Asymptotic Optimality of Order-up-to Policies in Lost Sales Inventory Systems

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Abstract

We study a single-product single-location inventory system under periodic review, where excess demand is lost and the replenishment lead time is positive. The performance measure of interest is the long run average holding cost and lost sales penalty. For a large class of demand distributions, we show that when the lost sales penalty becomes large compared to the holding cost, the relative difference between the cost of the optimal policy and the best order-up-to policy converges to zero. For any given cost parameters, we establish a bound on this relative difference. Numerical experiments show that the best order-up-to policy performs well, yielding an average cost that is within 1.5% of the optimal cost even when the ratio between the lost sales penalty and the holding cost is just 100.

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1. Introduction

We study the optimal replenishment policy in a periodic-review single-stage inventory system that procures inventory from a source with ample supply. There is a replenishment lead time of τ periods between placing an order and its delivery. Demands in different periods are independent and identically distributed. In the event that demand in a period exceeds the available on-hand inventory, excess demand is lost and we incur a *lost sales penalty cost* of \$$b$ per unit. We also charge holding costs on inventory on hand at the end of each period at the rate of \$$h$ per unit per period. We wish to find an ordering policy that minimizes the long run average holding cost and lost sales penalty.

When demand is backordered instead of lost, Karlin and Scarf (1958) show that an order-up-to policy is optimal; that is, there is an order-up-to-level to which we raise the inventory position – defined as inventory available for immediate sales plus the amount of inventory that has been ordered and not yet delivered – in each period. They also show that this simple policy fails to be optimal in the lost sales model. In many business environments, lost sales occur frequently when customer demands are not met immediately. Hence, finding the optimal replenishment policy, characterizing its structural properties, and developing heuristics that work well in practical settings are important. Moreover, in many important inventory systems, we observe that the lost sales penalty b is generally much larger than the holding cost h, as shown in the following examples from retail and service parts environments. In this paper, we propose simple inventory policies that are guaranteed to perform well in such systems.

- *Retail:* Consider a product whose procurement cost is \$1 per unit to the retailer. Assume the retailer reviews the system and replenishes its inventory once a week, and sells the product at \$$(1 + m)$ per unit, where m represents the mark-up. The lost sales cost in this case is at least \$$m$ per unit, not including any loss in customer goodwill due to unfulfilled demand . The holding cost per unit per period is simply the cost of holding \$1 in inventory for a week. At a cost of capital of 15% per year, this is approximately \$0.0025 per unit per period. So, the ratio between the lost sales penalty cost and the holding cost in this example is at least $400m$. At a 25% mark-up, which is quite common in many retail environments, this ratio is at least 100.

- *Service Parts Maintenance:* Consider, for example, the business of maintaining service parts for personal computers, photocopiers, or telecommunication equipments. Most corporate clients purchase *service-level agreements* that require the manufacturer, in the event of a failure, to bring the equipment back to service within a specified time window within stipulated minimum probabilities (for example, within 2 hours for 95% of failures and within 6 hours for 99% of failures). To meet these agreements, the equipment manufacturers frequently expedite service parts to customer locations when the closest stocking locations do not have the necessary parts. Consider a \$100 part that has to be expedited at an additional cost of \$14. These systems are typically reviewed once a day. Assuming a cost of capital of 25% per year, the cost of holding this part in inventory for one day is about \$0.07. Here, the ratio between the lost sales penalty cost – in this case, the expediting premium of \$14 – and the holding cost is 200.

Our main result is that, under mild assumptions on the demand distribution, the class of order-up-to policies is asymptotically optimal for these systems as the lost sales penalty increases. In fact, we show asymptotic optimality for a specific order-up-to policy that is computed using the newsvendor formula with appropriate parameters. For any given cost parameters, we also establish an upper bound on the increase in the total cost from using this specific order-up-to policy instead of the optimal policy. Finally, we present several computational results to evaluate the performance of the optimal policy, the best order-up-to policy, and the specific order-up-to policy mentioned above, for a wide range of demand distributions and cost parameters.

1.1 Notation and Problem Description

To facilitate the discussion of our main results, let us introduce the notation and the problem description. We will consider both the lost sales and the backorder systems. In both systems, the lead time between the placement of a replenishment order and its delivery is denoted by τ. The index for time periods is t and D_t is the demand in period t. We assume $D_1, D_2, \ldots$ are independent and identically distributed random variables and we use D to denote a generic random variable with the same distribution as D_t. Also, let $\mathbf{D} = \sum_{t=1}^{\tau+1} D_t$ denote the total demand over $\tau + 1$ periods, representing the total demand over the lead time including the period when we place the order. Let F denote the distribution function of $\mathbf{D}$.

At the beginning of period t, the replenishment order placed in period $t - \tau$ is received. Let $X_t^{\mathcal{L}} \in [0, \infty)$ denote the inventory on hand at this instant in the lost sales system. For the backorder system, let $X_t^{\mathcal{B}} \in (-\infty, \infty)$ denote the *net-inventory* in period t, that is, the inventory on hand minus backorders at the instant after receiving the delivery due in period t. After receiving deliveries, a new replenishment order is placed after which the demand D_t is observed.

For any $h \geq 0$ and $b \geq 0$, we denote by $\mathcal{L}(h, b)$ the lost sales system and by $\mathcal{B}(h, b)$ the backorder system, which is identical to $\mathcal{L}(h, b)$ except that excess demand is backordered. In both the lost sales $\mathcal{L}(h, b)$ and backorder $\mathcal{B}(h, b)$ systems, we charge holding costs on inventory on hand at the end of each period at the rate of $\$h$ per unit per period. While we incur lost sales penalty of $\$b$ per unit of unmet demand in the lost sales model $\mathcal{L}(h, b)$, the shortage costs in the backorder systems $\mathcal{B}(h, b)$ are charged at the rate of $\$b$ per unit of backordered demand per period.

Given the holding cost h and lost sales penalty b, we denote by $C^{\mathcal{L},S}(h, b)$ and $C^{\mathcal{L}*}(h, b)$ the long run average cost in the lost sales system $\mathcal{L}(h, b)$ under an order-up-to-S policy and under an optimal policy, respectively. The corresponding quantities $C^{\mathcal{B},S}(h, b)$, and $C^{\mathcal{B}*}(h, b)$ are defined similarly, with the interpretation of b as the backorder cost per unit per period. We denote by $S^{\mathcal{L}*}(h, b)$ and $S^{\mathcal{B}*}(h, b)$ the best order-up-to levels in the lost sales $\mathcal{L}(h, b)$ and the backorder $\mathcal{B}(h, b)$ systems, respectively. We note that in the backorder system $\mathcal{B}(h, b)$, order-up-to policies are optimal, and the best order-up-to level is given by the newsvendor formula under the distribution function F of $\mathbf{D}$.

1.2 Contributions and Organization of the Paper

Our analysis provides some important insights about both lost sales and backorder inventory systems, in addition to the main result on the asymptotic optimality of order-up-to policies in lost sales systems. We now describe the organization of the paper and discuss the main contributions of the individual sections. Table 1 provides a summary of the main results.

In Section 2, we provide a brief literature review and indicate how our research contributes to the current research on lost sales systems. In Section 3, we describe the assumption on the demand distribution that we will use throughout the paper. We then show in Theorem 4 that our assumption encompasses a broad class of demand distributions that commonly occur in many inventory settings.

Category	Description	Results
Backorder System (Section 4)	Robustness of Optimal Cost and Newsvendor Solution (Theorem 6)	For any $\nu > 0$, $\lim_{b\to\infty} \frac{C^{\mathcal{B}*}(h,\nu b)}{C^{\mathcal{B}*}(h,b)} = 1$, and $\lim_{b\to\infty} \frac{C^{\mathcal{B},S_{\nu b}}(h,b)}{C^{\mathcal{B}*}(h,b)} = \lim_{b\to\infty} \frac{C^{\mathcal{B},S_b}(h,\nu b)}{C^{\mathcal{B}*}(h,\nu b)} = 1$, where $S_b = S^{\mathcal{B}*}(h,b)$ and $S_{\nu b} = S^{\mathcal{B}*}(h,\nu b)$
Connections Between Lost Sales and Backorder Systems (Section 5)	Asymptotic Equivalence of the Optimal Costs (Theorem 8)	$\lim_{b\to\infty} \frac{C^{\mathcal{B}*}(h,b)}{C^{\mathcal{L}*}(h,b)} = 1$
	Bounds on the Cost of Any Order-up-to Policy (Lemma 13)	For any S, $C^{\mathcal{B},S}(h, b/(\tau+1)) \le C^{\mathcal{L},S}(h,b) \le C^{\mathcal{B},S}(h, b+\tau h)$
	Bounds on the Best Order-up-to Levels (Theorem 14)	$S^{\mathcal{B}*}(2h(\tau+1), b - h(\tau+1)) \le S^{\mathcal{L}*}(h,b) \le S^{\mathcal{B}*}(h, b+\tau h)$
Main Results (Section 6)	Asymptotic Optimality of Order-up-to Policies in Lost Sales Systems (Theorem 15)	$\lim_{b\to\infty} \frac{\min_S C^{\mathcal{L},S}(h,b)}{C^{\mathcal{L}*}(h,b)} = \lim_{b\to\infty} \frac{C^{\mathcal{L},S_{b+\tau h}}(h,b)}{C^{\mathcal{L}*}(h,b)} = 1$, where $S_{b+\tau h} = S^{\mathcal{B}*}(h, b+\tau h)$.

Table 1: A summary of the main results in the paper. All asymptotic results assume that the distribution of the demand over lead time satisfies Assumption 1, which is discussed in detail in Section 3.

Robustness of the Optimal Cost and the Optimal Policy in Backorder Systems (Section 4): As our first contribution, we show that the optimal cost in the backorder system is robust against changes in the backorder cost parameter b for large b. More precisely, the increase in total cost resulting from *incorrectly* estimating b becomes negligible for large b; that is, for any $h \ge 0$ and $\nu > 0$,

$$\lim_{b\to\infty} \frac{C^{\mathcal{B},S_{\nu b}}(h,b)}{C^{\mathcal{B}*}(h,b)} = \lim_{b\to\infty} \frac{C^{\mathcal{B}*}(h,\nu b)}{C^{\mathcal{B}*}(h,b)} = \lim_{b\to\infty} \frac{C^{\mathcal{B},S_b}(h,\nu b)}{C^{\mathcal{B}*}(h,\nu b)} = 1,$$

where $S_b = S^{\mathcal{B}*}(h,b)$ and $S_{\nu b} = S^{\mathcal{B}*}(h,\nu b)$ denote the optimal order-up-to levels in the backorder systems $\mathcal{B}(h,b)$ and $\mathcal{B}(h,\nu b)$, respectively.

Estimating the backorder cost can be difficult in many applications because we have to assess the long-term impact of a stockout and account for losses in customer goodwill from delays in order fulfillment. Suppose we *mistakenly* estimate the backorder parameter to be νb (instead of b) and use the order-up-to-$S_{\nu b}$ policy in the $\mathcal{B}(h,b)$ systems. The above result shows that the relative increase in the total cost from using such a policy converges to zero as b increases. We thus make precise the "folk theorem" that the cost function in a typical inventory control problem is "flat" around the optimal solution. Interestingly, the above result holds only for demand distributions satisfying Assumption 1 (see Section 3 for more details). In Section 4.1, we provide a counterexample that does not satisfy Assumption 1 and where the above result fails.

Connections Between Lost Sales and Backorder Systems (Section 5): As our second contribution, we also establish relationships between lost sales and backorder systems, in terms of the optimal cost, the cost of any order-up-to policy, and the best order-up-to level. In Theorem 8 in Section 5, we show the asymptotic equivalence between the optimal cost in the lost sales and the backorder systems as the parameter b increases; that is, for any $h \geq 0$,

$$\lim_{b\to\infty} \frac{C^{\mathcal{B}*}(h,b)}{C^{\mathcal{L}*}(h,b)} = 1.$$

When the parameter b is large, this result enables us to use the (easily computed) optimal cost of the backorder system $\mathcal{B}(h,b)$ as an approximation for the optimal cost in the corresponding lost sales system.

