Optimal Pricing and Inventory Planning with Charitable Donations

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This paper investigates firms' optimal operational decisions and after-tax profits with regard to tax deduction for charitable donations. Motivated by the steady growth in non-cash donations from U.S. companies, our work is the first to provide theoretical guidance on operational planning under tax deduction for both pre-committed donations and end-of-season donations. We analyze the impact of tax deduction for a profit-driven firm under a two-period price-markdown newsvendor model and characterize the firm's optimal price and quantity decisions. The firm's optimal donation behavior is driven by two factors: fixed cost and demand uncertainty. Specifically, a positive fixed cost can induce pre-committed donation during the regular selling season, and demand uncertainty can induce end-of-season donation during the clearance period. The enhanced tax deduction that is designed to encourage charitable donations results in unexpected behavior by the firm. For example, the firm's optimal clearance price can increase with the amount of leftover inventory, and the firm's optimal after-tax profit can increase as the tax rate increases. While the value of the deduction is tied to the fair market value (and the price) of the product, surprisingly, the firm may find it more profitable to charge a lower price, because the lower price may scale up the demand uncertainty and consequently increase the expected tax subsidy under the enhanced tax deduction. Our analysis reveals important insights about the impact of the tax law on a monopolist's optimal operational decisions and profit.

Key words: donation, tax deduction, price-setting newsvendor, integrated operational and tax planning

1. Introduction

Charitable giving has bounced back from the last recession, and companies have donated more products to charities in recent years. Rose (2011) finds that corporate non-cash donations increased from \$6 billion in 2007 to \$8 billion to 2010 in a matched set of 110 companies over this time period. Stroik (2013) estimates that more than 95% of the increase in the total aggregate giving from 2007 to 2012 came from non-cash giving.

Non-cash charitable donations are encouraged by Congress, especially when the donated products are used for the care of the ill, needy, or infants. In such cases, instead of claiming tax deductions based on the cost of goods sold, the firms can claim the enhanced tax deduction under Section 170(e)(3) of the U.S. Internal Revenue Code (IRC), which allows a deduction equal to the lesser of twice the cost basis and the average of the product's cost basis and its fair market value (FMV) at the time of donation. The intention is to provide financial incentives so that property donation becomes a preferred alternative to disposing of the inventory as waste, while preventing the situation where a company would be better off, after tax, by donating the inventory than it would be by selling the donated property and retaining the proceeds of the sale. Section 170(b)(2) of the U.S. IRC further limits the aggregate deductions associated with charitable contributions to 10% of the taxpayer's aggregate net income for the tax year. For contributions of food inventory, the Protecting Americans from Tax Hikes (PATH) Act of 2015 increased the charitable contribution percentage limitation to 15% for tax years beginning after 2015.

Since the enhanced tax deduction became law in 1976, gifts-in-kind intermediaries that match the donated goods to nonprofits that serve the ill, needy, or infants have enjoyed a robust growth. In 1977, the National Association for the Exchange of Industrial Resources (NAEIR) was among the first nonprofit intermediaries founded to help match the donated goods to charities in a manner that would maximize the new tax incentive under Section 170(e)(3) of the US tax law. Since its founding, NAIER has collected and redistributed product donations worth billions of dollars to tens of thousands of qualified nonprofits and public schools nationwide. Founded in 1983, Good360 (formerly Gifts In Kind International) has enjoyed an even more rapid growth and distributed goods worth three times as much as NAEIR. These matching and logistic services are also offered by smaller and more specialized intermediaries and some large multi-service nonprofits.

Coinciding with the development of the intermediaries that greatly lowers the transaction costs of product donation and relieves firms of the responsibility of finding a specific matching recipient (Ross and McGiverin-Bohan 2012), firms have increased inventory donations to charities over the years. In 1982, 11% of corporate America's charitable contributions took the form of non-cash donations, while in 1986, non-cash gifts were 20% of the total giving (Ross 1987), and in 2012, the proportion of non-cash donations rose to 69% (Stroik 2013).

Health care, consumer staples, and technology are the leading industries in terms of the percentage they give in non-cash contributions (Garton 2010, Perez 2015). Stroik (2013, 2014) estimates that about 90% of the total donations by pharmaceutical firms is in the form of product giving. For example, almost 95% of the donation by Merck is product donation (Table 1). Each year, Merck donates inventory worth about \$1.5 billion, while its annual operating income before tax is about \$6 billion. Because Merck discloses neither the method for determining the value of inventory donation (e.g., whether the donation value is calculated using retailing prices or adjusted for drugs

Fiscal Year	Income Before Tax ($ in millions)	Operating Income ($ in millions)	Product Donation ($ in millions)	Product Donation as % of Total Donation	Product Donation as % of Operating Income	Product Donation as % of Sales Revenue
2003	9,052	8,496	789	93.6	9.3	3.5
2004	7,975	6,622	921	94.1	13.9	4.0
2005	7,364	5,537	979	94.2	17.7	4.4
2006	6,221	3,544	768	93.0	21.7	3.4
2007	3,371	6,010	766	92.5	12.7	3.2
2011	7,334	7,670	1,194	94.2	15.6	2.5
2012	8,739	9,213	1,626	95.9	17.6	3.4
2013	5,545	5,956	1,751	94.1	29.4	4.0
2014	17,283	5,670	1,430	92.7	25.2	3.4
2015	5,401	6,928	1,684	92.5	24.3	4.3

Table 1 Contributions summary by Merck & Co., Inc.
Sources from Merck's Annual Reports and Merck & Co. Corporate Responsibility Report (2008, 2016).

past the expiration date) nor the amount of the enhanced tax deduction claimed in its tax return, it is not clear whether Merck's enhanced tax deduction is limited by 10% of its aggregate net income. The gap in the reported years presented in Table 1 is due to the fact that Merck has only revealed the worth of its product donation twice during the past 15 years, in its biennial corporate responsibility report.

Inventory donations may be established through formal donation programs between donors and recipients. The pharmaceutical industry is well-known for its long-term, structured donation programs that typically target specific diseases (Tzeneva 2014). The leading company in this regard is Merck, which provides dozens of drugs free of charge to eligible adults, primarily the uninsured who could not afford the needed health solutions. For example, Merck pledged to donate three million doses of GARDASIL® for HPV vaccination through its GARDASIL Access Program in 2007 (Merck & Co., Inc., 2008). In 2003, Pfizer committed to donate 135 million doses of its antibiotic azithromycin, marketed as Zithromax in the U.S., to the global effort to fight trachoma (Brown 2003). These formal donation programs, through which companies often donate their blockbuster drugs, are important means by which lower income countries access medicines (Davis 2017).

Inventory donations can also be occasional, especially in response to emergency situations. However, anecdotal evidence shows that, in such cases, matching the donors with the recipients can be challenging. For example, thousands of size-12 shoes were shipped to China after an earthquake (Andrews 2012), despite the fact that people in that region rarely need shoes of this size (Ross and McGiverin-Bohan 2012). It is estimated that 60% of the donated items after a disaster are not usable (Fessler 2013). Hechmann and Bunde-Birouste (2007) find that "Almost every time an emergency situation occurs, affected countries experience an influx of medications and equipment often not relevant for the emergency situation, as well as expired drugs or medications labeled in other languages." They argue that the tax benefit is one of the main reasons for the unsolicited donations from for-profit firms.

The benefit of the enhanced tax deduction depends on the FMV and the cost basis of the contribution. Section 20.2031-1(b) of the U.S. IRC defines FMV as the price at which the property would change hands between a willing buyer and a willing seller, neither being under any compulsion to buy or to sell and both having reasonable knowledge of relevant facts. The PATH Act of 2015 further clarifies that for food contribution that cannot or will not be sold solely by reason of internal standards of the company, lack of market, or similar circumstances, the fair market value may be determined by the *sale price* in the recent past, if there are no comparable sales at the time of contribution.

The cost basis of inventory consists of three components: direct material, direct labor, and manufacturing overhead. Direct material and direct labor are the costs that can be traced to individual units of product, while manufacturing overhead includes cost items like indirect materials, indirect labor, utility, maintenance and repair, depreciation, insurance, and property taxes. As a result, the cost basis of inventory includes both the fixed costs and the variable costs involved in the production or procurement of the product. The fixed cost component can play a major role in determining the cost basis, especially for firms with high operating leverage (e.g., software and pharmaceutical companies), whose variable costs are insignificant compared to their fixed costs.

In this paper, we investigate firms' optimal operational decisions and profits under the enhanced tax deduction. Building on Cachon and Kök (2007)'s two-period price-markdown newsvendor model, in which a firm first sets the regular price in the regular period and then sets the clearance price in the clearance period based on the realized demand, we allow the firm to donate and claim the enhanced tax deduction in either period. Specifically, the donation in the regular period, or the so-called "pre-committed donation," captures the inventory donation established through formal donation programs when the firm commits a pre-specified donation quantity before the realization of the demand. The donation in the clearance period, or the so-called "end-of-season donation," captures the occasional donation behavior when the firm chooses donation as a preferred alternative to salvaging after the demand realization. We analyze both the pre-committed donation and the end-of-season donation under the integrated operational and tax planning. Our analysis reveals the following important insights about the impact of the tax law on a monopolist's optimal operational decisions and profit.

First, the enhanced tax deduction induces unexpected behavior relating to a firm's optimal operational decisions and profit. For example, the firm's optimal clearance price can increase in the amount of leftover inventory (Section 4.3) and the firm's optimal after-tax profit can be an increasing function of the tax rate (Section 4.2).

Second, we identify the two driving forces for the two different types of donation behavior under the enhanced tax deduction: fixed cost and demand uncertainty. Specifically, it is profitable for the firm to pre-commit charitable donation during the regular season only if there is a positive fixed cost associated with production, and it is profitable to donate during the clearance period only if the demand is uncertain.

Third, we characterize when the enhanced tax deduction induces a higher or a lower optimal regular price and optimal production quantity under various settings. While the value of the deduction is tied to the FMV (and the price) of the product, surprisingly, the firm may find it more profitable to charge a lower price, when the lower price scales up the demand uncertainty and increases the expected tax subsidy under the enhanced tax deduction.

Last, our numerical analysis shows that integrated operational and tax planning can yield significant benefits to a firm when compared to the profits generated without the enhanced tax deduction. We also observe that the after-tax profit is a supermodular function of the fixed cost and the demand uncertainty with the enhanced tax deduction (i.e., the combined effect of the two driving factors on profit is greater than the sum of the individual effects of the two factors).

In summary, our analysis reveals the importance of integrated operational and tax planning and the surprising implications of the enhanced tax deduction law, which was originally intended to promote charitable donations. We hope that our work will encourage further analysis of this issue within the operations management community.

In the next section, we review the literature. Section 3 proposes the two-period price-markdown newsvendor model with donation. Section 4 characterizes the structure of the optimal solution under the enhanced tax deduction, and investigates how the tax law and the tax rate may impact the firm's optimal price, production quantity, and profit. Section 5 numerically illustrates the importance of integrated operational and tax planning, and examines the impact of the fixed cost and the demand uncertainty. We conclude and comment on future research directions in Section 6.

2. Literature Review

Our work is related to three streams of the literature: tax management in accounting, tax-efficient supply chain, and the price-setting newsvendor model. Ross and McGiverin-Bohan (2012) are among the first in accounting to provide a systematic evaluation of the benefit of product donation over salvaging. Nevertheless, their study focuses on how to dispose of products when a firm has excess inventory. Arya and Mittendorf (2015) consider a deterministic environment under which the enhanced tax deduction alone does not introduce price or quantity distortion for a profit-driven firm. However, when firms also have societal objectives, they show that the tax law can induce a higher price. In general, the tax management literature does not provide guidance on operational planning under the enhanced tax deduction.

Our work can also be considered as part of the literature on tax-efficient supply chain management, which has a growing body of publications; see Webber (2011) for an overview. Cohen and Mallik (1997) note the potential benefits of incorporating tax considerations into supply chain decisions. However, much of the work on this topic has focused on multinational enterprises and the issue of transfer pricing; see, for example, Huh and Park (2013), Shunko and Gavirneni (2007), Shunko et al. (2014), Xiao et al. (2015). The issue of charitable donation is largely ignored in the literature.

The model used in our paper is related to the price-setting newsvendor model, where a firm decides the price and production quantity simultaneously to maximize the expected profit; see Porteus (1990) for an excellent review on the classical newsvendor problems and Chan et al. (2004) for a comprehensive review on joint pricing and inventory decisions in newsvendor models. Our paper is particularly related to Cachon and Kök (2007), who consider a two-period setting where the market clearance price depends on the match between supply and demand. Similarly, in our setting, the FMV is tied to the actual transaction price of the product at the time of the donation.

