_j}]$ . Due to the  $K$ -Lipschitz continuity of  $h(\cdot)$ , we have  $h(x) \leq h(\mu) + K(x - \mu)$  for  $x > \mu$ . Also, we have  $h(x) \geq h(\mu) - K(\mu - x)$  for  $x < \mu$ . Hence, the variance of  $h$  can be written as follows.

$$\begin{aligned} \mathbf{Var}[h(X_{i_j})] &= \mathbf{E} \left[ (h(X_{i_j}) - \mathbf{E}[h(X_{i_j})])^2 \right] \leq \mathbf{E} \left[ (h(X_{i_j}) - h(\mu))^2 \right] \\ &\leq \int_{x \in \Omega_{i_j}} K^2(x - \mu)^2 g_{i_j}(x) dx = K^2 \mathbf{Var}[X_{i_j}]. \end{aligned}$$

Note that the second inequality holds because for every random variable  $X$ , the function  $\nu(c) = \mathbf{E}[(X - c)^2]$  is minimized at  $c = \mathbf{E}[X]$ . This completes the proof of lemma.  $\square$

Now, we are ready to bound the variance of  $f(\Theta_S)$ .

**Lemma 14** *If  $f$  is  $K$ -Lipschitz continuous, and  $\mathcal{V} = \max_{X_i \in S} \mathbf{Var}[X_i]$ , then  $\mathbf{Var}[f(\Theta_S)] \leq |S|^2 K^2 \mathcal{V}$ .*

*Proof* Define  $Y_j = f(\Theta_{S_j}) - f(\Theta_{S_{j-1}})$ . Hence, we have  $f(\Theta_S) = \sum_{j=1}^m Y_j$ . By Lemma 13, for all  $j$ ,  $\mathbf{Var}[Y_j] \leq K^2 \mathcal{V}$ . We have,

$$\begin{aligned} \mathbf{Var}[f(\Theta_S)] &= \sum_{j: X_j \in S} \mathbf{Var}[Y_j] + \sum_{i < j: \{X_i, X_j\} \subset S} 2\text{Cov}[Y_i, Y_j] \\ &\leq |S| K^2 \mathcal{V} + \sum_{i < j: \{X_i, X_j\} \subset S} 2\sqrt{K^2 \mathbf{Var}[Y_i] \cdot K^2 \mathbf{Var}[Y_j]} \\ &\leq |S| K^2 \mathcal{V} + |S|(|S| - 1) K^2 \mathcal{V} = |S|^2 K^2 \mathcal{V}. \end{aligned}$$

$\square$