In addition to asymptotic equivalence of the optimal costs, the long run average cost of *any* order-up-to policy in the lost sales $\mathcal{L}(h,b)$ system is bounded above and below by the cost of the same policy in the backorder systems $\mathcal{B}(h,b+\tau h)$ and $\mathcal{B}(h,b/(\tau+1))$, respectively. Lemma 13 shows that for any order-up-to level S,

$$C^{\mathcal{B},S}(h,b/(\tau+1)) \leq C^{\mathcal{L},S}(h,b) \leq C^{\mathcal{B},S}(h,b+\tau h).$$

We also develop bounds on the best order-up-to level in the lost sales system, as shown in Theorem 14, that

$$S^{\mathcal{B}*}(2h(\tau+1), b-h(\tau+1)) \leq S^{\mathcal{L}*}(h,b) \leq S^{\mathcal{B}*}(h,b+\tau h).$$

The above bounds represent the first such results that relate the cost of any order-up-to policy and the best order-up-to level in the lost sales system with the corresponding quantities in the well-studied backorder system. These bounds are easily computable since they are simply represented by newsvendor formulas.

Main Results (Section 6): The results from Section 4 and 5 set the stage for the main result of the paper (Theorem 15): order-up-to policies are asymptotically optimal in the lost sales system, or

$$\lim_{b \to \infty} \frac{\min_S C^{\mathcal{L},S}(h,b)}{C^{\mathcal{L}*}(h,b)} = \lim_{b \to \infty} \frac{C^{\mathcal{L},S_{b+\tau h}}(h,b)}{C^{\mathcal{L}*}(h,b)} = 1,$$

where $S_{b+\tau h} = S^{\mathcal{B}*}(h, b + \tau h)$. The above result shows that, in fact, the optimal order-up-to level for the *backorder system* $\mathcal{B}(h, b + \tau h)$ is asymptotically optimal for the lost sales system $\mathcal{L}(h, b)$. Theorem 15 also provides an explicit and computable bound on the rate of convergence for any finite value of b.

Computational Investigation (Section 7): In addition to establishing asymptotic optimality of base stock policies in lost sales inventory systems, we also discuss extensive computational investigation. In our experiment, we indicate the performance of base stock policies under different problem parameters. We show how order-up-to policies perform against other replenishment heuristics (Section 7.2), determine the impact of increasing demand on total cost (Section 7.3), and study how well order-up-to policies would do when the demand exhibits high variance-to-mean ratios (Section 7.4). Our computational results show that the cost of the best order-up-to policy is within 1.5% of the optimal cost even when the ratio between the lost sales penalty and the holding cost is just 100. Moreover, our order-up-to policy continues to perform well even for demands with large means or high variance. This result suggests that such a policy should perform well in many practical inventory applications.

2. Brief Literature Review

There are three main streams of research on lost sales inventory systems: the analysis of the optimal inventory policy, the analysis of these systems under an arbitrary policy or under policies of specific kinds, and the computational investigation of the performance of easily implementable heuristics. We now briefly review these three research streams in that order.

Karlin and Scarf (1958) first study the lost sales inventory system with a lead time of one period. They demonstrate that order-up-to policies are not optimal for these systems; the optimal order quantity is a decreasing function of the amount of inventory on hand, with the rate of decrease being smaller than one. For the general lead time case, they analyze the system under order-up-to policies and exponentially distributed demands, and derive an expression for the steady state distribution of on-hand inventory level. Morton (1969) extends Karlin and Scarf's results on the optimal ordering policy to the general lead time case. He also derives upper and lower bounds on the optimal order quantity in a period as functions of the state vector. Recently, Zipkin (2006b) presents an elegant derivation of Morton's structural results and extends the results to more general lost sales inventory systems (for example, allowing capacity restrictions).

Levi et al. (2005) develop a heuristic based on the dual balancing technique introduced originally for backorder systems by Levi et al. (2004). They show that this heuristic attains an expected cost per period that is at most twice that achieved by an optimal policy for a large class of demand models. Janakiraman et al. (2005) show analytically that the optimal cost of managing a lost sales inventory system is smaller than that of managing a backorder system when the backorder cost parameter in the latter system has the same magnitude as the lost sales cost parameter in the former system. Under varying assumptions, Karush (1957), Downs et al. (2001) and Janakiraman and Roundy (2004) all consider lost sales inventory systems under order-up-to policies and show the convexity of the expected cost per period with respect to the order-up-to level. Reiman (2004) studies the class of order-up-to policies and the class of constant order policies (policies that order a constant quantity every period regardless of the state of the system). He derives expressions for the order-up-to level and for the constant order-quantity that are asymptotically optimal *within the respective classes of policies* as the penalty cost becomes large. He also investigates the comparative performance of the two policies as the lead time grows.

Morton (1971) computationally investigates the performance of the myopic policy as a heuristic for problems with a lead time of one or two periods. For lost sales problems with additional features (for example, a set-up cost), Nahmias (1979) derives an intuitively appealing heuristic and investigates its performance for the cases of one and two period lead times. Recently, for problems with lead times ranging from one to four periods, Zipkin (2006a) investigates the performance of the optimal order-up-to policies, the myopic policy, a modified myopic policy that is based on the costs incurred in two periods, the dual balancing policy, a generalization of base-stock policies suggested by Morton (1971), and the optimal constant-order policy. To reduce the computational effort involved in evaluating each of these policies, he presents elegant analytical bounds on the size of the effective state space under any given policy.

Our paper has elements of all three research streams. Our asymptotic optimality results contribute to an understanding of the structure of the optimal policy by establishing conditions under which the optimal cost is asymptotically equal to the cost of the best base stock policy. Our bounds on the performance of a specific order-up-to policy and the analysis leading to such bounds illuminate the structural properties of base stock policies and establish connections between lost sales and backorder inventory systems. Finally, we complement Zipkin (2006a) by investigating the computational performance of order-up-to policies over a larger class of problem instances, especially when the lost sales penalty is significantly higher than the holding cost. We show that when the ratio b/h is large, order-up-to policies perform extremely well, with an average cost that is within 1.5% of the optimal.

3. Assumption on the Demand Distribution

Recall that $\mathbf{D} \equiv \sum_{t=1}^{\tau+1} D_t$ denotes the total demand over $\tau + 1$ periods, representing the total demand over the lead time including the period when we place the order. Let F denote the distribution function of $\mathbf{D}$. For any $t \geq 0$, we define the *mean residual life* $m_{\mathbf{D}}(t)$ as follows:

$$m_{\mathbf{D}}(t) = \begin{cases} E\left[\mathbf{D} - t \middle| \mathbf{D} > t\right], & \text{if } t < \sup\left\{x : F(x) < 1\right\}, \\ 0, & \text{otherwise.} \end{cases}$$

Through out this paper, we make the following assumption on the distribution of $\mathbf{D}$.

Assumption 1. $\lim_{t\to\infty} m_{\mathbf{D}}(t)/t = 0$.

The above assumption implies that the expected mean residual life of $\mathbf{D}$ at t does not grow faster than t. Before proceeding with examples of demand distributions satisfying Assumption 1, let us recall the following definition.

Definition 2. *A continuous (resp. discrete) random variable Y with a distribution function F and a density function f (resp. probability mass function p) has an* increasing failure rate *(IFR) property if $f(x)/(1-F(x))$ (resp. $p(x)/(1-F(x))$) is nondecreasing in x.*

The following result provides an equivalent characterization of an IFR random variable. The proof appears in Section 1.B.1 of Shaked and Shanthikumar (1994).

Lemma 3. *A random variable Y is IFR if and only if for any $0 \leq t_1 < t_2$, the residual life of Y at t_2 is stochastically smaller than the residual life of Y at t_1; that is, for any $s \geq 0$,*

$$\mathcal{P}\left\{Y - t_2 > s \;\middle|\; Y > t_2\right\} \leq \mathcal{P}\left\{Y - t_1 > s \;\middle|\; Y > t_1\right\}.$$

The following theorem, whose proof appears in Appendix A, shows that Assumption 1 encompasses a large class of discrete and continuous demand distributions that commonly occur in many inventory systems.

Theorem 4. *If any of the following conditions holds, then* $\mathbf{D}$ *satisfies Assumption 1.*

(a) The demand D *in each period (either discrete or continuous) is bounded; that is, there exists* M *such that* $\mathcal{P}\{D \leq M\} = 1$.

(b) The demand D *in each period (either discrete or continuous) has an increasing failure rate (IFR) distribution.*

(c) $\mathbf{D}$ *has a finite variance and the distribution* F *of* $\mathbf{D}$ *has a density function* f *and a failure rate function* $r(t)$ *of* F *that does not decrease to zero faster than* $1/t$*; that is,*

$$\lim_{t \to \infty} t \cdot r(t) = \infty,$$

where for any $t \geq 0$, $r(t) = f(t)/(1 - F(t))$.

The above theorem shows that Assumption 1 encompasses a very large of demand distributions used in many supply chain models, including any bounded demand random variables. For unbounded demand, part (b) of Theorem 4 shows that many commonly used distributions also satisfy Assumption 1. Examples include geometric distributions, Poisson distributions (see Corollary 5.2 in Ross et al. (2005)), negative binomial distributions with parameter $r > 0$ and $0 < p < 1$, exponential distributions, and Gaussian distributions. When the demand distribution does not exhibit an IFR property, part (c) of the above theorem shows that Assumption 1 remains satisfied as long as the failure rate does not decrease to zero too quickly. The following example shows a distribution that is not IFR, yet still satisfies part (c) of Theorem 4.

Example 1. *Suppose that* $\mathbf{D}$ *follows a Weibull distribution with scale parameter* $\lambda > 0$ *and shape parameter* $0 < k < 1$*. Then,* $\mathbf{D}$ *has the following distribution, density, and failure rate functions: for any* $x > 0$,

$$f(x) = \frac{k x^{k-1} e^{-(x/\lambda)^k}}{\lambda^k}, \quad F(x) = 1 - e^{-(x/\lambda)^k}, \quad \text{and} \quad r(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1},$$

and the first two moments of $\mathbf{D}$ *are*

$$E[\mathbf{D}] = \lambda \Gamma\left(1 + \frac{1}{k}\right) \quad \text{and} \quad E\left[\mathbf{D}^2\right] = \lambda^2 \Gamma\left(1 + \frac{2}{k}\right),$$

where $\Gamma(\cdot)$ *denotes the Gamma function. Since* $0 < k < 1$*, it is easy to verify that* $\mathbf{D}$ *is not IFR, but the failure rate function* r *still satisfies part (c) of Theorem 4.*

4. Asymptotic Properties for Backordered Systems

We start our analysis by showing asymptotic properties of the optimal policy in the backorder system as the backordere cost parameter b becomes large. To facilitate our discussion, let us introduce the following notation. For any $y \geq 0$, let $\psi(y; h, b)$ denote the ratio between the expected backorder and holding costs given the inventory position y in the $\mathcal{B}(h, b)$ system, that is,

$$\psi(y; h, b) = \frac{bE\left[(\mathbf{D} - y)^+\right]}{hE\left[(y - \mathbf{D})^+\right]}.$$

The following lemma establishes upper and lower bounds on the increase in the total cost from using sub-optimal policies.