Our paper differs from the existing newsvendor literature in the following dimensions. First, in the literature, the overage cost is typically independent of the price. However, in our model, the effective overage cost decreases with the price and the tax rate. The price-and-tax-dependent overage cost complicates the analysis significantly and leads to different pricing and production strategies. Moreover, in contrast with the extant literature, we want to understand the implication of the enhanced tax deduction for a profit-driven firm. As a result, we analyze the trajectories of the optimal profit, the optimal price, and the optimal production quantity decision as a function of the tax rate, and we obtain managerial insights from the structural properties of the optimal solutions under the tax law.

3. Model Setup

In this section, we introduce a two-period price-markdown newsvendor model with donation and analyze the structure of the firm's optimal strategy under the enhanced tax deduction. We present the model and the formulation of the firm's integrated operational and tax planning in Sections 3.1 and 3.2, respectively.

3.1. Model

We consider a risk-neutral firm that sells a product over a two-period selling season. The production lead time is long, and the firm needs to commit to the production quantity q and stock up the inventory prior to the selling season (i.e., during period 0). The selling season is composed of periods 1 and 2. Period 1 is the regular period, and period 2 is the clearance period. The firm sets price p_t for period t, and the resulting random demand in period t is $D_t(p_t, \xi)$ $(t = 1, 2)$, where the bounded random variable $\xi \in [\underline{\xi}, \bar{\xi}]$ captures the underlying demand state. Let $F(\cdot)$ and $f(\cdot)$ be the cumulative and density functions of the random variable ξ, respectively. Both $D_1(p, \xi)$ and $D_2(p, \xi)$ are weakly increasing in ξ, and the firm learns ξ at the end of period 1. $D_t(p, \xi)$ is non-negative, differentiable, and decreasing in p when $D_t(p, \xi) > 0$, and the inverse demand function is $d \mapsto p_t(d, \xi)$ and is well defined for d in the appropriate domain. We assume that, for all t and ξ, the revenue function $d \mapsto d \cdot p_t(d, \xi)$ is strictly concave in d in the domain. When demand exceeds available inventory, excess demand is lost. The firm adopts a markdown strategy, that is, the price in period 2 is bounded above by the price in period 1 (i.e., $p_2 \le p_1$), and the price in period 1 is bounded above by the maximum willingness-to-pay of the customers (i.e., $p_1 \le \bar{p}_1 \equiv p_1(d, \bar{\xi})|_{d \downarrow 0}$). We focus on the case that $\bar{p}_1 > v_c$ (when $\bar{p}_1 \le v_c$, producing zero unit is optimal for the firm). The leftover inventory at the end of period 2 has zero value. This set-up is essentially the same as the model used in Cachon and Kök (2007).

Not only does the firm need to make pricing and inventory decisions, as in Cachon and Kök (2007), the firm also needs to decide how to donate inventory over time. The sequence of events is as follows:

- In period 0 (before the selling season begins), the firm determines the production quantity q.
- At the beginning of period 1 (the regular period), the firms sets price p_1 and sets aside quantity r_1 to be donated during period 1. The sales amount s_1, where $s_1 \le q - r_1$, and the demand state ξ are realized at the end of period 1.
- At the beginning of period 2 (the clearance period), the firm decides price p_2. The sales amount s_2 is then realized, where $s_2 \le q - r_1 - s_1$. The firm then donates quantity r_2 at the end of period 2, where $r_2 \le q - r_1 - s_1 - s_2$. The leftover inventory at the end of period 2 has zero value and is salvaged.

To build up the inventory before the season starts, the firm incurs a non-negative fixed cost F_c and a non-negative per-unit variable cost v_c. With total production quantity q, the average unit cost of inventory is $c(q) \equiv v_c + \frac{F_c}{q}$. Although the unit cost of inventory depends on the total production quality in period 0, for ease of exposition, we will suppress the dependence on q and just write c to denote the unit cost. Under the current tax law, the allowable per-unit deduction amount depends on the cost and the FMV of the product. The definition of FMV ties it to the actual transaction price of the product at the time of the donation. The firm makes two donation decisions: the pre-committed donation r_1 at the beginning of period 1 and the end-of-season donation r_2 at the end of period 2. We thus have the following FMV assignment.

Assumption 3.1 (FMV Modeling Assumption). *The FMV of the pre-committed donation r_1 in period 1 is the regular price p_1, while the FMV of the end-of-season donation r_2 is the clearance price p_2, which is less than or equal to p_1.*

Under the current tax law, the per-unit deduction amount allowed is the cost basis c plus one-half of the appreciation $FMV - c$, capped at twice the cost basis, when the FMV exceeds the cost (i.e., $\min\{2c, \frac{FMV+c}{2}\} = \min\{2c, c + \frac{FMV-c}{2}\}$). To better understand the impact of the tax law, we adopt a general formula for the deduction amount. That is, for each unit of inventory donated, the firm may reduce the taxable income by $\min\{c + ac, c + b(FMV - c)\} = c + \min\{ac, b(FMV - c)\}$. We focus on the case that $a, b \in [0, \frac{1-\tau}{\tau})$, where τ is the marginal federal tax rate. Notice that, under the current tax law, $a = 1$, $b = \frac{1}{2}$, and $\tau < 40\%$, so that $a, b \in [0, \frac{1-\tau}{\tau})$.

3.2. The Firm's Integrated Operational and Tax Planning Problem

The firm produces q units of inventory at cost c and gradually sells and donates its inventory with price updates over the two periods. When the firm donates r_t units and sells s_t units at price p_t in period t ($t = 1, 2$), the resulting revenue is $p_t s_t$, the cost of goods sold is cs_t, the cost of inventory donated is cr_t, and the enhanced cost deduction amount is $(c + h(p_t))r_t$, where $h(p) = \min\{ac, b(p - c)\}$. The leftover inventory amount is $(q - r_1 - s_1 - r_2 - s_2)$ at the end of period 2, which contributes zero to the revenue and $c(q - r_1 - s_1 - r_2 - s_2)$ to the cost. The firm's aggregate revenue is $p_1 s_1 + p_2 s_2$, and the aggregate cost deduction amount is $h(p_1)r_1 + h(p_2)r_2 + cq$. The resulting tax is $\tau(p_1 s_1 + p_2 s_2 - h(p_1)r_1 - h(p_2)r_2 - cq)$ at tax rate τ, and the after-tax profit of the firm is $(1 - \tau)(p_1 s_1 + p_2 s_2) + \tau(h(p_1)r_1 + h(p_2)r_2) - (1 - \tau)cq$.

The firm's problem can be set up as a dynamic programming problem. Treating the initial inventory production cost as sunk, let $U_2(x|p_1, \xi)$ denote the maximum expected after-tax profit for period 2 given the regular period price p_1, demand parameter ξ, and x units of inventory available at the beginning of period 2. Also, let $U_1(x)$ denote the maximum expected after-tax profit for periods 1 and 2, given x units of inventory available at the beginning of period 1. We have the following dynamic programming equations:

$$U_2(x|p_1, \xi) = \max_{\substack{0 \le p_2 \le p_1 \\ 0 \le r_2 \le x \\ s_2 \le \min\{D_2(p_2, \xi), x - r_2\}}} \{(1 - \tau)p_2 s_2 + \tau h(p_2) r_2\},$$

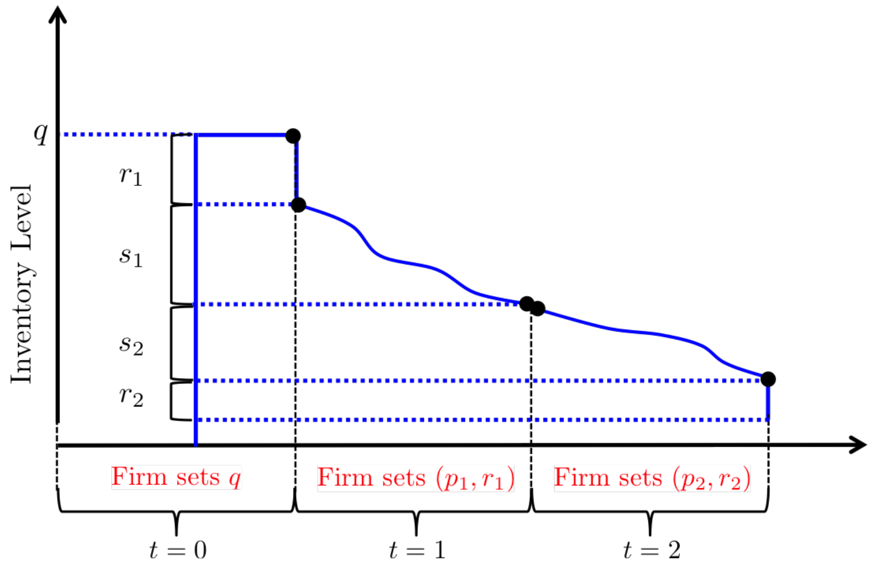


Figure 1 **Schematic representation of the two-period model with donation occurring prior to sales realization. The black dots represent the sequence of events.**

$$U_1(x) = \max_{\substack{0 \le p_1 \le \bar{p}_1 \\ 0 \le r_1 \le x \\ s_1 \le \min\{D_1(p_1,\xi),\, x - r_1\}}} \{(1-\tau)p_1 E_\xi[s_1] \;+\; \tau h(p_1) r_1 \;+\; E_\xi[U_2(x - s_1 - r_1 | p_1, \xi)]\}.$$

For both equations, the first constraint states that the price is positive and that the clearance price is bounded by the regular price and the regular price is bounded by the maximum willingness-to-pay; the second constraint indicates that the donation quantity r_t is constrained by x, the available inventory at the beginning of the period t; the last constraint shows that the sales quantity s_t is limited by both the demand $D_t(p_t, \xi)$ and the available inventory $x - r_t$ for $t = 1, 2$, because the donation quantity is deducted from the inventory before the sales realization in the regular period and the demand uncertainty is resolved in the clearance period. The firm's optimal production quantity decision at the beginning of the selling season in period 0, is obtained by maximizing the after-tax profit, corresponding to the following optimization problem:

$$U \equiv \max_q \; U_1(q) - (1-\tau)[v_c q + F_c] \; .$$

The firm's optimal policy consists of the optimal production quantity q^* in period 0, the optimal price p_1^* and donation amount r_1^* in the regular period (period 1), and for every realization ξ of the demand state, the optimal price $p_2^*(\xi)$ and donation amount $r_2^*(\xi)$ in the clearance period (period 2). Let U^* denote the corresponding total optimal profit. If multiple solutions generate the same optimal profit, we assume that the firm picks a solution with the least amount of donation according to a lexicographic order; that is, the firm first picks the solution with the least expected donations in period 2, then the least expected donations in period 1. If we still have a tie, we assume that the firm picks the solution with the highest production quantity. Without loss of generality, we assume that $p_1^* > c$. Also, s_1^* and $s_2^*(\xi)$ denote the optimal sales in period 1 and in period 2, respectively. Let $I^* = q^* - r_1^* - s_1^*$ denote the remaining inventory level at the end of period 1.

As shown in the numerical experiments in Section 5, the optimal policy can be computed numerically by solving the dynamic program for the value functions $U_1(\cdot)$ and $U_2(\cdot)$ for each value of q. Notice that the choice of q affects the average unit cost, which in turns affects the enhanced deduction amount $h(\cdot)$. Then, we can conduct a one-dimensional search for the optimal production quantity q^*. However, the main goal of our manuscript is to establish structural properties of the optimal policy and derive managerial insights about the drivers of donation behavior.

4. Optimal Policy Structure and Implications

In this section, we evaluate the optimal policy of the firm, and investigate how the enhanced tax deduction and the tax rate τ impact the firm's optimal profit and optimal decisions. In Section 4.1, we establish the general properties of the profit function. Section 4.2 illustrates the impact of the enhanced tax deduction through an example. We then investigate the structure of the optimal policy and the impact of the two driving factors on the firm's optimal price and production quantity in Sections 4.3 and 4.4. The proofs of the theoretical results are deferred to the Appendix and the Online Appendix.

4.1. General Properties of the Profit Function

We investigate how the enhanced tax deduction and the tax rate impact the firm's optimal regular price and production quantity. Recall that $(q^*, p_1^*, r_1^*, p_2^*(\xi), r_2^*(\xi))$ is the firm's optimal decision with the enhanced tax deduction. We write the optimal profit U^* as $U^*(\tau)$ to highlight the dependency on the tax rate. The next theorem states that the optimal after-tax profit is in fact convex in τ.

Theorem 4.1 (Convexity). *$U^*(\tau)$ is a convex function of τ.*

As the tax rate increases, the firm needs to pay a higher tax for the given profit and can also better take advantage of the enhanced tax deduction. It turns out that the firm may be better off with a higher tax rate, because a higher tax rate can create a greater tax subsidy under the enhanced deduction. To study the government tax subsidy due to enhanced deduction, let us define $\mathsf{EATD}(\tau) \equiv \mathbb{E}_\xi[h(p_1^*)r_1^* + h(p_2^*(\xi))r_2^*(\xi)]$ to be the expected additional tax deduction amount (in dollars) when the firm donates excess inventory, under the optimal production quantity, prices, and donation amounts. For notational convenience, we will suppress the dependence on prices, donations, and production quantity, and focus primarily on how $\mathsf{EATD}(\tau)$ changes with τ.