Lemma 5. *For any $h \geq 0$, $b \geq 0$, and $\nu > 0$, let $S_b = S^{\mathcal{B}*}(h, b)$ and $S_{\nu b} = S^{\mathcal{B}*}(h, \nu b)$ denote the optimal policies in the $\mathcal{B}(h, b)$ and $\mathcal{B}(h, \nu b)$ backorder systems, respectively. Then, the relative difference between the optimal costs of $\mathcal{B}(h, b)$ and $\mathcal{B}(h, \nu b)$ backorder systems can be bounded as follows:*

$$\frac{1 + \psi\left(S_{\nu b}; h, \nu b\right)}{1 + (1/\nu)\psi\left(S_{\nu b}; h, \nu b\right)} = \frac{C^{\mathcal{B}*}(h, \nu b)}{C^{\mathcal{B}, S_{\nu b}}(h, b)} \leq \frac{C^{\mathcal{B}*}(h, \nu b)}{C^{\mathcal{B}*}(h, b)} \leq \frac{C^{\mathcal{B}, S_b}(h, \nu b)}{C^{\mathcal{B}*}(h, b)} = \frac{1 + \nu\psi\left(S_b; h, b\right)}{1 + \psi\left(S_b; h, b\right)}$$

Proof. The first and second inequalities follows from the fact that $C^{\mathcal{B}*}(h, b) \leq C^{\mathcal{B}, S_{\nu b}}(h, b)$ and $C^{\mathcal{B}*}(h, \nu b) \leq C^{\mathcal{B}, S_b}(h, \nu b)$, respectively. To establish the first equality, note that

$$\frac{C^{\mathcal{B}*}(h, \nu b)}{C^{\mathcal{B}, S_{\nu b}}(h, b)} = \frac{\nu b E\left[\mathbf{D} - S_{\nu b}\right]^+ + hE\left[S_{\nu b} - \mathbf{D}\right]^+}{bE\left[\mathbf{D} - S_{\nu b}\right]^+ + hE\left[S_{\nu b} - \mathbf{D}\right]^+} = \frac{1 + \psi\left(S_{\nu b}; h, \nu b\right)}{1 + (1/\nu)\psi\left(S_{\nu b}; h, \nu b\right)},$$

where the last inequality follows from dividing the numerator and denominator by $hE\left[S_{\nu b} - \mathbf{D}\right]^+$. The proof of the second equality of the lemma is similar. □

The bounds in Lemma 5 lead directly to the main asymptotic result of this section, which is stated in the following theorem.

Theorem 6. *Under Assumption 1, the following results hold for any $h \geq 0$.*

(a) The ratio between the expected backorder cost per period to the expected holding cost per period under the optimal policy converges to zero as the backordere cost b increases, that is,

$$\lim_{b \to \infty} \psi\left(S^{\mathcal{B}*}(h, b); h, b\right) = 0.$$

(b) *For large values of b, the optimal cost and the optimal policy are robust against changes in the backorder cost; that is, for any $\nu > 0$,*

$$\lim_{b\to\infty} \frac{C^{\mathcal{B},S_{\nu b}}(h,b)}{C^{\mathcal{B}*}(h,b)} = \lim_{b\to\infty} \frac{C^{\mathcal{B}*}(h,\nu b)}{C^{\mathcal{B}*}(h,b)} = \lim_{b\to\infty} \frac{C^{\mathcal{B},S_b}(h,\nu b)}{C^{\mathcal{B}*}(h,\nu b)} = 1,$$

where $S_b = S^{\mathcal{B}*}(h,b)$ and $S_{\nu b} = S^{\mathcal{B}*}(h,\nu b)$.

Proof. To establish the result in part (a), note that the optimal order-up-to level $S^{\mathcal{B}*}(h,b)$ is given by the newsvendor formula:

$$S^{\mathcal{B}*}(h,b) = \inf\left\{y : \mathcal{P}\left\{\mathbf{D} \le y\right\} \ge \frac{b}{b+h}\right\},$$

which implies that $\mathcal{P}\left\{\mathbf{D} \le S^{\mathcal{B}*}(h,b)\right\} \ge b/(b+h)$ and $\mathcal{P}\left\{\mathbf{D} > S^{\mathcal{B}*}(h,b)\right\} \le h/(b+h)$ by the left continuity of the distribution function. Therefore,

$$\begin{aligned}
\psi\left(S^{\mathcal{B}*}(h,b);h,b\right) &= \frac{b\cdot\mathcal{P}\left\{\mathbf{D} > S^{\mathcal{B}*}(h,b)\right\} E\left[\mathbf{D} - S^{\mathcal{B}*}(h,b) \;\middle|\; \mathbf{D} > S^{\mathcal{B}*}(h,b)\right]}{h\cdot\mathcal{P}\left\{\mathbf{D} \le S^{\mathcal{B}*}(h,b)\right\} E\left[S^{\mathcal{B}*}(h,b) - \mathbf{D} \;\middle|\; \mathbf{D} \le S^{\mathcal{B}*}(h,b)\right]} \\
&\le \frac{E\left[\mathbf{D} - S^{\mathcal{B}*}(h,b) \;\middle|\; \mathbf{D} > S^{\mathcal{B}*}(h,b)\right]}{E\left[S^{\mathcal{B}*}(h,b) - \mathbf{D} \;\middle|\; \mathbf{D} \le S^{\mathcal{B}*}(h,b)\right]} \\
&= \left(\frac{m_{\mathbf{D}}\left(S^{\mathcal{B}*}(h,b)\right)}{S^{\mathcal{B}*}(h,b)}\right)\left(\frac{S^{\mathcal{B}*}(h,b)}{S^{\mathcal{B}*}(h,b) - E\left[\mathbf{D} \;\middle|\; \mathbf{D} \le S^{\mathcal{B}*}(h,b)\right]}\right).
\end{aligned}$$

Since $E\left[\mathbf{D} \;\middle|\; \mathbf{D} \le S^{\mathcal{B}*}(h,b)\right] \le E\left[\mathbf{D}\right]$, the desired result follows from Assumption 1 and the definition of $S^{\mathcal{B}*}(h,b)$.

To prove part (b), note that it follows from part (a) and the bounds in Lemma 5 that

$$\lim_{b\to\infty} \frac{C^{\mathcal{B}*}(h,\nu b)}{C^{\mathcal{B},S_{\nu b}}(h,b)} = \lim_{b\to\infty} \frac{C^{\mathcal{B}*}(h,\nu b)}{C^{\mathcal{B}*}(h,b)} = \lim_{b\to\infty} \frac{C^{\mathcal{B},S_b}(h,\nu b)}{C^{\mathcal{B}*}(h,b)} = 1,$$

and the desired result follows from the fact that

$$\frac{C^{\mathcal{B},S_{\nu b}}(h,b)}{C^{\mathcal{B}*}(h,b)} = \frac{C^{\mathcal{B}*}(h,\nu b)/C^{\mathcal{B}*}(h,b)}{C^{\mathcal{B}*}(h,\nu b)/C^{\mathcal{B},S_{\nu b}}(h,b)} \quad \text{and} \quad \frac{C^{\mathcal{B},S_b}(h,\nu b)}{C^{\mathcal{B}*}(h,\nu b)} = \frac{C^{\mathcal{B},S_b}(h,\nu b)/C^{\mathcal{B}*}(h,b)}{C^{\mathcal{B}*}(h,\nu b)/C^{\mathcal{B}*}(h,b)}.$$

□

Theorem 6 shows that, for distributions satisfying Assumption 1, the newsvendor solution and the optimal cost are robust against inaccurate estimation of the backorder parameter when the backorder parameter b is large. However, when Assumption 1 fails, the result of Theorem 6 many no longer hold, as we now demonstrate.

4.1 Pareto Distributions: An Example Where Theorem 6 Fails

For any $\theta > 1$, let the density function f_θ be defined by: for any $x \geq 0$,

$$f_\theta(x) = \frac{\theta}{(1+x)^{1+\theta}},$$

and let $F_\theta(x) = \int_0^x f_\theta(u)du$ denote the corresponding distribution function. The following proposition shows that F_θ does not satisfy Assumption 1 and the optimal cost is sensitive to the backorder parameter, even for large values of b.

Proposition 7. *For any $\theta > 1$, $b \geq 0$, $h \geq 0$, and $\nu > 0$, if $\mathbf{D}$ has a distribution function F_θ, then*

$$\lim_{t\to\infty} \frac{m_{\mathbf{D}}(t)}{t} = \frac{1}{\theta - 1} \quad \text{and} \quad \lim_{b\to\infty} \frac{C^{\mathcal{B}*}(h, \nu b)}{C^{\mathcal{B}*}(h, b)} = \nu^{1/\theta} \ .$$

Proof. Please see Appendix B. □

5. Connections between Lost Sales and Backorder Inventory Systems

The following result provides an intuitive basis for conjecturing that the optimal policy in the backorder system is also asymptotically optimal in the lost sales system.

Theorem 8. *Under Assumption 1, as b increases, the optimal cost in the lost sales $\mathcal{L}(h, b)$ system converges to the optimal cost in the backorder system $\mathcal{B}(h, b)$; that is, for any $h \geq 0$,*

$$\lim_{b\to\infty} \frac{C^{\mathcal{L}*}(h, b)}{C^{\mathcal{B}*}(h, b)} \ = \ 1.$$

Proof. Janakiraman et al. (2005) established the following bounds on the optimal cost in the lost sales system: $C^{\mathcal{B}*}(h, b/(\tau+1)) \ \leq \ C^{\mathcal{L}*}(h, b) \ \leq \ C^{\mathcal{B}*}(h, b)$. From Theorem 6(b), $\lim_{b\to\infty} C^{\mathcal{B}*}(h, b)/C^{\mathcal{B}*}(h, b/(\tau+1)) \ = \ 1$, which gives the desired result. □

Next, we establish connections between the dynamics in both the lost sales and the backorder systems under the same order-up-to policy. Let $X_t^{\mathcal{L},S}$ and $X_t^{\mathcal{B},S}$ denote the on-hand inventory in the lost sales system and the net inventory in the backorder system, respectively, at the beginning of period t under an order-up-to-S policy. Similarly, we use $LOST_t^{\mathcal{L},S}$ and $BACK_t^{\mathcal{B},S}$ to denote the lost sales incurred in period t and the backorders existed at the end of period t, respectively, under the order-up-to-S policy. By definition, $X_t^{\mathcal{L},S}$, $LOST_t^{\mathcal{L},S}$ and $BACK_t^{\mathcal{B},S}$ are non-negative random variables. The following lemma establishes the relationship among these random variables.

Lemma 9. *Assume both the lost sales and the backorder systems start at the same state in period 1 and the inventory position in this state is S or less. Then, for every demand sample path and for every $t \geq \tau + 1$,*

$$X_t^{\mathcal{B},S} \leq X_t^{\mathcal{L},S} \quad \text{and} \quad BACK_t^{\mathcal{B},S} - \sum_{i=t-\tau}^{t-1} LOST_i^{\mathcal{L},S} \ \leq \ LOST_t^{\mathcal{L},S} \ \leq \ BACK_t^{\mathcal{B},S}.$$

Proof. It follows from the dynamics of lost sales systems under the order-up-to-S policy (see Janakiraman and Roundy (2004)) that

$$X_t^{\mathcal{L},S} \ = \ S - \sum_{i=t-\tau}^{t-1} D_i + \sum_{i=t-\tau}^{t-1} LOST_i^{\mathcal{L},S} = X_t^{\mathcal{B},S} + \sum_{i=t-\tau}^{t-1} LOST_i^{\mathcal{L},S},$$

for any $t \geq \tau + 1$, which proves the first part of the lemma. Since $x^+ - y \leq (x-y)^+$ for any $x \in \Re$ and $y \geq 0$, we have

$$\begin{aligned} BACK_t^{\mathcal{B},S} - \sum_{i=t-\tau}^{t-1} LOST_i^{\mathcal{L},S} \ &= \ \left(D_t - X_t^{\mathcal{B},S}\right)^+ - \sum_{i=t-\tau}^{t-1} LOST_i^{\mathcal{L},S} \leq \left(D_t - X_t^{\mathcal{B},S} - \sum_{i=t-\tau}^{t-1} LOST_i^{\mathcal{L},S}\right)^+ \\ &= \ \left(D_t - X_t^{\mathcal{L},S}\right)^+ = LOST_t^{\mathcal{L},S} \leq \left(D_t - X_t^{\mathcal{B},S}\right)^+ = BACK_t^{\mathcal{B},S}, \end{aligned}$$

where the last inequality follows from the fact that $X_t^{\mathcal{B},S} \leq X_t^{\mathcal{L},S}$. □

The result of Lemma 9 relates the net inventory in the backorder system with the on-hand inventory in the lost sales system for any finite time period. To use this result for studying the long run average cost in $\mathcal{L}(h,b)$, this result should be extended to the steady-state on-hand inventory, which we will denote by $X_\infty^{\mathcal{L},S}$. This is our next step.

Before proving properties of $X_\infty^{\mathcal{L},S}$ and $X_\infty^{\mathcal{B},S}$, it is important to establish their existence. It is well known (and trivial to verify) that $X_\infty^{\mathcal{B},S}$ exists and

$$X_\infty^{\mathcal{B},S} \ \sim_d \ S - \sum_{t=1}^{\tau} D_t \ .$$

However, in general, for any given starting state and order-up-to level S, it is *n*ot true that the distribution of $X_t^{\mathcal{L},S}$ converges to a stationary distribution. In fact, Huh et al. (2006) give such an example. Interestingly, we are able to show two results that help us resolve this difficulty.

Lemma 10. *For every S and any starting state in period 1, the sequence of the expected cost per period over the interval $[1,T]$ given by*

$$\frac{\sum_{t=1}^{T} E[h \cdot (X_t^{\mathcal{L},S} - D_t)^+ + b \cdot (D_t - X_t^{\mathcal{L},S})^+]}{T}$$

converges to a limit that is independent of the starting state.

Proof. Please see Appendix C. □

Since the long run average cost is the quantity of interest to us in this paper, Lemma 10 implies that we can limit our analysis to any specific starting state. In the next lemma, we show that for a specific starting state we choose, the stationary distribution of the on hand inventory, $\{X_t^{\mathcal{L},S}\}$, exists.

Lemma 11. *Assume the starting state (in period* 1*) is such that there are* $S/(\tau+1)$ *units on hand and* $S/(\tau+1)$ *units due to be delivered in each of the periods* $2,\ldots,\tau$*. Then, the sequence of the distributions of the random variables* $\{X_t^{\mathcal{L},S}\}$ *converges.*

Proof. Please see Appendix D. □

We will use $X_\infty^{\mathcal{L},S}$ to denote a random variable whose distribution is the limiting distribution from Lemma 11. We can now define $C^{\mathcal{L},S}(h,b)$ mathematically as follows:

$$C^{\mathcal{L},S}(h,b) = hE\left[\left(X_\infty^{\mathcal{L},S} - D\right)^+\right] + bE\left[\left(D - X_\infty^{\mathcal{L},S}\right)^+\right],$$

where the random variable D denotes the demand in a single period.

Corollary 12. *The random variable* $X_\infty^{\mathcal{B},S}$ *is stochastically smaller than the random variable* $X_\infty^{\mathcal{L},S}$ *and the random variable* $LOST_\infty^{\mathcal{L},S}$ *is stochastically smaller than the random variable* $BACK_\infty^{\mathcal{B},S}$*, i.e. for any* $z \geq 0$*,*

$$\mathcal{P}\left\{X_\infty^{\mathcal{B},S} > z\right\} \leq \mathcal{P}\left\{X_\infty^{\mathcal{L},S} > z\right\} \quad \text{and} \quad \mathcal{P}\left\{LOST_\infty^{\mathcal{L},S} > z\right\} \leq \mathcal{P}\left\{BACK_\infty^{\mathcal{B},S} > z\right\}.$$

Proof. Notice that we assumed that $X_\infty^{\mathcal{L},S}$ represents the limiting distribution of $X_t^{\mathcal{L},S}$ when the starting state vector has $S/(\tau+1)$ units in each component. So, this starting state satisfies the assumption of Lemma 9. The result follows directly from this lemma. □

The next result establishes upper and lower bounds on the cost of any order-up-to policy in the lost sales system in terms of the costs in the backorder system.

Lemma 13. *The long run average cost of the order-up-to-*S *policy in the lost sales system* $\mathcal{L}(h,b)$ *is bounded above (resp. below) by the cost of the same policy in the backorder system* $\mathcal{B}(h,b+\tau h)$ *(resp.* $\mathcal{B}(h,b/(\tau+1))$*), i.e.*

$$C^{\mathcal{B},S}\left(h,b/(\tau+1)\right) \leq C^{\mathcal{L},S}(h,b) \;\leq\; C^{\mathcal{B},S}(h,b+\tau h).$$

Proof. Recall that the random variable $\mathbf{D}$ denotes the total demand over $\tau+1$ periods and $X_t^{\mathcal{L},S} = S - \sum_{i=t-\tau}^{t-1} D_i + \sum_{i=t-\tau}^{t-1} LOST_i^{\mathcal{L},S}$. Then, for any t, we have

$$
\begin{aligned}
& hE\left[\left(X_t^{\mathcal{L},S} - D_t\right)^+\right] + bE\left[\left(D_t - X_t^{\mathcal{L},S}\right)^+\right] \\
&= hE\left[X_t^{\mathcal{L},S} - D_t\right] + hE\left[\left(D_t - X_t^{\mathcal{L},S}\right)^+\right] + bE\left[\left(D_t - X_t^{\mathcal{L},S}\right)^+\right] \\
&= hE\left[S - \sum_{i=t-\tau}^{t-1} D_i + \sum_{i=t-\tau}^{t-1} LOST_i^{\mathcal{L},S} - D_t\right] + hE\left[LOST_t^{\mathcal{L},S}\right] + bE\left[LOST_t^{\mathcal{L},S}\right] \\
&= hE\left[S - \sum_{i=t-\tau}^{t} D_i\right] + h\sum_{i=t-\tau}^{t} E\left[LOST_i^{\mathcal{L},S}\right] + bE\left[LOST_t^{\mathcal{L},S}\right] \\
&= hE\left[\left(S - \sum_{i=t-\tau}^{t} D_i\right)^+\right] - hE\left[\left(\sum_{i=t-\tau}^{t} D_i - S\right)^+\right] + h\sum_{i=t-\tau}^{t} E\left[LOST_i^{\mathcal{L},S}\right] + bE\left[LOST_t^{\mathcal{L},S}\right] \\
&= hE\left[(S - \mathbf{D})^+\right] - hE\left[BACK_t^{\mathcal{B},S}\right] + h\sum_{i=t-\tau}^{t} E\left[LOST_i^{\mathcal{L},S}\right] + bE\left[LOST_t^{\mathcal{L},S}\right].
\end{aligned}
$$

Since the stochastic process $\left\{X_t^{\mathcal{L},S} : t \geq 1\right\}$ converges to $X_\infty^{\mathcal{L},S}$, we have

$$
\begin{aligned}
C^{\mathcal{L},S}(h,b) &= hE\left[(S-\mathbf{D})^+\right] - hE\left[BACK_\infty^{\mathcal{B},S}\right] + (b + (\tau+1)h)E\left[LOST_\infty^{\mathcal{L},S}\right] \\
&\leq hE\left[(S-\mathbf{D})^+\right] + (b + \tau h)E\left[BACK_\infty^{\mathcal{B},S}\right] = C^{\mathcal{B},S}(h, b+\tau h),
\end{aligned}
$$

where the inequality follows from Corollary 12 which implies that $E\left[LOST_\infty^{\mathcal{L},S}\right] \leq E\left[BACK_\infty^{\mathcal{B},S}\right]$. This establishes the upper bound in the statement of the lemma.

Next, we establish the lower bound. Since $BACK_t^{\mathcal{B},S} - \sum_{i=t-\tau}^{t-1} LOST_i^{\mathcal{L},S} \leq LOST_t^{\mathcal{L},S}$ with probability one by Lemma 9, taking the expectation on both sides and taking the limit as t increases to infinity, it follows that $E\left[LOST_\infty^{\mathcal{L},S}\right] \geq E\left[BACK_\infty^{\mathcal{B},S}\right] \Big/ (\tau+1)$. Therefore, it follows from the above expression for $C^{\mathcal{L},S}(h,b)$ that

$$
C^{\mathcal{L},S}(h,b) \geq hE\left[(S-\mathbf{D})^+\right] + \left(\frac{b + (\tau+1)h}{\tau+1} - h\right) E\left[BACK_\infty^{\mathcal{B},S}\right] = C^{\mathcal{B},S}(h, b/(\tau+1)).
$$

□

We now relate the best order-up-to level in the lost sales and backorder systems.

Theorem 14. *For any $h \geq 0$ and $b \geq 0$, the best order-up-to level in the lost sales system $\mathcal{L}(h,b)$ is*

(a) *bounded above by the best order-up-to level in the backorder system* $\mathcal{B}(h, b+\tau h)$ *with a backorder penalty cost parameter of* $b+\tau h$*, that is,*

$$S^{\mathcal{L}*}(h,b) \leq S^{\mathcal{B}*}(h, b+\tau h) \ ;$$

(b) *bounded below by the best order-up-to level in the backorder system* $\mathcal{B}(2h(\tau+1), b-h(\tau+1))$ *with a holding cost parameter of* $2h(\tau+1)$ *and a backorder penalty cost parameter of* $b-h(\tau+1)$*, that is,*

$$S^{\mathcal{L}*}(h,b) \geq S^{\mathcal{B}*}(2h(\tau+1), b-h(\tau+1)) \ .$$

Proof. Please see Appendix E and Appendix F. □

6. Performance Bounds and Asymptotic Results for Lost Sales Systems

We will now establish the asymptotic optimality of order-up-to policies in the lost sales system. We will, in fact, show the asymptotic optimality of the order-up-to-$S^{\mathcal{B}*}(h, b+\tau h)$ policy, corresponding to the optimal policy in the backorder system $\mathcal{B}(h, b+\tau h)$. Notice that this is the upper bound we derived for the best order-up-to level for the lost sales system (Theorem 14(a)). For any finite b, we also derive an upper bound on the loss in performance from using this policy relative to the optimal policy. Recall that for any $y \geq 0$,

$$\psi(y; h, b) = \frac{bE\left[(\mathbf{D}-y)^+\right]}{hE\left[(y-\mathbf{D})^+\right]}.$$

The main result of this section is stated in the following theorem.