Theorem 4.2 (Monotonicity of Tax Deduction Amount). *At optimal, the expected additional tax deduction amount* $\mathsf{EATD}(\tau)$ *is a weakly increasing function of* τ.

Theorem 4.2 shows that the firm can claim a higher amount of deduction and utilize the tax law to a greater extent when the tax rate is higher. Given that the after-tax government subsidy due to the enhanced tax deduction is $\tau\mathsf{EATD}(\tau)$, this result helps explain the convexity of the firm's after-tax profit. The next theorem explains that if the tax rate is sufficiently high, the firm can adopt a profitable strategy of producing for donation and the resulting profit is only confined by the aggregate charitable contribution deduction limit.

Theorem 4.3 (Tax Rate Threshold). *When* $\tau > \frac{1}{1+a}$, $U^*(\tau) \geq \left(\tau - \frac{1}{1+a}\right) M$ *if* $M/(1+a) > F_c$ *and* $\bar{p}_1 \geq \frac{a+b}{b} v_c \frac{M}{M-(1+a)F_c}$, *where* M *is the maximum deduction amount under the tax law based on the aggregate income and deduction of the firm's other offerings.*

For any taxable year, a firm can deduct up to 10% of its aggregate net income and carry over the remaining amount. Theorem 4.3 shows that when τ is larger than $\frac{1}{1+a}$, producing solely for the donation purpose can be a profitable strategy for the firm when the fixed cost is relatively small compared to the potential deduction limit ($F_c < M/(1+a)$) and the firm can establish FMV at $\frac{a+b}{b} v_c \frac{M}{M-(1+a)F_c}$. The donation becomes profit-enhancing because it reduces the tax liability of the firm's other offerings. In this case, the benefit of donation is only limited by the aggregate deduction amount. Notice that Theorem 4.3 only establishes a lower bound on the firm's optimal profit, and the firm's optimal strategy may be different from producing solely for the donation purpose, especially when the maximum deduction amount M is small. In the ensuing analysis, we ignore the deduction limit, because here we focus on a single product while a firm can have many products and services. For a single product, the deduction can exceed 10% of its income, and, in fact, it would be beneficial for some products to exceed the limit to take advantage of the income brought in by other offerings.

The analysis in Theorem 4.3 provides us with a natural upper bound $\frac{1}{1+a}$ for the tax rate, which is 50% under the current law with $a = 1$.

4.2. A Two-Period Example with Random Demand

In this section, we illustrate how tax rate τ impacts the firm's optimal profit and decisions through a multiplicative random demand example.

Example 1 (Random Demand). $D_1(p,\xi) = D_2(p,\xi) = \xi D(p)$, where $D(p) = \mathsf{WTP} - p$, $\mathsf{WTP} = 10$, and ξ follows a log-normal distribution $\ln\mathcal{N}(\mu,\sigma^2)$, with $\mu = -0.5$ and $\sigma = 1$ (so that $\mathbb{E}[\xi] = 1$). $F_c = 20$ and $v_c = 1$, while $a = 1$ and $b = \frac{1}{2}$ according to the current law.

When the tax rate is zero, salvaging and donating each provides the same profit to the firm. The enhanced tax deduction does not provide the firm an incentive for inventory donation, and the firm's optimal solution would be the optimal solution when the enhanced tax deduction is prohibited by the tie-breaking rule. We use superscript 0 to denote the firm's decision when the tax rate is zero, or, equivalently, when the enhanced tax deduction is prohibited. So, let q^0 and p_1^0 denote the production quantity and the regular price, respectively, and let $p_2^0(\xi)$ denote the clearance price given the observed demand state ξ. Also, let $U^0(\tau)$ denote the firm's profit under $q^0, p_1^0, p_2^0(\xi)$, when evaluated under the tax rate τ. Our goal is to compare $(q^0, p_1^0, p_2^0(\xi))$ and $U^0(\tau)$ with the optimal decisions $(q^*, p_1^*, p_2^*(\xi))$ and $U^*(\tau)$ under different tax rates.

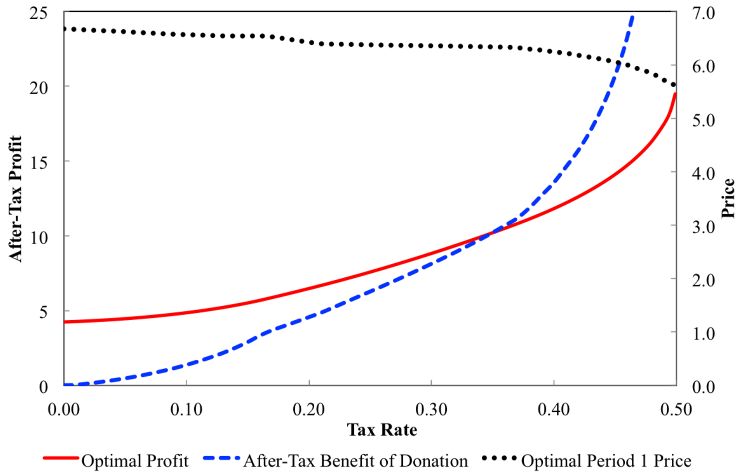
Figure 2 shows how the optimal operational decisions and the after-tax profit vary as a function of tax rate τ. The top graph shows the optimal price p_1^* in period 1, the after-tax optimal profit $U^*(\tau)$, and the tax subsidy $\tau\mathsf{EATD}(\tau)$ due to the enhanced deduction. The bottom graph illustrates the optimal production quantity q^* and expected donation $\mathbb{E}_\xi[r_2^*(\xi)]$ in the clearance period. Column 1 in Table 2 lists the tax rate $\tau \in \{0, 0.15, 0.35, 0.50\}$. Columns 2 to 4 present the optimal price, production quantity, and after-tax profit with the enhanced tax deduction, respectively. Columns 5 to 7 report the optimal price, production quantity, and after-tax profit without the enhanced tax deduction, respectively. We note that p^0 and q^0 are always the same regardless of the tax rate, so we only list them once in Columns 5 and 6. Columns 8 and 9 show the absolute improvement and the percentage improvement in profit due to the enhanced tax deduction, respectively.

τ	p_1^*	q^*	U^*	p_1^0	q^0	U^0	$U^* - U^0$	$\frac{U^*-U^0}{U^0}$
0%	6.67	12.10	4.24	6.67	12.10	4.24	0	0
15%	6.54	16.15	5.51			3.61	1.91	52.9%
35%	6.34	22.57	10.21			2.76	7.45	270.0%
50%	5.50	∞	20.25			2.12	18.13	854.3%

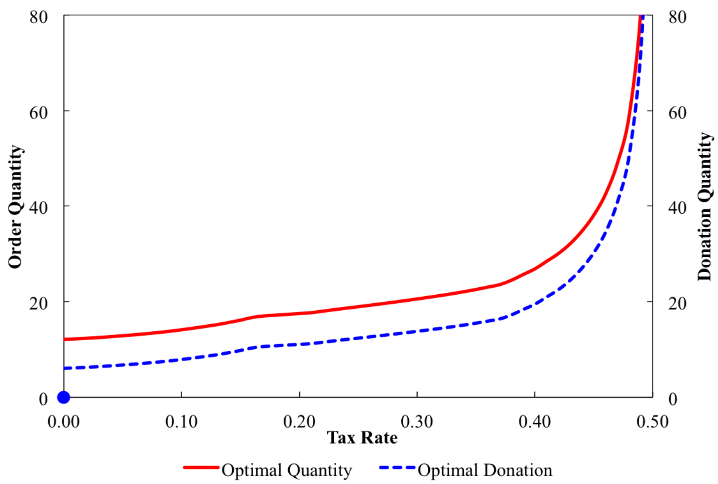
Table 2 Expected after-tax profit and optimal decisions at different tax rates for the example in Section 4.2.

Several phenomena deserve our attention. First, note that the after-tax profit with the enhanced tax deduction is a convex function of the tax rate, as specified by Theorem 4.1. Surprisingly, the after-tax profit is also an increasing function of the tax rate, as illustrated in Figure 2. With a higher tax rate, the firm loses on the profit earned through sales, but gains on the tax subsidy. The tax subsidy is rather valuable, especially under higher tax rates when the firm gets a higher subsidy for each unit donated, and donates more. Table 2 reports that the tax law may have a profound impact on the firm's optimal after-tax profit.

Second, the firm's optimal production quantity increases with respect to the tax rate, while the firm's optimal regular price decreases with respect to the tax rate, which means that the enhanced



Plot of the optimal regular price p_1^* in period 1 (dotted black), the optimal after-tax profit $U^*(\tau)$ (solid red) and the expected tax subsidy $\tau\mathsf{EATD}(\tau)$ (dashed blue) for different tax rates.



Plot of the optimal production quantity q^* (solid red) and the expected optimal donation quantity $\mathbb{E}_\xi[r_2^*(\xi)]$ in the clearance period (dashed blue) for different tax rates. At $\tau = 0$, there is no donation.

Figure 2 **Plots of the optimal decisions, after-tax profit and benefit for Example 1.**

tax deduction can result in a higher production quantity and a lower regular price despite the fact that a lower regular price implies a lower FMV and potentially a lower deduction. This motivates us to study the impact of the enhanced tax deduction on the firm's optimal price and quantity decisions in later sections.

Finally, the optimal solution here does not engage in pre-committed donation. That is, in this example, $r_1^* = 0$. At the same time, $E_\xi[r_2^*(\xi)] > 0$ for all $\tau > 0$ due to the demand uncertainty. As a result, we separate two driving factors — fixed cost and demand uncertainty — in the next two sections and study their impacts on the optimal policy structure.

4.3. Positive Fixed Cost with Deterministic Demand

In this section, we consider the case that $\xi = \underline{\xi} = \overline{\xi}$ is a constant and demand is deterministic. First we study how the firm's optimal strategy varies with the fixed cost in Section 4.3.1. Next we illustrate the profitability of the pre-committed donation in the regular period in Section 4.3.2. Finally, we analyze the impact of the enhanced tax deduction on the firm's operational decisions in Section 4.3.3.

4.3.1. Optimal Donation Under Deterministic Demand

Recall that $I = q - r_1 - s_1$ is the remaining inventory level at the end of period 1, and the addition of an asterisk sign represents that the variable is at its optimal level. Theorem 4.4 shows that the optimal solution structure under deterministic demand depends on the value of the fixed cost F_c.

Theorem 4.4 (Optimal Solution Structure under Deterministic Demand). *When $\xi = \underline{\xi} = \overline{\xi}$, $r_2^* = 0$, $s_1^* = D_1(p_1^*, \xi)$, and $s_2^* = I^*$. Moreover, there exists $0 \le \hat{F} \le \infty$ such that the firm's optimal strategy has following form:*

(I) *When $0 \le F_c \le \hat{F}$, the firm does not donate and perfectly plans the demand; that is, $r_1^* = r_2^* = 0$ and $q^* = s_1^* + s_2^*$. Furthermore, $\frac{\partial(s_1 p_1(s_1,\xi))}{\partial s_1}|_{s_1=s_1^*} = v_c$ and $\frac{\partial(s_2 p_2(s_2,\xi))}{\partial s_2}|_{s_2=s_2^*} = v_c$ if markdown constraint $p_1 \ge p_2$ is not binding; otherwise $p_2^* = p_1^*$.*

(II) *When $F_c > \hat{F}$, the firm engages in pre-committed donation; that is, $r_1^* > 0$ and $q^* = r_1^* + s_1^* + s_2^*$. Moreover, $q^* = F_c/(c^* - v_c)$, $h(p_1^*) = ac^*$ (i.e., $p_1^* \ge \frac{a+b}{b} c^*$), and $\frac{\tau}{1-\tau} ac^* > v_c \Leftrightarrow \frac{F_c}{F_c + v_c q^*} > \frac{1-\tau-\tau a}{1-\tau}$.*

(i) *If $p_1^* > \frac{a+b}{b} c^*$, $q^* = \sqrt{\frac{\tau a F_c (s_1^* + s_2^*)}{(1-\tau-\tau a) v_c}} > \frac{(a+b) F_c}{b p_1^* - (a+b) v_c}$. Furthermore, $\frac{\partial(s_1 p_1(s_1,\xi))}{\partial s_1}|_{s_1=s_1^*} = \frac{\tau}{1-\tau} ac$ and $\frac{\partial(s_2 p_2(s_2,\xi))}{\partial s_2}|_{s_2=s_2^*} = \frac{\tau}{1-\tau} ac$ if markdown constraint $p_1 \ge p_2$ is not binding; otherwise $p_2^* = p_1^*$;*

(ii) *Otherwise, $p_1^* = \frac{a+b}{b} c^*$, $q^* = \frac{(a+b) F_c}{b p_1^* - (a+b) v_c} \ge \sqrt{\frac{\tau a F_c (s_1^* + s_2^*)}{(1-\tau-\tau a) v_c}}$. Furthermore, $\frac{\partial(s_2 p_2(s_2,\xi))}{\partial s_2}|_{s_2=s_2^*} = \frac{\tau}{1-\tau} ac$ if markdown constraint $p_1 \ge p_2$ is not binding; otherwise $p_2^* = p_1^*$.*

Furthermore, $\hat{F} = \infty$ if $\bar{p}_1 \le \frac{(1-\tau)(a+b)}{\tau ab} v_c$ and $\hat{F} < \infty$ if $\bar{p}_1 > \frac{(1-\tau)(a+b)}{\tau ab} v_c$.