Theorem 15. *For any* $h \geq 0$ *and* $b \geq 0$*, let* $S_{b+\tau h} = S^{\mathcal{B}*}(h, b+\tau h)$ *and* $S_{b/(\tau+1)} = S^{\mathcal{B}*}(h, b/(\tau+1))$ *denote the optimal order-up-to policies in the backorder systems* $\mathcal{B}(h, b+\tau h)$ *and* $\mathcal{B}(h, b/(\tau+1))$*, respectively. Then,*

(a) *The ratio between the cost of the order-up-to-*$S_{b+\tau h}$ *and the optimal policies in the lost sales system* $\mathcal{L}(h,b)$ *can be bounded as follows:*

$$\frac{C^{\mathcal{L}, S_{b+\tau h}}(h,b)}{C^{\mathcal{L}*}(h,b)} \ \leq \ \frac{1+\left(\frac{(b+\tau h)(\tau+1)}{b}\right)\psi\left(S_{b/(\tau+1)}; h, b/(\tau+1)\right)}{1+\psi\left(S_{b/(\tau+1)}; h, b/(\tau+1)\right)}.$$

(b) *Under Assumption 1, the order-up-to-*$S_{b+\tau h}$ *policy is asymptotically optimal in the lost sales system* $\mathcal{L}(h,b)$*, i.e.*

$$\lim_{b\to\infty} \frac{\min_{S\geq 0} C^{\mathcal{L},S}(h,b)}{C^{\mathcal{L}*}(h,b)} \ = \ \lim_{b\to\infty} \frac{C^{\mathcal{L}, S_{b+\tau h}}(h,b)}{C^{\mathcal{L}*}(h,b)} \ = \ 1 \ .$$

Proof. We know from Janakiraman et al. (2005) that $C^{\mathcal{B}*}(h, b/(\tau+1)) \le C^{\mathcal{L}*}(h,b)$ and it follows from Lemma 13 that

$$\frac{C^{\mathcal{L},S_{b+\tau h}}(h,b)}{C^{\mathcal{L}*}(h,b)} \le \frac{C^{\mathcal{B},S_{b+\tau h}}(h,b+\tau h)}{C^{\mathcal{B}*}(h,b/(\tau+1))} = \frac{C^{\mathcal{B}*}(h,b+\tau h)}{C^{\mathcal{B}*}(h,b/(\tau+1))} \le \frac{1+\nu_b \psi\left(S_{b/(\tau+1)}; h, b/(\tau+1)\right)}{1+\psi\left(S_{b/(\tau+1)}; h, b/(\tau+1)\right)},$$

where $\nu_b = (b+\tau h)(\tau+1)/b$. Note that the equality follows from the definition of $S_{b+\tau h}$ and the last inequality follows from Lemma 5. This proves part (a). Since $\lim_{b\to\infty} \nu_b = \tau + 1$ and by Theorem 6(a)

$$\lim_{b\to\infty} \psi\left(S_{b/(\tau+1)}; h, b/(\tau+1)\right) = 0,$$

it follows that

$$\lim_{b\to\infty} \frac{C^{\mathcal{L},S_{b+\tau h}}(h,b)}{C^{\mathcal{L}*}(h,b)} = 1,$$

which is the desired result. To complete the proof, note that $C^{\mathcal{L}*}(h,b) \le \min_S C^{\mathcal{L},S}(h,b) \le C^{\mathcal{L},S_{b+\tau h}}(h,b)$. □

7. Computational Investigation

In this section, we compare the total cost of the optimal policy, the best base stock policy, and the base stock policy suggested in Theorem 15. In Section 7.1, we describe the methodologies for our experiments. In Section 7.2, we consider the performance of our proposed order-up-to policies on problem instances considered by Zipkin (2006b), enabling us to benchmark our performance against other replenishment policies. Then, in Section 7.3, we study the performance of base stock policies as the expected demand increases, focusing on the commonly used Poisson demand models. Finally, in Section 7.4, by considering negative binomial distributions, we explore the impact of increasing variance-to-mean ratios on the performance of base stock policies.

7.1 Methodologies

To compute the long run average cost of the optimal replenishment policy for a given discrete demand distribution, we consider the average cost dynamic programming formulation. The state space $\mathcal{S}$ for our dynamic program consists of τ-dimensional vectors given by

$$\mathcal{S} = \{(z_0, z_1, \ldots, z_{\tau-1}) : z_i \in \mathbb{Z}_+ \cup \{0\}\},$$

where z_0 denotes the on-hand inventory after receiving the replenishment order and for $1 \le i \le \tau-1$, z_i denotes the replenishment quantities that will arrive i periods from now, corresponding to the order placed $\tau - i$ periods in the past. In our experiment, the demand D in each period has the property that $\mathcal{P}\{D = 0\} > 0$. It follows that from Proposition 2.6 in Bertsekas (1995) that for any $h \geq 0$ and $b \geq 0$, the optimal cost $C^{\mathcal{L}*}(h, b)$ is the unique solution of the following average cost dynamic program: for any $x \in \mathcal{S}$,

$$C^{\mathcal{L}*}(h,b) + g(x) = \min_{u \geq 0} \left\{ hE\left[z_0 - D\right]^+ + bE\left[D - z_0\right]^+ + E\left[g\left((z_0 - D)^+ + z_1, z_2, \ldots, z_{\tau-1}, u\right)\right] \right\},$$

where $g(\cdot)$ denotes the differential cost vector and u represents the ordering quantity.

Since we will consider unbounded demand in our experiment, we apply the elegant state-space reduction technique introduced by Zipkin (2006b), enabling us to consider a dynamic program with only a finite number of states (although the size of the state space still increases exponentially with the lead time). Then, to determine the optimal replenishment policy, we then apply the relative value iteration method for 1000 iterations or until the change between iterations is less than 0.001 (see Bertsekas (1995); Zipkin (2006b) for more details).

We note that for any $S \geq 0$, a similar dynamic programming formulation can be used to determine the long run average cost of the order-up-to-S policy. In this case, for any $x \in \mathcal{S}$, instead of minimizing over all possible ordering quantities as in optimal dynamic program above, the ordering quantity is given by $u = \left[S - \sum_{i=0}^{\tau-1} z_i\right]^+$. We can then apply the same relative value iteration method to determine the long run average cost $C^{\mathcal{L},S}(h, b)$.

To determine the best order-up-to level, we use the fact that the total cost $C^{\mathcal{L},S}(h, b)$ is convex in S (Downs et al. (2001) and Janakiraman and Roundy (2004)) and the best order-up-to level is bounded above by $S^{\mathcal{B}*}(h, b + \tau h)$ (Theorem 14(a)) and below by $S^{\mathcal{B}*}(2h(\tau + 1), b - h(\tau + 1))$ (Theorem 14(b)).

7.2 Representative Problems

In this section, we report the computational results for representative problems considered in Zipkin (2006b), enabling us to compare the cost of our base stock policies with other replenishment heuristics. We consider Poisson and Geometric demand distributions, both with mean 5. The lead time ranges from 1 to 4 periods. Assuming a holding cost of \$1, we consider the lost sales penalty ranging from \$1 to \$199. We compare the cost of the optimal policy, the best base stock policy, and the order-up-to-$S^{\mathcal{B}*}(h, b + \tau h)$ policy, which is shown to be asymptotically optimal (Theorem 15). Table 2 and 3 show the costs of these polices for Poisson and geometric distributions, respectively. (We remark that Zipkin (2006b) reports the cases where the lost sales penalty is \$4, \$9 or \$19.)

From Table 2 and 3, we observe that as the lost sales penalty increases, the cost of the best base stock and the order-up-to-$S^{\mathcal{B}*}(h, b + \tau h)$ policies converge to the optimal cost as predicted by Theorem 15. For $b = 199$, the costs of both base stock policies differ from the optimal by at most 5%. However, for a specific cost parameter, the performance of our base stock policies tends to degrade as the lead time increases. When comparing with the performance of other heuristics on the same problem instances (as reported in Zipkin (2006b)), the performance of our base stock policies are comparable with other heuristics.

7.3 Impact of Varying Mean Demand

In this section, we explore the performance of base stock policies as the mean demand changes. To facilitate our discussion, we assume a lead time of two periods ($\tau = 2$) and consider a Poisson demand distribution whose mean varies from 1 to 10. Table 4 shows a comparison among the optimal cost, the cost of the best base stock policy, and the order-up-to-$S^{\mathcal{B}*}(h, b + \tau h)$ policy.

From Table 4, we observe that, for a specific lost sales penalty, the relative difference between the optimal cost and the cost of the order-up-to-$S^{\mathcal{B}*}(h, b + \tau h)$ policy remains pretty small even when the mean demand increases. We observe a similar pattern for different values of lead times as well. This observation has an important practical implication. When the mean demand is small, computing the best order-up-to level is relatively easy since the range of base stock levels to consider is small; in fact, the optimal policy itself might be computationally feasible.

On the other hand, for large means, computing the optimal policy (or even the best order-up-to level) is computationally more difficult because the search space is larger. Nonetheless, our experimental results indicate that the simple and easily computable base stock policy (with the order-up-to level of $S^{\mathcal{B}*}(h, b + \tau h)$) continues to perform well even this setting, yielding total cost that is within 2% of the optimal (for $b = 199$). This result suggests a practical and effective replenishment heuristic: compute the optimal order-up-to policy exactly for small demand, and use the base stock level $S^{\mathcal{B}*}(h, b + \tau h)$ as an approximation for larger demand.

7.4 Impact of Increasing Variance-to-Mean Ratio

In this section, we explore the impact of the variance-to-mean ratio on the performance of base stock policies. As in the previous section, we assume the lead time is 2 ($\tau = 2$) and the demand D in each period follows a negative binomial demand distribution with parameter (r, p) where $r \in \{1, 2\}$ and

Lead Time	Lost Sales Penalty	Optimal Cost	Best Base Stock			$S^{\mathcal{B}*}(h, b+\tau h)$		
			Level	Cost	% Diff From Optimal Cost	Level	Cost	% Diff From Optimal Cost
1	1	1.97	8	2.08	5.91%	11	2.61	32.55%
	4	4.04	12	4.16	3.01%	13	4.39	8.63%
	9	5.44	13	5.55	2.00%	14	5.56	2.19%
	19	6.68	15	6.73	0.78%	16	6.95	4.18%
	49	8.17	17	8.22	0.54%	17	8.22	0.54%
	99	9.18	18	9.20	0.27%	18	9.20	0.27%
	199	10.13	19	10.14	0.14%	19	10.14	0.14%
2	1	2.03	12	2.23	9.73%	18	4.11	102.21%
	4	4.40	16	4.64	5.55%	19	5.35	21.76%
	9	6.09	19	6.32	3.65%	20	6.55	7.54%
	19	7.66	21	7.84	2.33%	22	8.15	6.31%
	49	9.52	23	9.63	1.13%	24	9.94	4.42%
	99	10.79	24	10.84	0.48%	25	11.03	2.18%
	199	11.99	25	12.03	0.33%	26	12.09	0.83%
3	1	2.06	15	2.31	12.09%	24	5.11	147.52%
	4	4.60	20	4.97	8.16%	25	6.29	36.67%
	9	6.53	23	6.86	5.10%	27	8.01	22.62%
	19	8.36	26	8.60	2.91%	28	9.19	9.87%
	49	10.55	28	10.73	1.73%	30	11.10	5.25%
	99	12.05	30	12.15	0.82%	31	12.30	2.13%
	199	13.41	32	13.52	0.81%	32	13.52	0.81%
4	1	2.08	18	2.37	14.13%	30	6.08	192.96%
	4	4.73	25	5.20	9.90%	31	7.21	52.51%
	9	6.84	28	7.27	6.36%	33	8.97	31.28%
	19	8.89	31	9.23	3.88%	34	10.16	14.31%
	49	11.38	34	11.60	1.97%	36	12.16	6.86%
	99	13.07	36	13.24	1.27%	37	13.44	2.81%
	199	14.62	38	14.77	1.04%	39	15.08	3.19%

Table 2: Performance of base stock policies for the Poisson distribution with mean 5.

Lead Time	Lost Sales Penalty	Optimal Cost	Best Base Stock			$S^{\mathcal{B}*}(h, b+\tau h)$		
			Level	Cost	% Diff From Optimal Cost	Level	Cost	% Diff From Optimal Cost
1	1	3.95	5	4.06	2.92%	12	6.02	52.49%
	4	9.82	12	10.04	2.30%	17	11.22	14.24%
	9	14.51	17	14.73	1.51%	21	15.46	6.60%
	19	19.22	22	19.40	0.96%	25	19.81	3.09%
	49	25.35	29	25.47	0.50%	31	25.79	1.74%
	99	29.88	33	29.99	0.37%	35	30.14	0.87%
	199	34.34	38	34.41	0.21%	40	34.68	0.99%
2	1	3.97	6	4.18	5.16%	20	8.71	119.25%
	4	10.24	15	10.71	4.54%	25	13.75	34.29%
	9	15.50	22	15.99	3.13%	29	18.02	16.24%
	19	20.89	28	21.31	2.00%	34	22.96	9.89%
	49	27.90	36	28.22	1.16%	40	29.25	4.84%
	99	33.04	41	33.28	0.73%	45	34.20	3.50%
	199	38.03	46	38.22	0.50%	49	38.72	1.81%
3	1	3.98	7	4.25	6.53%	28	11.47	187.85%
	4	10.47	18	11.13	6.38%	33	16.38	56.53%
	9	16.14	26	16.87	4.55%	37	20.59	27.63%
	19	22.06	33	22.73	3.00%	42	25.62	16.11%
	49	29.83	42	30.34	1.71%	48	32.07	7.50%
	99	35.50	48	35.90	1.13%	53	37.21	4.83%
	199	40.97	54	41.30	0.80%	58	42.33	3.30%
4	1	3.99	8	4.29	7.45%	36	14.25	256.99%
	4	10.61	21	11.44	7.84%	40	18.38	73.27%
	9	16.58	30	17.54	5.82%	45	23.22	40.05%
	19	22.95	38	23.85	3.91%	49	27.63	20.40%
	49	31.38	48	32.09	2.26%	56	34.76	10.77%
	99	37.54	54	38.10	1.49%	62	40.57	8.09%
	199	43.45	61	43.91	1.05%	67	45.79	5.39%

Table 3: Performance of base stock policies for the Geometric distribution with mean 5.