Theorem 4.4 shows that the pre-committed donation in the regular period can be profitable for the firm when the fixed cost F_c is large. The larger the fixed cost, the larger the difference between the unit cost and the marginal cost, and, consequently, the higher the benefit for donation under the enhanced tax deduction. Furthermore, the pre-committed donation is profitable only if the ratio $\frac{F_c}{F_c + v_c q^*}$ exceeds $\frac{1-\tau-\tau a}{1-\tau}$. The term $\frac{Fc}{F_c + v_c q^*} = \frac{c^* - v_c}{c^*}$ measures the company's fixed cost as a percentage of its total cost and relates to the concept of operating leverage. This threshold holds for both the deterministic demand case and the stochastic demand case.

Under general demand functions, it is challenging to write down the closed-form solution even for the deterministic demand case because first, the underlying optimization problem is not convex (e.g., when the markdown constraint $p_1 \geq p_2$ is binding, multiple solutions may maximize the firm's profit); and second, the optimal solution is not continuous with respect to F_c (e.g., at $F_c = \hat{F}$).

To better understand the optimal solution structure, we consider a special case under which $D_1(p,\xi) = D_2(p,\xi) = \mathsf{WTP} - p$. The following theorem characterizes different regimes of the optimal decision.

Theorem 4.5 (Optimal Solution Structure under Deterministic Linear Demand). *When $D_1(p,\xi) = D_2(p,\xi) = \mathsf{WTP} - p$, there exists $0 \leq \hat{F} \leq \breve{F} \leq \bar{F}$ such that the firm's optimal strategy has following form:*

(I) *When $0 \leq F_c \leq \hat{F}$, $p_1^* = p_2^* = \frac{\mathsf{WTP}+v_c}{2}$, $s_1^* = s_2^* = \frac{\mathsf{WTP}-v_c}{2}$, $r_1^* = r_2^* = 0$ and $q^* = s_1^* + s_2^*$.*

(II) *When $\hat{F} < F_c \leq \breve{F}$, $p_1^* = p_2^* = \frac{\mathsf{WTP}+\frac{\tau}{1-\tau}ac^*}{2}$, $s_1^* = s_2^* = \frac{\mathsf{WTP}-\frac{\tau}{1-\tau}ac^*}{2}$, $q^* = \frac{F_c}{c^*-v_c}$, $r_1^* = q^* - s_1^* - s_2^*$, and $r_2^* = 0$, where c^* is the unique solution to $\tau a\left(\mathsf{WTP} - \frac{\tau}{1-\tau}ac\right)(c-v_c)^2 - F_c(1-\tau-\tau a)v_c = 0$ on $\left(v_c, \max\left\{v_c, \frac{\mathsf{WTP}}{\frac{2(a+b)}{b} - \frac{\tau a}{1-\tau}}\right\}\right]$.*

(III) *When $\breve{F} < F_c \leq \bar{F}$, $p_1^* = \frac{a+b}{b}c^*$, $p_2^* = \frac{\mathsf{WTP}+\frac{\tau}{1-\tau}ac^*}{2}$, $s_1^* = \mathsf{WTP} - \frac{a+b}{b}c^*$, $s_2^* = \frac{\mathsf{WTP}-\frac{\tau}{1-\tau}ac^*}{2}$, $q^* = \frac{F_c}{c^*-v_c}$, $r_1^* = q^* - s_1^* - s_2^*$, and $r_2^* = 0$, where c^* is the unique solution to $\frac{1}{2}\tau a\left(\mathsf{WTP} - \frac{\tau}{1-\tau}ac\right)(c-v_c)^2 - (1-\tau)\frac{a+b}{b} - \tau a)\left(\mathsf{WTP} - \frac{2(a+b)}{b}c\right)(c-v_c)^2 - F_c(1-\tau-\tau a)v_c = 0$ on $\left(\max\left\{v_c, \frac{\mathsf{WTP}}{\frac{2(a+b)}{b} - \frac{\tau a}{1-\tau}}\right\}, \frac{b\mathsf{WTP}}{a+b}\right]$.*

(IV) *When $F_c > \bar{F}$, $p_1^* = \frac{a+b}{b}c^*$, $p_2^* = \frac{\mathsf{WTP}+\frac{\tau}{1-\tau}ac^*}{2}$, $s_1^* = 0$, $s_2^* = \frac{\mathsf{WTP}-\frac{\tau}{1-\tau}ac^*}{2}$, $q^* = \frac{F_c}{c^*-v_c}$, $r_1^* = q^* - s_1^* - s_2^*$, and $r_2^* = 0$, where $c^* = \frac{b\mathsf{WTP}}{a+b}$.*

Furthermore, $\hat{F} = \infty$ if and only if $\tau \leq \frac{(a+b)v_c}{(a+b)v_c + ab\mathsf{WTP}}$ and when $F_c > \hat{F}$, p_1^*, p_2^*, and c^* are continuous (weakly) increasing functions of F_c.

Theorem 4.5 offers detailed structural results under the deterministic linear demand setting. When the fixed cost is small, the firm does not engage in pre-committed donation. When the fixed cost is large, producing for donation becomes profitable. In this case, both the optimal unit cost and the price increase with the fixed cost. When the fixed cost is moderately high, it is optimal to have the same prices in the regular period and in the clearance period given that both periods face the same deterministic demand pattern. When the fixed cost is sufficiently high, it is optimal to have different prices in the two periods to better take advantage of the pre-committed donation. We illustrate the pre-committed donation behavior using the ensuing example.

4.3.2. An Illustrative Example

Example 2 (Deterministic Demand). $D_1(p,\xi) = D_2(p,\xi) = \mathsf{WTP} - p$, where $\mathsf{WTP} = 10$. $F_c = 20$, $v_c = 1$, and $\tau = 35\%$, while $a = 1$ and $b = 0.5$ according to the current law.

Without the enhanced tax deduction, the optimal production quantity and regular price are $(q^0, p_1^0) = (9.00, 5.50)$, and the firm's after-tax profit is 13.33. With the enhanced tax deduction, the optimal production quantity and regular price are $(q^*, p_1^*) = (18.86, 6.18)$, $p_2^* = 5.55$, and the firm's after-tax profit is 13.77. Moreover, the optimal donation quantities are $r_1^* = 10.60$ and $r_2^* = 0$, i.e., the firm pre-commits all donations in the regular period.

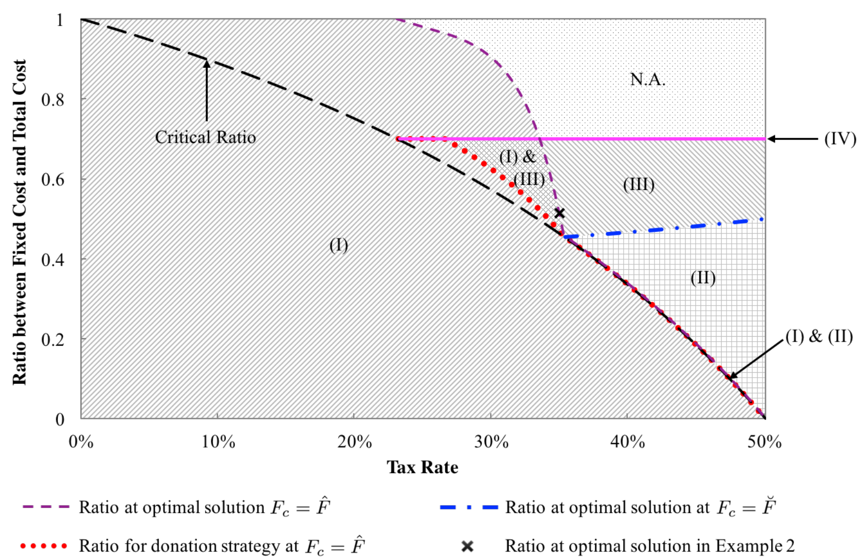


Figure 3 **Ratio between fixed cost and total cost versus tax rate for all realizations of the fixed cost. The regions (I)-(IV) correspond to the four forms of the firm's optimal strategies in Theorem 4.5, respectively.**

Figure 3 illustrates that as the tax rates and the fixed cost change, how the optimal ratio between the fixed cost and the total cost varies with the tax rate under the linear deterministic demand setting outlined in Example 2. The black long dashed line represents the critical ratio $\frac{1-\tau-\tau a}{1-\tau}$, above which pre-committed donation may be profitable.

When the tax rate is small (specifically when $\tau \leq 0.23$ by Theorem 4.4), $\hat{F} = \infty$. Thus, the no donation strategy (i.e., strategy (I)) is optimal and the optimal ratio varies from 0 to 1 as F_c varies from 0 to infinity. When the tax rate is large, $\hat{F}$ is finite and approaches 0 as the tax rate approaches 50%. Therefore, as the tax rate increases, the region of no donation shrinks. When $F_c \leq \hat{F}$, the optimal ratio is increasing in F_c for the given tax rate. When F_c increases beyond $\hat{F}$, the optimal strategy switches to one of the pre-committed donation strategies (i.e., strategies

(II), (III), or (IV)), the optimal production quantity jumps upwards and the optimal ratio jumps downwards. When $F_c > \hat{F}$, the optimal ratio is increasing in F_c for a given tax rate, resulting in the overlapping areas in Figure 3. When $F_c > \bar{F}$, the optimal ratio takes on a single value independent of the tax rate, which results in a single line for region (IV).

For the most common tax rate of 35%, the critical ratio (on the long dashed line) is 0.46, which provides a lower bound for the optimal ratio at which pre-committed donation is profitable. When $\tau = 0.35$, $\hat{F} = \breve{F} = 9.2$, and the optimal strategy switches from (I) to (III) at $F_c = \hat{F}$. The actual gap between the dotted and dashed lines is small, as the ratios between the fixed cost and the total cost are 0.505 and 0.457 for these two lines at $\tau = 0.35$. When $F_c = 20$, as in Example 2, strategy (III) is optimal. The ratio between the fixed and the total costs is 0.515, and is marked by the black cross in region (III) in Figure 3.

The software and pharmaceutical industries are notable with low variable costs and high operating leverages. Theorems 4.4 and 4.5 imply that it might be in the best interest of these firms to offer structured donation programs and produce additional inventory for the purpose of donation. This prediction is consistent with the anecdotal evidence and examples to this effect cited in the introduction section.

4.3.3. Impact of the Enhanced Tax Deduction on Optimal Solution

When $\xi = \underline{\xi} = \overline{\xi}$, the analysis of Theorems 4.4 and 4.5 shows that the firm's optimal decision can be rewritten as:

$$\begin{aligned} \max_{p_1, p_2, r_1 \geq 0} \quad & (1-\tau)[p_1 D_1(p_1,\xi) + p_2 D_2(p_2,\xi) - (F_c + v_c(r_1 + D_1(p_1,\xi) + D_2(p_2,\xi))] + \tau h(p_1) r_1, \\ \text{s.t.} \quad & p_2 \leq p_1 \leq \bar{p}_1 \ . \end{aligned}$$

When the markdown constraint is not binding, either the formulation becomes two separable one-period problems — if donation is not part of the optimal strategy — or the two periods are linked via the enhanced tax deduction benefit $h(p_1)$. Primarily, the firm needs to consider the margin-volume tradeoff in deciding the optimal prices. The potential complication that we have seen in Sections 4.3.1 and 4.3.2 is that the optimal solution is not continuous with respect to the parameters.

Despite the discontinuity, we investigate the impact of the enhanced tax deduction on the optimal regular price. In Example 2, the enhanced tax deduction induces both a higher optimal price and a positive pre-committed donation. This provides a sharp contrast to Example 1 in Section 4.2, in which the enhanced tax deduction induces a lower optimal price and the optimal solution does not engage in pre-committed donation. The key difference between Examples 1 and 2 is demand uncertainty. Formally, we show that the enhanced tax deduction (weakly) increases the optimal regular price when the firm does not face demand uncertainty.

Theorem 4.6 (Price Increase with Deterministic Demand). *When demand is deterministic, the optimal regular price under the enhanced tax deduction is at least as large as the optimal price without the enhanced tax deduction; that is, if* $\xi = \underline{\xi} = \overline{\xi}$*, then* $p_1^* \geq p_1^0$.