Lost Sales Penalty	Mean Demand	Optimal Cost	Best Base Stock			$S^{\mathcal{B}*}(h, b + \tau h)$		
			Level	Cost	% diff from optimal	Level	Cost	% diff from optimal
9	1	2.79	4	2.91	4.15%	6	3.39	21.50%
	2	3.92	8	4.02	2.53%	10	4.60	17.37%
	3	4.75	12	4.91	3.36%	13	5.16	8.58%
	4	5.47	15	5.64	3.05%	17	6.10	11.35%
	5	6.09	19	6.32	3.65%	20	6.55	7.54%
	6	6.67	22	6.88	3.14%	24	7.37	10.54%
	7	7.19	25	7.43	3.34%	27	7.76	7.92%
	8	7.67	29	7.95	3.57%	31	8.52	11.06%
	9	8.13	32	8.40	3.28%	34	8.85	8.77%
	10	8.56	35	8.84	3.31%	38	9.59	12.02%
19	1	3.60	5	3.68	2.41%	6	3.72	3.27%
	2	4.96	9	5.09	2.64%	10	5.10	2.75%
	3	6.02	13	6.12	1.67%	14	6.22	3.25%
	4	6.89	17	7.01	1.78%	18	7.21	4.72%
	5	7.66	21	7.84	2.33%	22	8.15	6.31%
	6	8.37	24	8.52	1.72%	25	8.66	3.40%
	7	9.01	28	9.21	2.25%	29	9.48	5.20%
	8	9.62	31	9.79	1.80%	33	10.30	7.15%
	9	10.17	34	10.38	2.10%	36	10.68	5.01%
	10	10.70	38	10.91	1.94%	40	11.46	7.04%
49	1	4.57	7	4.63	1.29%	7	4.63	1.29%
	2	6.21	11	6.26	0.83%	12	6.57	5.79%
	3	7.50	15	7.56	0.81%	16	7.80	4.02%
	4	8.57	19	8.65	0.92%	20	8.91	3.95%
	5	9.52	23	9.63	1.13%	24	9.94	4.42%
	6	10.39	26	10.50	1.06%	27	10.54	1.48%
	7	11.17	30	11.26	0.87%	31	11.43	2.33%
	8	11.89	34	12.03	1.14%	35	12.30	3.43%
	9	12.59	37	12.70	0.87%	38	12.80	1.62%
	10	13.22	41	13.36	1.05%	42	13.58	2.73%
99	1	5.23	7	5.24	0.27%	8	5.41	3.56%
	2	7.08	12	7.11	0.49%	12	7.11	0.49%
	3	8.53	16	8.56	0.44%	17	8.76	2.73%
	4	9.74	20	9.78	0.43%	21	9.94	2.05%
	5	10.79	24	10.84	0.48%	25	11.03	2.18%
	6	11.74	28	11.81	0.57%	29	12.05	2.61%
	7	12.62	32	12.71	0.71%	32	12.71	0.71%
	8	13.46	35	13.55	0.64%	36	13.59	0.98%
	9	14.21	39	14.29	0.54%	40	14.45	1.69%
	10	14.94	43	15.04	0.71%	43	15.04	0.71%
199	1	5.81	8	5.82	0.22%	8	5.82	0.22%
	2	7.89	13	7.92	0.37%	13	7.92	0.37%
	3	9.48	17	9.50	0.24%	18	9.69	2.25%
	4	10.83	21	10.86	0.26%	22	10.94	1.00%
	5	11.99	25	12.03	0.33%	26	12.09	0.83%
	6	13.04	29	13.07	0.25%	30	13.16	0.95%
	7	13.98	33	14.02	0.30%	34	14.17	1.37%
	8	14.87	37	14.92	0.36%	38	15.14	1.85%
	9	15.71	41	15.79	0.46%	41	15.79	0.46%
	10	16.53	44	16.60	0.42%	45	16.64	0.65%

Table 4: Performance of base stock policies for different Poisson distributions when lead time is 2.

$0.1 \le p \le 0.5$. Thus, $E[D] = r(1-p)/p$ and $Var[D] = r(1-p)/p^2$, leading to a variance-to-mean ratio of $1/p$. Table 5 compares the costs of different policies for different variance-to-mean ratios.

We observe from the table that our base stock policies are quite robust. The relative difference between the optimal cost and the cost of order-up-to policies seems to be independent of the variance-to-mean ratio. We observe similar results even for larger lead times (not included in the paper due to space constraint), suggesting that our policies should perform well in many practical settings where the demand exhibits significant variance.

A. Proof of Theorem 4

Proof. Since $\mathbf{D} = \sum_{t=1}^{\tau+1} D_t$, if the demand in each period has a bounded support, so is $\mathbf{D}$. Part (a) then follows immediately from the definition of $m_{\mathbf{D}}(t)$. To prove part (b), if the demand in each period has an IFR distribution, it follows from Corollary 1.B.20 on page 23 in Shaked and Shanthikumar (1994) that $\mathbf{D}$ also has an IFR distribution. Then, it follows from Lemma 3 that for any $s \ge 0$,

$$\mathcal{P}\left\{D - t_2 > s \;\middle|\; D > t_2\right\} \le \mathcal{P}\left\{D - t_1 > s \;\middle|\; D > t_1\right\},$$

which implies that the mean residual life $m_{\mathbf{D}}(t)$ is a decreasing function in t, giving us the desired result. To prove part (c), we can assume without loss of generality that $F(x) < 1$ for all $x \ge 0$. For any x, let $\bar{F}(x) = 1 - F(x)$. It then follows from the definition of $m_{\mathbf{D}}(t)$ that

$$\frac{m_{\mathbf{D}}(t)}{t} \quad = \quad \frac{\int_t^\infty \bar{F}(u)du}{t\bar{F}(t)}.$$

Since $E\left[\mathbf{D}^2\right] < \infty$, it follows that $E[\mathbf{D}] = \int_0^\infty \bar{F}(u)du < \infty$. Therefore,

$$\lim_{t\to\infty} \int_t^\infty \bar{F}(u) = 0.$$

Moreover, we have that $E\left[\mathbf{D}^2\right] = \int_0^\infty 2u\bar{F}(u)du$. Since $\mathbf{D}$ has a finite second moment, it follows that

$$\lim_{t\to\infty} t\bar{F}(t) = 0,$$

implying that both the numerator and the denominator in the expression for $m_{\mathbf{D}}(t)/t$ converge to zero at t increases to infinity. Since $\mathbf{D}$ is assumed to be a continuous random variable, we can apply

Lost Sales Penalty	Negative Binomial Parameter r	p	Variance-to-Mean Ratio	Optimal Cost	Best Base Stock Level	Cost	% diff from optimal	$S^{\mathcal{B}*}(h, b+\tau h)$ Level	Cost	% diff from optimal
9	1	0.1	10.00	26.85	39	27.71	3.17%	52	31.61	17.71%
	1	0.2	5.00	12.66	18	13.06	3.18%	24	15.02	18.63%
	1	0.3	3.33	7.89	10	8.13	3.06%	14	9.11	15.36%
	1	0.4	2.50	5.48	7	5.62	2.53%	10	6.67	21.67%
	1	0.5	2.00	3.99	5	4.10	2.86%	7	4.86	21.94%
	2	0.1	10.00	37.86	73	39.16	3.43%	88	43.19	14.06%
	2	0.2	5.00	17.85	33	18.46	3.41%	40	20.35	13.98%
	2	0.3	3.33	11.13	19	11.52	3.45%	24	12.70	14.07%
	2	0.4	2.50	7.73	13	7.99	3.25%	16	8.85	14.42%
	2	0.5	2.00	5.65	9	5.83	3.13%	11	6.41	13.41%
19	1	0.1	10.00	36.18	49	36.92	2.05%	60	39.81	10.03%
	1	0.2	5.00	17.06	23	17.41	2.06%	27	18.50	8.46%
	1	0.3	3.33	10.64	14	10.86	2.04%	17	11.86	11.41%
	1	0.4	2.50	7.40	9	7.52	1.54%	11	8.01	8.24%
	1	0.5	2.00	5.42	6	5.49	1.36%	8	5.98	10.28%
	2	0.1	10.00	49.74	87	50.81	2.14%	98	53.46	7.47%
	2	0.2	5.00	23.46	39	23.96	2.14%	45	25.32	7.94%
	2	0.3	3.33	14.64	24	14.95	2.15%	27	15.75	7.61%
	2	0.4	2.50	10.18	16	10.39	2.10%	18	10.93	7.44%
	2	0.5	2.00	7.44	11	7.58	1.86%	13	8.20	10.10%
49	1	0.1	10.00	48.31	63	48.87	1.15%	70	50.53	4.60%
	1	0.2	5.00	22.79	29	23.04	1.11%	32	23.72	4.09%
	1	0.3	3.33	14.22	17	14.39	1.18%	20	15.02	5.62%
	1	0.4	2.50	9.89	12	9.99	1.06%	13	10.22	3.36%
	1	0.5	2.00	7.27	8	7.31	0.62%	10	7.84	7.92%
	2	0.1	10.00	64.75	103	65.50	1.16%	112	67.35	4.01%
	2	0.2	5.00	30.54	47	30.89	1.15%	51	31.63	3.56%
	2	0.3	3.33	19.06	28	19.30	1.22%	31	19.80	3.84%
	2	0.4	2.50	13.27	19	13.40	1.00%	21	13.84	4.32%
	2	0.5	2.00	9.72	13	9.83	1.10%	15	10.23	5.24%
99	1	0.1	10.00	57.20	72	57.63	0.75%	79	59.23	3.55%
	1	0.2	5.00	26.98	33	27.18	0.73%	36	27.73	2.77%
	1	0.3	3.33	16.85	20	16.96	0.67%	22	17.32	2.75%
	1	0.4	2.50	11.72	14	11.81	0.76%	15	12.07	2.97%
	1	0.5	2.00	8.59	10	8.65	0.69%	11	8.97	4.47%
	2	0.1	10.00	75.51	115	76.08	0.75%	122	77.46	2.58%
	2	0.2	5.00	35.62	53	35.89	0.75%	56	36.51	2.49%
	2	0.3	3.33	22.24	32	22.40	0.70%	34	22.80	2.50%
	2	0.4	2.50	15.48	21	15.61	0.84%	23	15.89	2.69%
	2	0.5	2.00	11.35	15	11.41	0.52%	16	11.55	1.81%
199	1	0.1	10.00	65.85	81	66.19	0.51%	87	67.45	2.43%
	1	0.2	5.00	31.06	37	31.23	0.53%	40	31.70	2.06%
	1	0.3	3.33	19.40	23	19.49	0.44%	25	20.00	3.07%
	1	0.4	2.50	13.51	16	13.60	0.64%	17	13.90	2.93%
	1	0.5	2.00	9.90	11	9.93	0.28%	12	10.12	2.27%
	2	0.1	10.00	85.82	126	86.26	0.51%	131	87.07	1.46%
	2	0.2	5.00	40.49	58	40.69	0.49%	61	41.29	2.00%
	2	0.3	3.33	25.29	35	25.40	0.45%	37	25.75	1.83%
	2	0.4	2.50	17.60	24	17.69	0.53%	25	17.92	1.82%
	2	0.5	2.00	12.90	17	12.96	0.49%	18	13.24	2.63%

Table 5: Performance of base stock policies for negative binomial distributions with different variance-to-mean ratio when the lead time is 2.

L'Hospital's Rule to conclude that

$$\lim_{t\to\infty}\frac{m_{\mathbf{D}}(t)}{t}=\lim_{t\to\infty}\frac{-\bar{F}(t)}{\bar{F}(t)-tf(t)}=\lim_{t\to\infty}\frac{1}{t\cdot r(t)-1}=0,$$

which is the desired result. □

B. Proof of Proposition 7

Proof. It is easy to verify that $1-F_\theta(x)=1/(1+x)^\theta$. It follows that $E[\mathbf{D}]=\int_0^\infty 1-F_\theta(x)dx=1/(\theta-1)$. Then, using the fact that $m_{\mathbf{D}}(t)=\int_t^\infty \mathcal{P}\{\mathbf{D}>z\}\,dz/\mathcal{P}\{\mathbf{D}>t\}$, we can also show that $m_{\mathbf{D}}(t)=(1+t)/(\theta-1)$, which proves the first part of Proposition 7.

To establish the second part, note that by definition $S^{\mathcal{B}*}(h,b)=F_\theta^{-1}(b/(b+h))$, which implies that $S^{\mathcal{B}*}(h,b)=\left(\frac{b+h}{h}\right)^{1/\theta}-1$. Then, we have that

$$\begin{aligned}E\left[\left(\mathbf{D}-S^{\mathcal{B}*}(h,b)\right)^+\right] &= \mathcal{P}\left\{\mathbf{D}>S^{\mathcal{B}*}(h,b)\right\}\cdot E\left[\mathbf{D}-S^{\mathcal{B}*}(h,b)\;\middle|\;\mathbf{D}>S^{\mathcal{B}*}(h,b)\right]\\ &= \frac{h}{b+h}m_{\mathbf{D}}\left(S^{\mathcal{B}*}(h,b)\right)=\frac{h\left(1+S^{\mathcal{B}*}(h,b)\right)}{(b+h)(\theta-1)},\end{aligned}$$

where the last equality follows from the formula for $m_{\mathbf{D}}(\cdot)$. Thus,

$$\begin{aligned}E\left[\left(S^{\mathcal{B}*}(h,b)-\mathbf{D}\right)^+\right] &= E\left[S^{\mathcal{B}*}(h,b)-\mathbf{D}\right]+E\left[\left(\mathbf{D}-S^{\mathcal{B}*}(h,b)\right)^+\right]\\ &= S^{\mathcal{B}*}(h,b)-\frac{1}{\theta-1}+\frac{h\left(1+S^{\mathcal{B}*}(h,b)\right)}{(\theta-1)(b+h)},\end{aligned}$$

and therefore,

$$\begin{aligned}C^{\mathcal{B}*}(h,b) &= hE\left[\left(S^{\mathcal{B}*}(h,b)-\mathbf{D}\right)^+\right]+bE\left[\left(\mathbf{D}-S^{\mathcal{B}*}(h,b)\right)^+\right]\\ &= h\left(S^{\mathcal{B}*}(h,b)-\frac{1}{\theta-1}\right)+\frac{h\left(1+S^{\mathcal{B}*}(h,b)\right)}{\theta-1}=\frac{h\theta S^{\mathcal{B}*}(h,b)}{\theta-1}.\end{aligned}$$

Thus,

$$\lim_{b\to\infty}\frac{C^{\mathcal{B}*}(h,\nu b)}{C^{\mathcal{B}*}(h,b)}=\lim_{b\to\infty}\frac{S^{\mathcal{B}*}(h,\nu b)}{S^{\mathcal{B}*}(h,b)}=\lim_{b\to\infty}\frac{\left(\frac{\nu b+h}{h}\right)^{1/\theta}-1}{\left(\frac{b+h}{h}\right)^{1/\theta}-1}=\nu^{1/\theta},$$

which is the desired result. □

C. Proof of Lemma 10

Proof. Let $\underline{M}=\sup\{x:P(D\le x)=0\}$ denote the lowest possible single period demand. Huh et al. (2006) show the convergence of the stochastic process $\{X_t^{\mathcal{L},S}\}$ for all $S>\underline{M}\cdot(\tau+1)$. This implies the result of the lemma for all such S. Next, we discuss the case of $S \leq \underline{M} \cdot (\tau + 1)$. We will show that

$$\lim_{T \to \infty} \frac{\sum_{t=1}^{T} E[h \cdot (X_t^{\mathcal{L},S} - D_t)^+ + b \cdot (D_t - X_t^{\mathcal{L},S})^+]}{T}$$

exists and equals $b \cdot (\mu - S/(\tau+1))$ for all $S \leq \underline{M} \cdot (\tau+1)$, where $\mu = E[D]$. We will first show that the lim sup of this sequence is bounded above by this quantity and the lim inf is bounded below by this quantity.

Let IP_1 denote the inventory position at the beginning of period 1. Consider the following policy π whose on hand inventory process and lost sales process will be denoted by $\{X_t^{\mathcal{L},\pi}\}$ and $\{LOST_t^{\mathcal{L},\pi}\}$, respectively. If $IP_1 > S$, π mimics the order-up-to S policy until the first period in which the inventory position falls below S before the ordering opportunity (and therefore, reaches S after ordering). Let us call this period as $\overline{T}$. It is easy to verify that $E[\overline{T}] < \infty$ if $E[D_t] > 0$. For all $t \in \{1, \ldots, \overline{T}\}$, $X_t^{\mathcal{L},\pi} = X_t^{\mathcal{L},S}$. For all $t \in \{1, \ldots, \overline{T} - 1\}$, $LOST_t^{\mathcal{L},\pi} = LOST_t^{\mathcal{L},S}$. The policy π deviates from the order-up-to S policy in the following sense from period $\overline{T}$ onwards.

The policy π orders-up-to S each period, as usual, but sells no units in the interval $[\overline{T}, \overline{T}+\tau-1]$ and artificially limits the sales to at most $S/(\tau+1)$ units in each period in the interval $[\overline{T}+\tau, \infty)$. We will now claim that this restriction actually implies that π sells exactly $S/(\tau+1)$ units in each of those periods.

First, the demand in each period exceeds $\underline{M}$, which, by assumption, exceeds $S/(\tau+1)$. Second, observe that the inventory position at the beginning of period $\overline{T}$ is S by definition. By construction, π does not sell any units in the interval $[\overline{T}, \overline{T}+\tau-1]$. This implies that $X_{\overline{T}+\tau}^{\mathcal{L},\pi} = S$. This, along with the restriction on sales, means that exactly $(\overline{T}+2\tau+1-t) \cdot (S/(\tau+1))$ units are available on hand at the beginning of each period in the interval $[\overline{T}+\tau, \overline{T}+2\tau]$. This quantity clearly exceeds $S/(\tau+1)$. Since demand and supply both exceed $S/(\tau+1)$, sales is dictated exclusively by the upper limit imposed on sales by π, thus proving the claim for this interval. Moreover, this also implies that $X_t^{\mathcal{L},\pi} = S/(\tau+1)$ for $t = \overline{T}+2\tau$. We now show the claim for the interval $[\overline{T}+2\tau+1, \infty)$.

Notice that the quantity ordered in any period t, $t > \overline{T}+\tau$, is the amount sold in the previous period. This implies that exactly $S/(\tau+1)$ units are received at the beginning of period $\overline{T}+2\tau+1$, thereby implying the availability of exactly $S/(\tau+1)$ units for sale at the beginning of that period. This implies that the amount sold in that period is also $S/(\tau+1)$. From this period onwards, the inventory on hand at the beginning of every period and the sales in every period are both exactly equal to $S/(\tau+1)$, thus proving the claim about the sales quantities. Since all available units are sold, no holding costs are incurred in these periods.

Based on the previous two paragraphs, the following facts can easily be verified for all $t \geq \overline{T}+2\tau+1$: (i) $X_t^{\mathcal{L},\pi} = S/(\tau+1)$, (ii) the ending inventory in period t is zero and so, the holding cost incurred in that period is zero, (iii) the expected lost sales cost in period t is $b\cdot(\mu - S/(\tau+1))$ and (iv) the cost incurred in the interval $[1,t]$ by the order-up-to S policy is smaller than the cost incurred by π in that interval for every sample path of demands.

Fact (iv) implies that

$$\lim_{T\to\infty}\sup \frac{\sum_{t=1}^{T} E[h\cdot(X_t^{\mathcal{L},S}-D_t)^+ + b\cdot(D_t - X_t^{\mathcal{L},S})^+]}{T} \;\leq\; \lim_{T\to\infty}\sup \frac{E[\text{cost incurred by } \pi \text{ in } [1,T]]}{T} \;.$$

Facts (ii) and (iii) above establish that

$$\lim_{T\to\infty} \frac{E[\text{cost incurred by } \pi \text{ in } [1,T]]}{T}$$

exists and equals $b\cdot(\mu - S/(\tau+1))$. Thus, we have proved that

$$\lim_{T\to\infty}\sup \frac{\sum_{t=1}^{T} E[h\cdot(X_t^{\mathcal{L},S}-D_t)^+ + b\cdot(D_t - X_t^{\mathcal{L},S})^+]}{T} \;\leq\; b\cdot(\mu - S/(\tau+1)) \;.$$

We will now show that $b\cdot(\mu - S/(\tau+1))$ is a lower bound on the lim inf of the average expected cost of the order-up-to S policy. By the definitions of $\overline{T}$ and the order-up-to S policy, we know that for all $t \geq \overline{T}$, the inventory position at the beginning of a period is exactly S. This means that the maximum number of units that can be sold in the interval $[t, t+\tau]$ is S. The expected demand in this interval is $\mu\cdot(\tau+1)$. So, $b\cdot(\mu\cdot(\tau+1) - S)$ is a lower bound on the lost sales penalty costs incurred in the interval $[t, t+\tau]$ for any $t \geq \overline{T}$. Recall that $\overline{T}$ has a finite expectation. Therefore,

$$\lim_{T\to\infty}\inf \frac{\sum_{t=1}^{T} E[h\cdot(X_t^{\mathcal{L},S}-D_t)^+ + b\cdot(D_t - X_t^{\mathcal{L},S})^+]}{T} \;\geq\; b\cdot(\mu - S/(\tau+1)) \;.$$

Thus, we have shown that

$$\lim_{T\to\infty} \frac{\sum_{t=1}^{T} E[h\cdot(X_t^{\mathcal{L},S}-D_t)^+ + b\cdot(D_t - X_t^{\mathcal{L},S})^+]}{T}$$

exists and equals $b\cdot(\mu - S/(\tau+1))$. □

D. Proof of Lemma 11

Proof. As mentioned earlier, Huh et al. (2006) show the convergence of the stochastic process $\{X_t^{\mathcal{L},S}\}$ for all $S > \underline{M} \cdot (\tau+1)$ independent of the starting state vector. For any $S \leq \underline{M} \cdot (\tau+1)$, it is easy to verify that

$$X_t^{\mathcal{L},S} \;=\; S/(\tau+1) \;\forall\, t$$

because $D_t \geq \underline{M} \;\forall\, t$. This implies the result with $X_\infty^{\mathcal{L},S}$ being the deterministic quantity $S/(\tau+1)$ for all $S \leq \underline{M}$. □

E. Proof of Theorem 14(a)

Proof. Since the long-run average cost $C^{\mathcal{L},S}(h,b)$ is convex in S (see Janakiraman and Roundy (2004)), it suffices to show that $\frac{d}{dS}C^{\mathcal{L},S}(h,b) \geq \frac{d}{dS}C^{\mathcal{B},S}(h,b+\tau h)$. (To simplify our exposition, we assume the demand is a continuous random variable, and thus, $C^{\mathcal{L},S}(h,b)$ and $C^{\mathcal{B},S}(h,b)$ are differentiable in S. When the demand is discrete, we can consider $\Delta C^{\mathcal{B},S}(h,b) = C^{\mathcal{B},S+1}(h,b) - C^{\mathcal{B},S}(h,b)$ and $\Delta C^{\mathcal{L},S}(h,b) = C^{\mathcal{L},S+1}(h,b) - C^{\mathcal{L},S}(h,b)$, and exactly the same proof technique still applies.)