Given the potential pre-committed donation behavior, it is natural to conjecture that the enhanced tax deduction induces a (weakly) higher production quantity. Under the linear deterministic demand setting in Example 2, we indeed observe that the enhanced tax deduction (weakly) drives up the optimal production quantity for all the tax rates and all the realizations of the fixed cost. Specifically, at $F_c = \hat{F}$, optimal production quantity jumps under pre-committed donation strategies compared to the optimal no donation strategy.

Nevertheless, due to the margin-volume tradeoff, the increased optimal price under the enhanced tax deduction (Theorem 4.6) implies a reduced optimal sales quantity. As a result, one can construct examples such that the enhanced tax deduction results in a reduced optimal production quantity when demand is deterministic (e.g., by tweaking the special case $(p - v_c)D(p, \xi) = C$ for some constant $C > 0$, the optimal production quantity without the enhanced tax deduction can be arbitrarily large).

4.4. Zero Fixed Cost with Uncertain Demand

In this section, we consider the case that $F_c = 0$ and demand is uncertain. We first study how the firm's optimal strategy varies with the underlying demand state in Section 4.4.1. Then we illustrate the pricing and donation behavior in the clearance period in Section 4.4.2. Finally, we analyze the impact of the enhanced tax deduction on the firm's operational decisions in Section 4.4.3.

4.4.1. Optimal Donation Under Zero Fixed Cost

We first show that the firm has no financial incentive to engage in pre-committed donation in the regular period.

Theorem 4.7 (No Pre-committed Donation). *When* $F_c = 0$*,* $r_1^* = 0$.

Now, to analyze the solution strategy in the clearance period, we solve the proposed dynamic program for $U_2(\cdot)$. Recall that $I = q - r_1 - s_1$ is the remaining inventory level at the end of period

1, and the addition of an asterisk sign represents that the variable is at its optimal level. We first analyze the optimal price, sales, and donations p_2^*, s_2^*, and r_2^*, respectively, in the clearance period.

Theorem 4.8 (Optimal Policy in the Clearance Period). *Given the optimal first period decisions* (q^*, p_1^*, r_1^*)*, we have the following properties when* $\frac{\partial^2 p_2(s,\xi)}{\partial s \partial \xi} \geq 0$*:*

(I) *The optimal remaining inventory* I^* *at the end of period 1 is a weakly decreasing function of* ξ*;*

(II) *Suppose that for all* ξ*,* $p_2(s,\xi)$ *is concave in* s *or* $h(p_2(s,\xi)) = ac$ *for all* s *in the domain. Then, the net profit in the clearance period* $(1-\tau)sp_2(s,\xi) + \tau h(p_2(s,\xi))(I^* - s)$ *is concave in* s*. Moreover, there exist* $\hat{\xi}, \breve{\xi}$ *such that* $\underline{\xi} \leq \hat{\xi} \leq \breve{\xi} \leq \overline{\xi}$*, and we can partition the demand state into three consecutive intervals: high* $(\breve{\xi}, \overline{\xi}]$*, medium* $[\hat{\xi}, \breve{\xi}]$*, and low* $[\underline{\xi}, \hat{\xi})$*. Depending on the realized state of the demand at the end of period 1, the optimal decisions in the clearance period (period 2) are as follows:*

(i) *In the high demand state, the firm does not donate but clears all remaining inventory, and the clearance price is equal to the regular selling price; that is, if* $\breve{\xi} < \xi \leq \overline{\xi}$*, then* $r_2^* = 0$*,* $s_2^* = I^*$*, and* $p_2^* = p_1^* < p_2(I^*, \xi)$*;*

(ii) *In the medium demand state, the firm does not donate and still clears all remaining inventory, but at a lower price than the regular price; that is, if* $\hat{\xi} \leq \xi \leq \breve{\xi}$*, then* $r_2^* = 0$*,* $s_2^* = I^*$*, and* $p_2^* = p_2(I^*, \xi) \leq p_1^*$ *with* p_2^* *(weakly) increasing in* ξ*;*

(iii) *In the low demand state, either the firm does not donate and sets the clearance price below unit cost or the firm donates and sets the clearance price above the unit cost; that is, if* $\underline{\xi} \leq \xi < \hat{\xi}$*, then either* $r_2^* = 0$*,* $s_2^* = \arg\max_s (sp_2(s,\xi)) < I^*$*, and* $p_2^* = p_2(s_2^*, \xi) \leq c$*, or* $r_2^* > 0$*,* $s_2^* = \max\{D_2(p_1^*, \xi), \arg\max_s ((1-\tau)sp_2(s,\xi) + \tau h(p_2(s,\xi))(I^* - s)\}$*, and* $p_2^* = p_2(s_2^*, \xi) > c$*.*

Theorem 4.8 and the upcoming Theorem 4.9 also apply to the case that $F_c \geq 0$. The technical condition that $p_2(s,\xi)$ is concave in s or $h(p_2(s,\xi)) = ac$ for all s in the domain ensures that the clearance period profit function is still concave in s_2 when the enhanced deduction benefit is included. Under the linear demand setting studied in Sections 4.3.1 and 4.3.2, $p_2(s,\xi)$ is, of course, concave in s. The condition $h(p_2(s,\xi)) = ac$ is equivalent to $p_2(s,\xi) \geq (1 + \frac{a}{b})c = 3c$, under the current law with $a = 1$ and $b = \frac{1}{2}$. This condition is generally met for high-profit-margin products such as brand-name drugs and medicines.

Theorem 4.8 offers some insights on the trajectory of the optimal policy in the clearance period as the demand state varies. Note that the firm's available inventory I at the beginning of period 2 is decreasing in ξ, because the stronger the demand, the less the remaining inventory. Without the enhanced tax deduction, the firm should clear the inventory if the remaining inventory is low.

In this case, the firm's optimal second period price would be increasing in ξ, which is equivalent to decreasing in I. That is, the higher the remaining inventory, the lower the price. If the remaining inventory is high, the firm would set the optimal price to maximize the revenue and salvage the leftovers without the enhanced tax deduction.

The firm's solution is much more complicated when it is possible to take the enhanced tax deduction. Theorem 4.8 reveals that, while the second period problem can be solved efficiently due to the concavity structure, the firm may switch back and forth between salvaging and donating the excess inventory, as ξ varies if the remaining inventory is high. As a result, the optimal second period decisions may not be monotonous or even continuous.

Note that the condition $\frac{\partial^2 p_2(s,\xi)}{\partial s \partial \xi} \geq 0$ is implied by either $D_2(p,\xi) = D(p)$ (because when $D_2(p,\xi)$ is independent of ξ, $\frac{\partial^2 p_2(s,\xi)}{\partial s \partial \xi} = 0$) or $D_2(p,\xi) = \xi D(p)$ (because when $p_2(s,\xi) = p(s/\xi)$, $\frac{\partial^2 p_2(s,\xi)}{\partial s \partial \xi} = \frac{\partial (p'(s/\xi)/\xi)}{\partial \xi} = -\frac{(s/\xi)p''(s/\xi)+p'(s/\xi)}{\xi^2}$, which is non-negative by the concavity of $d_2 p_2(d_2,\xi)$).

Let $p(d)$ be the inverse function of $D(p)$. With a stronger condition, we can subdivide the "low demand" interval $[\underline{\xi}, \hat{\xi})$ from Theorem 4.8 into two sub-intervals and provide explicit characterization of the optimal decision in each of these intervals.

Theorem 4.9 (Refinement of the Optimal Policy under Multiplicative Demand). *Given the optimal first period decisions (q^*, p_1^*, r_1^*), when either $D_2(p,\xi) = D(p)$ or $D_2(p,\xi) = \xi D(p)$, all the results from Theorem 4.8 (I) and (II) parts (i) and (ii) continue to hold. In addition, there exist $\check{\xi}$ such that $\underline{\xi} \leq \check{\xi} \leq \hat{\xi}$, and the low demand interval $[\underline{\xi}, \hat{\xi})$ can be further subdivided into two intervals: moderately low $[\check{\xi}, \hat{\xi})$ and very low $[\underline{\xi}, \check{\xi})$, with the following properties:*

- *In the moderately low demand state, the firm does not donate and sets the clearance price below the unit cost; that is, if $\check{\xi} \leq \xi < \hat{\xi}$, then $r_2^* = 0$, $s_2^* = \xi D(\hat{p}) < I^*$, and $p_2^* = \hat{p} \equiv \arg\max_p (pD(p)) \leq c$.*

- *In the very low demand state, the firm donates, and the clearance price is above the unit cost; that is, if $\underline{\xi} \leq \xi < \check{\xi}$, then $r_2^* > 0$, $s_2^* = \max\{D_2(p_1^*, \xi), \arg\max_s ((1-\tau) s p_2(s,\xi) + \tau h(p_2(s,\xi))(I^* - s)\}$, and $p_2^* = p_2(s_2^*, \xi) > c$ with p_2^* and r_2^* (weakly) decreasing in ξ and s_2^* (weakly) increasing in ξ.*

With the assumption of multiplicative demand, the result of Theorem 4.9 provides a full picture of the optimal policy in the clearance period. As demand state ξ decreases from $\overline{\xi}$ to $\underline{\xi}$ (and remaining inventory I increases), the firm first chooses to clear the inventory (for high and medium demand states), next the firm may salvage the excess inventory (for moderately low demand), and then the firm may donate the excess inventory (for very low demand states). As the available inventory increases, the price p_2 is first decreasing when the firm clears the inventory, then it may remain constant at $\hat{p}$ when salvaging the excess inventory, and jump when the firm switches from salvaging to donating. Furthermore, as available inventory continues to increase, the optimal second period price and donation quantity are weakly increasing in the available inventory I.

4.4.2. An Illustrative Example

Example 3 (Clearance Period Price). $D_1(p,\xi) = D_2(p,\xi) = \xi D(p)$, where $D(p) = \mathsf{WTP} - p$, $\mathsf{WTP} = 1.9$, and ξ follows a log-normal distribution $\ln \mathcal{N}(\mu, \sigma^2)$, with $\mu = -0.5$ and $\sigma = 1$ (so that $\mathbb{E}[\xi] = 1$). $F_c = 0, v_c = 1$, and $\tau = 35\%$, while $a = 1$ and $b = \frac{1}{2}$ according to the current law.

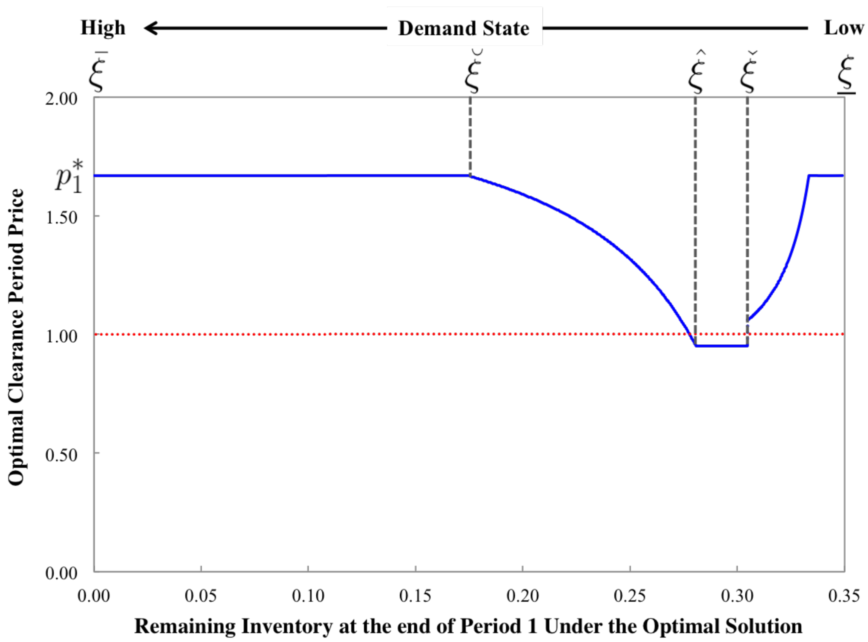


Figure 4 **Plot of the optimal clearance period price p_2^* versus I^*, the optimal remaining inventory at the end of period 1.**

Figure 4 provides an illustration of how the optimal period 2 price p_2^* varies with I^*, the optimal remaining inventory at the end of period 1. Under our setting, $p_1^* = 1.67$ and $q^* = 0.35$. Notice that there is a jump in the firm's optimal clearance period price p_2^* . That is, as I^* increases, p_2^* first weakly decreases (when the firm clears the inventory), then stays constant below the unit cost (when the firm salvages the inventory) and finally jumps above the unit cost and increases to the optimal regular period price p_1^* (when the firm donates the inventory). That is, the firm may adopt a price maintenance strategy to establish a higher FMV for donation when it has excess inventory. Notice that while pre-committed donation is part of the optimal strategy in Example 1 when demand is deterministic, the optimal solution here does not engage in pre-committed donation; that is, $r_1^* = 0$.