Since $X_t^{\mathcal{L},S} = S - \sum_{i=t-\tau}^{t-1} D_i + \sum_{i=t-\tau}^{t-1}\left(D_i - X_i^{\mathcal{L},S}\right)^+$, it follows that

$$E\left[X_\infty^{\mathcal{L},S}\right] = S - \tau E\left[D\right] + \tau E\left[\left(D - X_\infty^{\mathcal{L},S}\right)^+\right],$$

where D denotes the demand in a single period. The above result implies that

$$\begin{aligned}
C^{\mathcal{L},S}(h,b) &= hE\left[\left(X_\infty^{\mathcal{L},S} - D\right)^+\right] + bE\left[\left(D - X_\infty^{\mathcal{L},S}\right)^+\right] \\
&= hE\left[X_\infty^{\mathcal{L},S} - D\right] + (b+h)E\left[\left(D - X_\infty^{\mathcal{L},S}\right)^+\right] \\
&= hS - (\tau+1)hE[D] + (b+(\tau+1)h)E\left[\left(D - X_\infty^{\mathcal{L},S}\right)^+\right],
\end{aligned}$$

and therefore,

$$\begin{aligned}
\frac{d}{dS}C^{\mathcal{L},S}(h,b) &= h - (b+(\tau+1)h)E\left[\frac{dX_\infty^{\mathcal{L},S}}{dS} \cdot \mathbf{1}\left(D > X_\infty^{\mathcal{L},S}\right)\right] \\
&\geq h - (b+(\tau+1)h)E\left[\mathbf{1}\left(D > X_\infty^{\mathcal{L},S}\right)\right] \\
&\geq h - (b+(\tau+1)h)E\left[\mathbf{1}\left(D > X_\infty^{\mathcal{B},S}\right)\right] \\
&= \frac{d}{dS}C^{\mathcal{B},S}(h,b+\tau h),
\end{aligned}$$

where the first inequality follows from Lemma 1 of Janakiraman and Roundy (2004) and related analysis which show that $\frac{dX_t^{\mathcal{L},S}}{dS} \in \{0,1\}$ for all t, and therefore, $\frac{dX_\infty^{\mathcal{L},S}(h,b)}{dS} \in [0,1]$. The second inequality follows from Corollary 12 which shows that $X_\infty^{\mathcal{B},S}$ is stochastically smaller than $X_\infty^{\mathcal{L},S}$. The final equality follows from the expression of the cost function in the backorder system $\mathcal{B}(h, b+\tau h)$, from which it follows that

$$\frac{d}{dS} C^{\mathcal{B},S}(h, b+\tau h) \;=\; h - (b + (\tau+1)h) E\left[\mathbf{1}\left(D > X_\infty^{\mathcal{B},S}\right)\right].$$

□

F. Proof of Theorem 14(b)

Proof. Let $\mathcal{A}(S)$ and $\mathcal{A}(S+\epsilon)$ denote two lost sales inventory systems that use order-up-to policies with parameters S and $S+\epsilon$, respectively. Let the starting state of $\mathcal{A}(S)$ (resp., $\mathcal{A}(S+\epsilon)$) be such that it has S (resp., $S+\epsilon$) units on hand and none on order. Let $X_t^{\mathcal{L},S}$ and $X_t^{\mathcal{L},S+\epsilon}$ denote the inventory on hand at the beginning of period t in $\mathcal{A}(S)$ and $\mathcal{A}(S+\epsilon)$, respectively. Let $LOST_t^{\mathcal{L},S}$ and $LOST_t^{\mathcal{L},S+\epsilon}$ denote the amounts of lost sales in period t in the two systems, respectively. That is,

$$LOST_t^{\mathcal{L},S} \;=\; (D_t - X_t^{\mathcal{L},S})^+ \;\text{ and }\; LOST_t^{\mathcal{L},S+\epsilon} \;=\; (D_t - X_t^{\mathcal{L},S+\epsilon})^+ \;.$$

Consider a third lost sales inventory system $\overline{\mathcal{A}}(S+\epsilon)$ with the following characteristics, which is operated in parallel to $\mathcal{A}(S)$ and $\mathcal{A}(S+\epsilon)$. That is, each system is experiencing the same sample path of demands. In the $\overline{\mathcal{A}}(S+\epsilon)$ system, each order raises the inventory position to $S+\epsilon$. Furthermore, in $\overline{\mathcal{A}}(S+\epsilon)$, we assume that for each period in the intervals $[(\tau+1)+1, 2\cdot(\tau+1)]$, $[3\cdot(\tau+1)+1, 4\cdot(\tau+1)]$, $[5\cdot(\tau+1)+1, 6\cdot(\tau+1)]$ etc., it does not make all its inventory on hand available for sale. Specifically, the amount of sales in any of these time periods cannot exceed the demand nor the amount of inventory on hand nor the amount of inventory on hand in the parallel system $\mathcal{A}(S)$. That is, the sales in $\overline{\mathcal{A}}(S+\epsilon)$ in period t are given by

$$\min\{D_t, X_t^{\mathcal{L},S}, \overline{X}_t^{\mathcal{L},S+\epsilon}\} \;.$$

In all other periods – periods within the intervals $[1, \tau+1]$, $[2\cdot(\tau+1)+1, 3\cdot(\tau+1)]$, $[4\cdot(\tau+1)+1, 5\cdot(\tau+1)]$ etc. – the system $\overline{\mathcal{L}}(S+\epsilon)$ behaves exactly like a lost sales inventory system, and the sales in period t are given by $\min\{D_t, \overline{X}_t^{\mathcal{L},S+\epsilon}\}$.

We now make the following claims:

(a) $X_t^{\mathcal{L},S} \leq \overline{X}_t^{\mathcal{L},S+\epsilon}$ for every t,

(b) $\sum_{u=1}^{t} LOST_u^{\mathcal{L},S} \geq \sum_{u=1}^{t} \overline{LOST}_u^{\mathcal{L},S+\epsilon} \geq \sum_{u=1}^{t} LOST_u^{\mathcal{L},S+\epsilon}$ for every t, and

(c) $\sum_{u=2k(\tau+1)+1}^{2(k+1)(\tau+1)} [LOST_u^{\mathcal{L},S} - \overline{LOST}_u^{\mathcal{L},S+\epsilon}] \geq \epsilon \cdot \mathbf{1}[\sum_{u=2k(\tau+1)+1}^{(2k+1)(\tau+1)} (D_u - X_u^{\mathcal{L},S})^+ > \epsilon]$ for every $k \geq 0$.

We will now verify statements (a)-(c). Statement (a) can be proved by induction and using the definition of the sales in period t in $\overline{\mathcal{A}}(S+\epsilon)$. This immediately implies that $\overline{\mathcal{A}}(S+\epsilon)$ incurs fewer lost sales than $\mathcal{A}(S)$ in every period, thus implying the first part of (b). Moreover, since $\overline{\mathcal{A}}(S+\epsilon)$ does not sell all the units it has available, the cumulative lost sales incurred by $\overline{\mathcal{A}}(S+\epsilon)$ in any interval $[1,t]$ exceeds the corresponding quantity in $\mathcal{A}(S+\epsilon)$. This proves the second part of (b). To show (c), divide the time line into cycles, each of length $(\tau+1)$ periods. That is, $[1,\tau+1]$ forms the first cycle, $[(\tau+1)+1, 2\cdot(\tau+1)]$ forms the second cycle and for any $k \geq 1$, $[(k-1)(\tau+1)+1, k(\tau+1)]$ forms the k^{th} cycle. In every period of every even cycle (i.e. $[(2k-1)(\tau+1)+1, 2k(\tau+1)]$), $\overline{\mathcal{A}}(S+\epsilon)$ sells exactly the same number of units as $\mathcal{A}(S)$, although it might have more units available. This ensures that at the beginning of every odd cycle, $\overline{\mathcal{A}}(S+\epsilon)$ has exactly ϵ units more on hand than $\mathcal{A}(S)$. This implies that if $\mathcal{A}(S)$ loses ϵ or more units of sales in an odd cycle, then $\overline{\mathcal{A}}(S+\epsilon)$ loses ϵ fewer units than $\mathcal{A}(S)$. This shows (c).

Let us now consider the following relations:

$$\lim_{T\to\infty} E\left(\sum_{u=1}^{T} \frac{LOST_u^{\mathcal{L},S+\epsilon} - LOST_u^{\mathcal{L},S}}{T}\right) = \lim_{k\to\infty} E\left(\sum_{u=1}^{2k(\tau+1)} \frac{(LOST_u^{\mathcal{L},S+\epsilon} - LOST_u^{\mathcal{L},S})}{2k(\tau+1)}\right),$$

$$\mathbf{1}\left(\sum_{u=2k(\tau+1)+1}^{(2k+1)(\tau+1)} D_u > S+\epsilon\right) \leq \left(\sum_{u=2k(\tau+1)+1}^{(2k+1)(\tau+1)} (D_u - X_u^{\mathcal{L},S})^+ > \epsilon\right).$$

Combining the above inequalities with Statements (b)-(c) and using the fact that demands are identical and independently distributed, we get

$$\lim_{T\to\infty} E\left(\frac{1}{T}\sum_{u=1}^{T} LOST_u^{\mathcal{L},S+\epsilon} - LOST_u^{\mathcal{L},S}\right) \leq \frac{-\epsilon}{2(\tau+1)} \mathcal{P}\left\{\mathbf{D} > S+\epsilon\right\} .$$

Therefore, we get

$$\lim_{\epsilon\downarrow 0} \frac{\lim_{T\to\infty} E\left(\sum_{u=1}^{T}(LOST_u^{\mathcal{L},S+\epsilon} - LOST_u^{\mathcal{L},S})/T\right)}{\epsilon} \leq -\frac{1}{2(\tau+1)} \cdot \overline{F}(S) .$$

Since $\lim_{\epsilon \downarrow 0}(C^{\mathcal{L},S+\epsilon}(h,b) - C^{\mathcal{L},S}(h,b))/\epsilon \geq 0$ holds for any $S \geq S^{\mathcal{L}*}(h,b) = \arg\min_{S\geq 0} C^{\mathcal{L},S}(h,b)$, and we know

$$C^{\mathcal{L},S+\epsilon} - C^{\mathcal{L},S} \; = \; h \cdot \epsilon \; + \; (b + h \cdot (\tau+1)) \cdot \lim_{T\to\infty} E[\sum_{u=1}^{T}(LOST_u^{\mathcal{L},S+\epsilon} - LOST_u^{\mathcal{L},S})]/T \; ,$$

it follows, for all $S \geq S^{\mathcal{L}*}(h,b) = \arg\min_{S\geq 0} C^{\mathcal{L},S}(h,b)$, that

$$0 \; \leq \; h - (b + h \cdot (\tau+1))\overline{F}(S)/[2(\tau+1)] \; .$$

This inequality applied at $S^{\mathcal{L}*}(h,b)$ implies

$$F^{-1}\left(\frac{b - h\cdot(\tau+1)}{b + h\cdot(\tau+1)}\right) \leq \arg\min_{S\geq 0} C^{\mathcal{L},S}(h,b) \; .$$

Thus, we obtain the required result since $(b - h\cdot(\tau+1))/(b + h\cdot(\tau+1))$ is the newsvendor fractile of the $S^{\mathcal{B}*}(2h(\tau+1), b - h(\tau+1))$ system. □

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