Notice that the marginal contribution of additional inventory in the second period, $\frac{U_2(x|p_1,\xi)}{\partial x}$, equals to $\tau h(p_2)$ when the firm opts to donate the excess inventory. With a lot of inventory available at the end of period 1, the firm chooses a higher price p_2 so as to better take advantage of the enhanced tax deduction. Theorem 4.9 implies that when the available inventory in the second period is high, the marginal contribution of additional inventory is an increasing function of the available inventory; therefore, $U_2(x|p_1,\xi)$ is not a concave function of x in general, which provides a stark contrast to Cachon and Kök (2007).

4.4.3. Impact of the Enhanced Tax Deduction on Optimal Solution

When the fixed cost is zero, $r_1 = 0$ at optimal by Theorem 4.7. Theroems 4.8 and 4.9 imply that with some regularity condition, the firm's after-tax profit can be represented by a function of (q, p_1):

$$\begin{aligned}\Pi(q,p_1) = &-(1-\tau)(F_c + v_c q) + (1-\tau)p_1 \int_{\underline{\xi}}^{\bar{\xi}} \min\{D_1(p_1,\xi), q\} dF(\xi) \\ &+ (1-\tau)\int_{\breve{\xi}}^{\bar{\xi}} p_1(q - D_1(p_1,\xi))dF(\xi) + (1-\tau)\int_{\hat{\xi}}^{\breve{\xi}} p_2(q - D_1(p_1,\xi),\xi)(q - D_1(p_1,\xi))dF(\xi) \\ &+ (1-\tau)\int_{\check{\xi}}^{\hat{\xi}} \max_p(pD_2(p,\xi))dF(\xi) + \int_{\underline{\xi}}^{\check{\xi}} \max_p((1-\tau)pD_2(p,\xi) + \tau h(p)(q - D_1(p_1,\xi) - D_2(p,\xi))dF(\xi)\ .\end{aligned}$$

The first term comes from the firm's production cost. The next term represents the after-tax sales revenue in the first period. The last four terms correspond to the potential scenarios in the second period described by Theorems 4.8 and 4.9.

We can glimpse the complexity of the problem by examining the first-order condition. Taking advantage of the envelope theorem, the first order derivative of the first two terms in $\Pi(q, p_1)$ with respect to p_1 would result in the standard margin-volume tradeoff in the first period, as discussed in Section 4.3.3. The first order derivative of the next term in $\Pi(q, p_1)$ captures the profit impact when p_2 is capped at p_1. The change of the regular price may further impact the leftover inventory quantity and the values of $\breve{\xi}$, $\hat{\xi}$, and $\check{\xi}$. Similarly, a small change in q at the optimal solution would introduce the standard newsvendor tradeoff in the first period and change the inventory availability in the second period. Under the interplay of these effects, Theorems 4.4-4.5 and Theorems 4.8-4.9 illustrate that the profit maximization formulation is no longer concave and the optimal strategy is no longer continuous. Therefore, it is extremely challenging if not impossible to pinpoint the impact of the tax law on the firm's optimal regular price and production quantity.

Despite the challenges, we show that the enhanced tax deduction (weakly) increases either the optimal regular price or the optimal production quantity when the fixed cost is zero. We first prove a lemma.

Lemma 1. *$U_2(x|p_1,\xi) - U_2(x|p_1,\xi,r_2=0)$ is (weakly) increasing in both p_1 and x for all ξ.*

Lemma 1 says that the benefit of the enhanced deduction in the clearance period is increasing in both the regular price and the on-hand inventory level for all realizations of the demand state. By decomposing the firm's profit into the profit without the enhanced tax deduction and the gain from the enhanced tax deduction, the next theorem shows that the enhanced tax deduction will not reduce the optimal price and the quantity simultaneously.

Theorem 4.10 (Price or Quantity Increase Under Zero Fixed Cost). *When there is no fixed cost, then either the regular price or the production quantity increases with the enhanced tax deduction; that is, if $F_c=0$, then either $p_1^* \geq p_1^0$ or $q^* \geq q^0$.*

To our surprise, our extensive numerical studies show that, typically, the enhanced tax deduction drives up the optimal production quantity but drives down the optimal regular price, especially when the regular price is high; that is, $b(p_1^*-c) > ac$, and the deduction benefit is capped at ac. This phenomenon is further reported in Section 5.

The enhanced tax deduction can induce a lower optimal price because a lower price can increase demand uncertainty and scale up the expected donation quantity and the tax subsidy (e.g., under a multiplicative demand model). A more rigid analysis is available from the authors upon request.

5. Numerical Analysis

In this section, we evaluate the implication of the tax law on the firm's operational planning. Specifically, we explore how the two driving forces — fixed cost and demand uncertainty — impact the firm's optimal regular price and production quantity and the resulting profit.

We employ the same multiplicative demand structures as in Section 4.2. Specifically, v_c is assumed to be 1 without loss of generality. The demand follows a multiplicative model, under which $D_1(p,\xi) = D_2(p,\xi) = \xi(\mathsf{WTP} - p)$, $\mathsf{WTP} = 10$, and ξ is a log-normal random variable with parameter (μ,σ) such that $\mu = -\sigma^2/2$ and $E[\xi]=1$. Under the current tax law, $a=1$ and $b=0.5$. We focus on the most common federal corporate tax rate $\tau = 35\%$.

We investigate the firm's operational decisions and profit with and without the enhanced tax deduction under $\sigma^2 \in \{0, 0.25, 0.50, 1.00\}$ and $F_c \in \{0, 10, 20\}$. In each problem instance, we compare the optimal solution and after-tax profit with the enhanced tax deduction to the optimal solution and after-tax profit without the enhanced tax deduction. We report the results in Table 3.

In Table 3, Columns 1 and 2 specify the test parameters σ^2 and F_c, respectively. Columns 3 to 5 summarize the firm's statistics with the enhanced tax deduction. Specifically, Column 3 reports the

σ^2	F_c	p_1^*	q^*	U^*	p_1^0	q^0	U^0	U^*-U^0	$\frac{U^*-U^0}{U^0}$
0	0	5.50	9.00	26.33	5.50	9.00	26.33	0	0
	10	5.50	9.00	19.83			19.83	0	0
	20	5.96	20.27	13.60			13.33	0.28	2.08%
0.25	0	6.00	14.22	23.26	6.24	11.27	21.68	1.58	7.28%
	10	5.95	15.90	18.33			15.18	3.15	20.77%
	20	5.95	20.36	13.35			8.70	4.67	53.80%
0.50	0	6.14	16.33	21.51	6.42	11.92	19.30	2.21	11.46%
	10	6.12	18.05	16.92			12.80	4.12	32.18%
	20	6.04	21.11	12.38			6.30	6.08	96.54%
1.00	0	6.36	18.62	18.66	6.67	12.10	15.76	2.90	18.38%
	10	6.35	20.27	14.39			9.26	5.14	55.47%
	20	6.34	22.57	10.21			2.76	7.45	270.02%

Table 3 Comparison of the optimal regular price, production quantity, and after-tax profit

optimal regular price p_1^*, Column 4 reports the production quantity q^*, and Column 5 shows the optimal after-tax profit under the enhanced tax deduction. Similarly, Columns 6 to 8 summarize the firm's statistics without the enhanced tax deduction. We report the absolute improvement and the percentage improvement in profit in Columns 9 and 10, respectively.

First, we observe that the enhanced tax deduction has no impact on the firm's operational decisions if and only if the demand is deterministic (i.e., $\sigma^2 = 0$) and the fixed cost is low, which is consistent with Theorem 4.4. When the enhanced tax deduction impacts the firm's operational decisions, the magnitude of change in the production quantity seems to be much larger than the magnitude of change in the regular price. The enhanced tax deduction increases the optimal production quantity. When the demand is deterministic, the enhanced tax deduction also increases the optimal regular price (Theorem 4.6); when the demand is uncertain, the enhanced tax turns out to reduce the optimal regular price under our test instances. As discussed in Section 4.4.3, a lower price can increase demand uncertainty and scale up the tax subsidy.

Second, a higher fixed cost F_c hurts the firm's profit with or without the enhanced tax deduction. Without the enhanced tax deduction, the firm's operational decisions are independent of F_c. When the firm optimally engages in donation with the enhanced tax deduction, we observe that the optimal production quantity is increasing in F_c, because a higher F_c increases the inventory cost c and the value of donation so that the firm faces a smaller overage cost and prefers to have more safety stock. We also observe that the optimal regular price is decreasing in F_c at a very slow rate, because the increased value of donation implies that the firm may prefer a slightly higher demand uncertainty to better take advantage of the tax subsidy.

Third, a higher demand variability σ^2 hurts the firm's profit with or without the enhanced tax deduction. We observe that both the optimal production quantity and the optimal regular price are increasing in σ^2. The optimal production quantity is increasing because a higher uncertainty translates into a higher safety level in our instances (the service rates are all greater than 50%). The optimal regular price is increasing because the firm prefers to increase the price to reduce demand uncertainty when the underlying demand state becomes more uncertain.

Last, while the after-tax profit is a modular function of the fixed cost and the demand uncertainty without the enhanced tax deduction, the after-tax profit appears to be a supermodular function of the fixed cost and the demand uncertainty with the enhanced tax deduction. That is, the combined effect of the two driving factors on profit is greater than the sum of the individual effects of the two factors. This statement is true both under the absolute performance improvement and under the percentage performance improvement.

6. Conclusion and Future Directions

Each year, companies donate goods worth billions of dollars. In this paper, we analyze the impact of the enhanced tax deduction on a firm's profit and operational decisions under a two-period price-markdown newsvendor model. We study the two driving forces — fixed cost and demand uncertainty — that induce the firm to donate inventories in a profitable way. Specifically, a positive fixed cost can induce pre-committed donation during the regular selling season, and the demand uncertainty can induce donation during the clearance period. These insights offer a potential explanation as to why health care, consumer staples, and technology are the leading industries in non-cash donation. While the value of deduction is tied to the FMV (and the price) of the product, surprisingly, the firm may find it more profitable to charge a lower regular price, because the lower price may scale up the demand uncertainty and the expected tax subsidy under the enhanced tax deduction.

Broadly speaking, we believe there is a significant opportunity to apply analytical models to understand the impact of tax laws, especially on product line design, supply chain, and revenue management. We hope that this paper will encourage more research at the interface of accounting and operations management.

Acknowledgements

The authors would like to thank the Editor-in-Chief Christopher Tang, the Associate Editor, and the two anonymous referees for their detailed and thoughtful comments and suggestions that notably improved the paper in many ways.

References

Andrews, E. 2012. The brave new world of philanthropy. *SPEA Magazine,* **Spring** 4–10. Available at https://spea.indiana.edu/magazine/Spring-2012/pdf/spring-2012.pdf. Date accessed: 06/29/2014.

Arya, A., B. Mittendorf. 2015. Supply chain consequences of subsidies for corporate social responsibility. *Production and Operations Management* **24**(8) 1346–1357.

Brown, D. 2003. Pfizer donating 135 million doses of anti-blindness drug. *The Washington Post*, Nov 12. Available at https://www.washingtonpost.com/archive/politics/2003/11/12/pfizer-donating-135-million-doses-of-anti-blindness-drug/22113d7c-4ce9-40ad-95bc-45ad7f6e8419/?utm_term=.ee38c0d04713. Date accessed: 08/01/2017.

Cachon, G., A. G. Kök. 2007. Implementation of the newsvendor model with clearance pricing: How to (and how not to) estimate a salvage value. *Manufacturing & Service Operations Management* **9**(3) 276–290.

Chan, L. M., Z. M. Shen, D. Simchi-Levi, J. L. Swann. 2004. Coordination of pricing and inventory decisions: A survey and classification. *Handbook of Quantitative Supply Chain Analysis*. Springer US, 335–392.

Cohen, M., S. Mallik. 1997. Global supply chains: Research and applications. *Production and Operations Management* **6**(3) 193–210.

Davis, N. 2017. Innovative access to medicines strategies in lower income countries. *The Pharma Letter*, January 10. Available at http://www.thepharmaletter.com/article/innovative-access-to-medicines-strategies. Date accessed: 03/03/2017.

Fessler, P. 2013. Thanks, but no thanks: When post-disaster donations overwhelm. *NPR Around the Nation*, January 09. Available at http://www.npr.org/2013/01/09/168946170/thanks-but-no-thanks-when-post-disaster-donations-overwhelm. Date accessed: 02/25/2017.

Garton, C. 2010. Companies donate employee's time, service instead of cash. *USA Today*, Aug 8. Available at http://usatoday30.usatoday.com/money/companies/2010-08-08-corporate-philanthropy_N.htm. Date accessed: 06/28/2014.

Hechmann, R., A. Bunde-Birouste. 2007. Drug donations in emergencies, the Sri Lankan post-tsunami experience. *Journal of Humanitarian Assistance.* September 26.

Huh, W. T., K. S. Park. 2013. Impact of transfer pricing methods for tax purposes on supply chain performance under demand uncertainty. *Naval Research Logistics* **60**(4) 269–293.

Merck & Co., Inc. 2008. Merck corporate responsibility report 2007. Available at `http://www.merck.com/`. Date accessed: 07/31/2017.

Perez, C. 2015. Giving in numbers: 2015 edition. CECP, in association with The Conference Board.

Porteus, E. L. 1990. Stochastic inventory theory. D. P. Heyman, M. J. Sobel, eds., *Handbooks in Operations Research and Management Science*, vol. 2. Elsevier, North Holland, 605–652.

Rose, A. P. 2011. Giving in numbers: 2011 edition. Committee Encouraging Corporate Philanthropy, New York, NY. Available at `http://www.corporatephilanthropy.org/pdfs/giving_in_numbers/GivinginNumbers2011.pdf`.

Ross, J. M., K. L. McGiverin-Bohan. 2012. The business case for product philanthropy. White Paper prepared for *The Home Depot Foundation*.

Ross, N. L. 1987. Giving products instead of money; more firms are donating inventory to charities. *The Washington Post*, July 27, F01.

Shunko, M., S. Gavirneni. 2007. Role of transfer prices in global supply chains with random demands. *Journal of Industrial and Managment Optimization* **3**(1) 99–117.

Shunko, M., L. Debo, S. Gavirneni. 2014. Transfer pricing and sourcing strategies for multinational firms. *Production and Operations Management* **23**(12) 2043–2057.

Stroik, M. 2013. Giving in numbers: 2013 edition. New York: Committee Encouraging Corporate Philanthropy.

Stroik, M. 2014. Giving in numbers: 2014 edition. New York: Committee Encouraging Corporate Philanthropy.

Tzeneva, G. 2015. Targeted drug donations: a necessary eveil in need of a global harness and a possible cure for TRIPS shortcomings. *Indiana Journal of Global Legal Studies* **21**(1) 369–388.

Webber, S. 2011. The tax-efficient supply chain. *Tax Notes International* **61**(2) 149–168.

Xiao, W., V. N. Hsu, Q. Hu. 2015. Manufacturing capacity decisions with demand uncertainty and tax cross-crediting. *Manufacturing & Service Operations Management* **17**(3) 384–398.

Appendix: Proofs of Selected Theorems in Section 4

Proof of Theorem 4.1: Let A be the set of all feasible policy $((p_t), q, (r_t)|\xi)$ and $U(((p_t), q, (r_t)|\xi), \tau)$ be the after-tax profit under policy $((p_t), q, (r_t)|\xi)$ and tax rate τ. For a given policy, $U(((p_t), q, (r_t)|\xi), \tau)$ is linear and convex with respect to τ. Therefore, $U^*(\tau) = \max_{((p_t),q,(r_t)|\xi)\in A} U(((p_t), q, (r_t)|\xi), \tau)$ is also a convex function of τ. □

Proof of Theorem 4.4 The optimal decisions can be written as $(q_1^*, p_1^*, r_1^*, p_2^*, r_2^*|F_c)$. Suppose $r_2^* > 0$. By donating r_2^* in period 1 rather than period 2 and keeping other decisions unchanged (that is, making decisions $(q_1^*, p_1^*, r_1^* + r_2^*, p_2^*, 0))$, the after-tax profit is weakly improved. This contradicts the optimality of $(q^*, p_1^*, r_1^*, p_2^*, r_2^*|F_c)$ by the tie-breaking rule. Thus, $r_2^* = 0$. The firm clears its inventory in period 2; that is, $s_2^* = I^*$.

Now we show that $p_1^* = p_1(s_1^*, \xi) \Leftrightarrow s_1^* = D_1(p_1^*, \xi)$. If $p_1^* = \bar{p}_1$, $s_1^* \in [0, D_1(p_1^*, \xi)]$ implies that $s_1^* = 0 = D_1(p_1^*, \xi)$. If $p_1^* < \bar{p}_1$ and $s_1^* < D_1(p_1^*, \xi)$, then the firm can improve its profit by raising p_1^* to $\min\{\bar{p}_1, p_1(s_1^*, \xi)\}$ and keeping all other variables the same. Thus, $p_1^* = p_1(s_1^*, \xi) \Leftrightarrow s_1^* = D_1(p_1^*, \xi)$.

Therefore, the firm's optimal decision can be rewritten as:

$$\begin{aligned} \max_{q, p_1, r_1} \quad & (1-\tau)[p_1 D_1(p_1, \xi) + p_2(q - r_1 - D_1(p_1, \xi), \xi)(q - r_1 - D_1(p_1, \xi)) - (F_c + v_c q)] + \tau h(p_1) r_1, \\ \text{s.t.} \quad & p_2(q - r_1 - D_1(p_1, \xi), \xi) \le p_1 \le \bar{p}_1 \ . \end{aligned}$$

By Theorem 4.7, $r_1^* = 0$ when $F_c = 0$.

Define $\hat{F} = \inf_{F_c \ge 0}\{F_c | r_1^* > 0\}$. We first show that when $F_c > \hat{F}$, $r_1^* > 0$ by contradiction.

Suppose that there exists $F_c'' > \hat{F}$ such that the optimal decision under F_c'' is $(q_1'', p_1'', r_1'', p_2'', r_2'')$ with $r_1'' = 0$. Recall that $(q^0, p_1^0, r_1^0, p_2^0, r_2^0)$ is the firm's optimal decision without the enhanced tax deduction. Therefore, when $F_c = 0$ or $F_c = F_c''$, the firm adopts the same optimal operational decisions and $U^*|_{F_c = F_c''} - U^*|_{F_c = 0} = -(1-\tau)F_c''$.

By the definition of $\hat{F}$, there exists $F_c' < F_c''$ such that the optimal decision under F_c' is $(q_1', p_1', r_1', p_2', r_2')$ with $r_1' > 0$. $U^*|_{F_c = F_c'} - U^*|_{F_c = 0} > -(1-\tau)F_c'$ due to the tie-breaking rule. Furthermore, $U^*|_{F_c = F_c''} - U^*|_{F_c = F_c'} > -(1-\tau)(F_c'' - F_c')$ because the firm under F_c'' can adopt $(q_1', p_1', r_1', p_2', r_2')$ as its decision and weakly increase its deduction. Therefore, $U^*|_{F_c = F_c''} - U^*|_{F_c = 0} > -(1-\tau)F_c''$ and we reach a contradiction.

Therefore, when $F_c < \hat{F}$, $r_1^* = 0$; when $F_c > \hat{F}$, $r_1^* > 0$. When $F_c = \hat{F}$, $r_1^* = 0$ due to the continuity of the profit function with respect to F_c and the tie-breaking rule.

When $F_c \le \hat{F}$ and $r_1^* = 0$, $q^* = s_1^* + s_2^*$ follows from $r_2^* = 0$. The firm's after-tax profit can be written as

$$\begin{aligned} \max_{s_1, s_2, p_2} \quad & (1-\tau)[s_1 p_1(s_1, \xi) + s_2 p_2 - (F_c + v_c(s_1 + s_2))], \\ \text{s.t.} \quad & p_2 \le p_1(s_1, \xi) \le \bar{p}_1, p_2 \le p_2(s_2, \xi) \ . \end{aligned}$$

When the markdown constraint is not binding, $p_2 = p_2(s_2, \xi)$ and we can convert the formulation into a convex optimization with a single constraint $p_1 \le \bar{p}_1$. Ignoring the constraint, the first-order conditions reveal that $\frac{\partial(s_1 p_1(s_1, \xi))}{\partial s_1}|_{s_1 = s_1^*} = \frac{\partial(s_2 p_2(s_2, \xi))}{\partial s_2}|_{s_2 = s_2^*} = v_c$. Notice that as $v_c < \bar{p}_1$, the constraint $p_1 \le \bar{p}_1$ is satisfied. The concavity of $s_i p_i(s_i, \xi)$ $(i = 1, 2)$ also implies that when the markdown constraint is binding, $p_1^* = p_2^*$. This proves (I).

When $F_c > \hat{F}$ and $r_1^* > 0$, $q^* = r_1^* + s_1^* + s_2^*$ follows from $r_2^* = 0$ and $q^* = F_c/(c^* - v)$ follows from the definition of c. Given $(p_1^*, s_1^*, p_2^*, s_2^*)$, the optimal production quantity $(\geq s_1^* + s_2^*)$ maximizes

$$-(1-\tau)v_c q + \tau h(p_1^*)(q - s_1^* - s_2^*). \tag{1}$$

Recall that $h(p_1) = \min\{ac, b(p_1 - c)\}$. Notice that $b(p_1 - c) = b\left(p_1 - v_c - \frac{F_c}{q}\right)$ is increasing in q and $ac = a\left(v_c + \frac{F_c}{q}\right)$ is decreasing in q. Thus, we have $b(p_1 - c) \leq ac$ if $q \in \left(0, \frac{(a+b)F_c}{bp_1 - (a+b)v_c}\right]$ and $b(p_1 - c) > ac$ if $q > \frac{(a+b)F_c}{bp_1 - (a+b)v_c}$ when $bp_1 > (a+b)v_c$. Otherwise, $b(p_1 - c) \leq ac$ for all $q > 0$.

When $h(p_1^*) = b(p_1^* - c)$, objective function (1) becomes

$$-(1-\tau)v_c q + \tau b(p_1^* q - v_c q - F_c) - \tau b\left(p_1^* - v_c - \frac{F_c}{q}\right)(s_1^* + s_2^*), \tag{2}$$

which is convex in q. When $h(p_1^*) = ac^*$, objective function (1) becomes

$$-(1-\tau)v_c q + \tau a(v_c q + F_c) - \tau a\left(v_c + \frac{F_c}{q}\right)(s_1^* + s_2^*), \tag{3}$$

which is concave in q.

Therefore, if $p_1^* \leq \frac{a+b}{b} v_c$, objective function (1) takes the form (2) for $q(\geq s_1^* + s_2^*)$. We show that, in this case, $q^* = s_1^* + s_2^*$ and $\hat{F} = \infty$. Due to the convexity, it suffices to compare the values at $q = s_1^* + s_2^*$ and $q \uparrow \infty$. Notice that as $-(1-\tau)v_c + \tau b(p_1^* - v_c) \leq -(1-\tau)v_c + \tau a v_c < 0$ because $a < \frac{1-\tau}{\tau}$, the objective value approaches negative infinity as $q \uparrow \infty$. Therefore, $q^* = s_1^* + s_2^*$ and $r_1^* = 0$, and we reach a contradiction.

Now consider $p_1^* > \frac{a+b}{b} v_c$. The objective function is convex in q when q is small and concave in q when q is large. Thus, q^* equals to either $\sqrt{\frac{\tau a F_c (s_1^* + s_2^*)}{(1-\tau-\tau a)v_c}}$, the maximizer for (3), or $\frac{(a+b)F_c}{bp_1^* - (a+b)v_c}$, the threshold at which the objective function switches from convex to concave if the maximizer is less than the threshold. As a result, $q^* = \max\{\sqrt{\frac{\tau a F_c (s_1^* + s_2^*)}{(1-\tau-\tau a)v_c}}, \frac{(a+b)F_c}{bp_1^* - (a+b)v_c}\}$ and $h(p_1^*) = ac^* \leq b(p_1^* - c^*)$, which holds if and only if $p_1^* \geq \frac{a+b}{b} c^*$. Furthermore, it must be the case that $\tau h(p_1^*) > (1-\tau)v_c \Leftrightarrow \frac{\tau}{1-\tau} ac^* > v_c$; otherwise, the firm would prefer to maintain the same prices and reduce the production quantity to reduce the donation quantity r_1^* to 0. Note that $\frac{\tau}{1-\tau} ac^* > v_c \Leftrightarrow \frac{F_c}{F_c + v_c q^*} = \frac{c^* - v_c}{c^*} > \frac{1-\tau-\tau a}{1-\tau}$.

Because $q^* = \frac{F_c}{c^* - v_c}$, $q^* = \frac{(a+b)F_c}{bp_1^* - (a+b)v_c}$ if and only if $p_1^* = \frac{a+b}{b} c^*$. When $p_1^* > \frac{a+b}{b} c^*$, $q^* = \sqrt{\frac{\tau a F_c (s_1^* + s_2^*)}{(1-\tau-\tau a)v_c}} > \frac{(a+b)F_c}{bp_1^* - (a+b)v_c}$. The firm's after-tax profit given q^* can be written as

$$\begin{aligned} \max_{s_1, s_2, p_2} \quad & (1-\tau)[s_1 p_1(s_1, \xi) + s_2 p_2 - (F_c + v_c q)] + \tau a c^* (q^* - s_1 - s_2), \\ \text{s.t.} \quad & p_2 \leq p_1(s_1, \xi) \leq \bar{p}_1, p_2 \leq p_2(s_2, \xi) \ . \end{aligned}$$

When the markdown constraint is not binding, $p_2 = p_2(s_2, \xi)$ and we can convert the formulation into a convex optimization with a single constraint $p_1 \leq \bar{p}_1$. Ignoring the constraint, the first-order conditions reveal that $\frac{\partial(s_1 p_1(s_1, \xi))}{\partial s_1}|_{s_1 = s_1^*} = \frac{\partial(s_2 p_2(s_2, \xi))}{\partial s_2}|_{s_2 = s_2^*} = \frac{\tau}{1-\tau} ac^*$. Therefore, the constraint $p_1 \leq \bar{p}_1$ would be binding if and only if $\frac{\tau}{1-\tau} ac^* > \bar{p}_1$, under which case $p_1^* = \bar{p}_1$. Nevertheless, $\frac{\tau}{1-\tau} ac^* < c^* \leq \frac{a+b}{b} c^* < p_1^* \leq \bar{p}_1$. Thus, the constraint $p_1 \leq \bar{p}_1$ is always satisfied at the solution

$\frac{\partial(s_1p_1(s_1,\xi))}{\partial s_1}|_{s_1=s_1^*} = \frac{\partial(s_2p_2(s_2,\xi))}{\partial s_2}|_{s_2=s_2^*} = \frac{\tau}{1-\tau}ac^*$ when $p_1^* > \frac{a+b}{b}c^*$. The concavity of $s_ip_i(s_i,\xi)$ $(i=1,2)$ also implies that when the markdown constraint is binding, $p_1^*=p_2^*$. This proves (II)(i).

When $p_1^* = \frac{a+b}{b}c^* \le \bar{p}_1$, the firm's after-tax profit given q^* can be written as

$$\max_{s_1,s_2,p_2} \quad (1-\tau)[s_1p_1(s_1,\xi)+s_2p_2-(F_c+v_cq)]+\tau ac^*(q^*-s_1-s_2),$$
$$\text{s.t.} \quad p_1(s_1,\xi) = \frac{a+b}{b}c^*, p_2 \le p_1(s_1,\xi), p_2 \le p_2(s_2,\xi)\ .$$

When the markdown constraint is not binding, $p_2 = p_2(s_2,\xi)$ and we can convert the formulation into an unconstrained convex optimization. The first-order conditions reveal that $\frac{\partial(s_2p_2(s_2,\xi))}{\partial s_2}|_{s_2=s_2^*} = \frac{\tau}{1-\tau}ac^*$. The concavity of $s_2p_2(s_2,\xi)$ also implies that when the markdown constraint is binding, $p_1^*=p_2^*$. This proves (II)(ii).

Consider the case $\bar{p}_1 > \frac{(1-\tau)(a+b)}{\tau ab}v_c$. Notice that both s_1^* and s_2^* are bounded ($\frac{\partial(s_1p_1(s_1,\xi))}{\partial s_1}|_{s_1=s_1^*} \ge v_c$ and $\frac{\partial(s_2p_2(s_2,\xi))}{\partial s_2}|_{s_2=s_2^*} \ge v_c$). When F_c is large, consider the strategy under which $p_1=\bar{p}_1$ and $q = \frac{F_c}{\frac{b}{a+b}p_1 - v_c}$. As $F_c \uparrow \infty$, objective function (3) becomes $\frac{\tau a\frac{b}{a+b}p_1-(1-\tau)v_c}{\frac{b}{a+b}p_1-v_c}F_c - \tau a\frac{b}{a+b}p_1(s_1^*+s_2^*)$, which approaches positive infinity. This implies that when F_c is large, the pre-committed donation is part of the optimal strategy. That is, $\hat{F} < \infty$ when $\bar{p}_1 > \frac{(1-\tau)a+b}{\tau ab}v_c$.

Now consider the case $\bar{p}_1 \le \frac{(1-\tau)(a+b)}{\tau ab}v_c$. If $\hat{F}<\infty$, when $F_c > \hat{F}$, it must be the case that $p_1^* > \frac{a+b}{b}v_c$ and $q^* = \max\{\sqrt{\frac{\tau aF_c(s_1^*+s_2^*)}{(1-\tau-\tau a)v_c}}, \frac{(a+b)F_c}{bp_1^*-(a+b)v_c}\}$. Notice that both $s_1^*+s_2^*$ and $bp_1^*-(a+b)v_c \le b\bar{p}_1 - (a+b)v_c$ are bounded from above; therefore, when F_c is sufficiently large, it must be the case that $q^* = \frac{(a+b)F_c}{bp_1^*-(a+b)v_c}$. As $F_c \uparrow \infty$, objective function (3) becomes $\frac{\tau a\frac{b}{a+b}p_1^*-(1-\tau)v_c}{\frac{b}{a+b}p_1^*-v_c}F_c - \tau a\frac{b}{a+b}p_1^*(s_1^*+s_2^*)$, while objective function (1) becomes $-(1-\tau)v_c(s_1^*+s_2^*)$ at $q=s_1^*+s_2^*$. Notice that the difference is $\left(\tau a\frac{b}{a+b}p_1^* - (1-\tau)v_c\right)\left(\frac{F_c}{\frac{b}{a+b}p_1^*-v_c} - (s_1^*+s_2^*)\right)$, which is non-positive when F_c is sufficiently large due to $p_1^* \le \bar{p}_1 \le \frac{(1-\tau)(a+b)}{\tau ab}v_c$. Therefore, we reach a contradiction and $\hat{F}=\infty$. □

Proof of Theorem 4.7 Suppose $r_1^*>0$ when $F_c=0$. We show that $(q^*-r_1^*, p_1^*, 0)$ is also an optimal solution for period 1. Note that if $F_c=0$, then we have $h(p_1) = \min\{ac, b(p_1-v_c)\}$ and $c=v_c$. Therefore,

$$\Pi(q^*-r_1^*,p_1^*,0) - \Pi(q^*,p_1^*,r_1^*) = (1-\tau)v_cr_1^* - \tau\min\{av_c, b(p_1^*-v_c)\}r_1^* \ge ((1-\tau)-\tau a)v_cr_1^* \ge 0,$$

where the last inequality follows from $a \le \frac{1-\tau}{\tau}$. Thus, $(q^*-r_1^*,p_1^*,0)$ is optimal. By the tie-breaking rule, we reach a contradiction. Therefore, $r_1^*=0$ if $F_c=0$. □

Proof of Theorem 4.8: Recall that $I^* = q^*-r_1^*-s_1^* = \max\{0, q^*-r_1^*-D_1(p_1^*,\xi)\}$. Because $D_1(p_1^*,\xi)$ is (weakly) increasing in ξ, I^* is (weakly) decreasing in ξ.

Notice that $D_2(p_1^*,\xi)$ is increasing in ξ and $D_2(p_1^*,\xi) > I^* \Leftrightarrow p_1^* < p_2(I^*,\xi)$. If $\{\xi|D_2(p_1^*,\xi)>I^*\} = \emptyset$, set $\breve{\xi} = \overline{\xi}$. Otherwise, set $\breve{\xi} = \inf\{\xi|D_2(p_1^*,\xi) > I^*\}$. When $\breve{\xi} < \xi \le \overline{\xi}$, it is optimal to sell the remaining inventory I^* at price p_1^* due to $p_2 \le p_1^*$, which is greater than c (otherwise, the firm would not make a positive profit). That is, $r_2^*=0$, $r_2^*=0$, $s_2^*=I^*$, and $p_2^*=p_1^*<p_2(I^*,\xi)$. Furthermore,

because $p_2(I^*,\xi)$ is decreasing in I^* and (weakly) increasing in ξ, and I^* is (weakly) decreasing in ξ, $p_2(I^*,\xi)$ is (weakly) increasing in ξ. This proves (i) in (II).

Now we consider the case that $\xi \le \breve{\xi}$, under which $p_2(I^*,\xi) \le p_1^*$.

To establish the threshold $\hat{\xi}$, we consider two demand states $\xi' < \xi''$. Let the optimal price, donation quantity, and sales in the second period be (p_2', r_2', s_2') and (p_2'', r_2'', s_2'') for these two states, respectively. Let the available inventory at the beginning of the second period be I' and I'' for these two states, respectively.

We first show that if $r_2' = 0$ and $s_2' = I'$, salvaging excess inventory (i.e., $r_2'' = 0$ and $s_2'' < I''$) is suboptimal at ξ''. If salvaging excess inventory is optimal, then it must be the case that $p_2 \le c < p_1^*$. The optimal solution under both ξ' and ξ'' can thus be characterized by the following formulation:

$$s_2 = \arg\max_{s \le I^*} sp_2(s,\xi).$$

Consider the relaxed formulation

$$s_2 = \arg\max_{s} sp_2(s,\xi).$$

The objective function is concave with respect to s. Let $\dot{s}_2'$ ($\dot{s}_2''$) denote the unique optimal solution for ξ' (ξ''). We show that $\dot{s}_2' \le \dot{s}_2''$.

Notice that $\frac{\partial^2(sp_2(s,\xi))}{\partial s \partial \xi} = \frac{\partial p_2(s,\xi)}{\partial \xi} + s\frac{\partial^2 p_2(s,\xi)}{\partial s \partial \xi} > 0$ because $\frac{\partial p_2(s,\xi)}{\partial \xi} \ge 0$ and $s\frac{\partial^2 p_2(s,\xi)}{\partial s \partial \xi} > 0$.

The optimality of $\dot{s}_2'$ implies that for any $\dot{s} < \dot{s}_2'$, $\dot{s}p_2(\dot{s},\xi') < \dot{s}_2' p_2(\dot{s}_2',\xi')$. Therefore, $\dot{s}_2' p_2(\dot{s}_2',\xi'') - \dot{s}p_2(\dot{s},\xi'') = \dot{s}_2' p_2(\dot{s}_2',\xi') - \dot{s}p_2(\dot{s},\xi') + \int_{\xi'}^{\xi''}\int_{\dot{s}}^{\dot{s}_2'} \frac{\partial^2(sp_2(s,\xi))}{\partial s \partial \xi} ds d\xi > 0$. Thus, $\dot{s}_2'' \ge \dot{s}_2'$.

Because $sp_2(s,\xi)$ is a concave function and $s_2'' < I''$ is the optimal solution to the original problem, it must be the case that $\dot{s}_2'' = s_2''$. Therefore, $\dot{s}_2' \le \dot{s}_2'' = s_2'' < I'' \le I'$ for $\xi' < \xi''$ so that the relaxed constraint is satisfied and $\dot{s}_2'(< I')$ is the unique optimal solution for ξ in the original formulation. Thus, we reach a contradiction. Therefore, if $r_2' = 0$ and $s_2' = I'$, salvaging excess inventory (i.e., $r_2'' = 0$ and $s_2'' < I''$) is suboptimal at ξ''.

We now show that if $r_2' = 0$ and $s_2' = I'$, donating excess inventory (i.e., $r_2'' = I'' - s_2''$, $s_2'' < I''$, and $p_2^* = p_2(s_2'',\xi'') \le p_1^*$) is suboptimal at ξ''. If donating excess inventory is optimal, the following formulation provides the optimal solution under both ξ' and ξ'':

$$s_2 = \arg\max_{s: s \le I^*, p_2(s,\xi) \le p_1^*} (1-\tau)sp_2(s,\xi) + \tau h(p_2(s,\xi))(I^* - s).$$

Now we consider the relaxed formulation

$$s_2 = \arg\max_{s} \ (1-\tau)sp_2(s,\xi) + \tau h(p_2(s,\xi))(I^* - s).$$

Recall that we assume either $p_2(s,\xi)$ is concave in s or $h(p_2(s,\xi)) = ac$. Therefore, the objective function $(1-\tau)sp_2(s,\xi) + \tau h(p_2(s,\xi))(I^* - s)$ is concave in s. Let $\ddot{s}_2'$ ($\ddot{s}_2''$) denote the unique optimal solution for ξ' (ξ''). We show that $\ddot{s}_2' \le \ddot{s}_2''$.

We have shown that $\frac{\partial^2 p_2(s,\xi)}{\partial s \partial \xi} > 0$ implies that $\frac{\partial^2 (s p_2(s,\xi))}{\partial s \partial \xi} > 0$. Recall that I^* is (weakly) decreasing in ξ and $p_2(s,\xi)$ is decreasing in s, so we have $\frac{\partial^2(((1-\tau)sp_2(s,\xi)+\tau ac(I^*-s))}{\partial s \partial \xi} = (1-\tau)\frac{\partial^2 (sp_2(s,\xi))}{\partial s \partial \xi} > 0$ and $\frac{\partial^2((1-\tau-\tau b)sp_2(s,\xi)+\tau b p_2(s,\xi) I^* - \tau bc(I^*-s))}{\partial s \partial \xi} = (1-\tau-\tau b)\frac{\partial^2 (sp_2(s,\xi))}{\partial s \partial \xi} + \tau b I^* \frac{\partial^2 p_2(s,\xi)}{\partial s \partial \xi} + \tau b \frac{dI^*}{d\xi}\frac{\partial p_2(s,\xi)}{\partial s} > 0$.

Using the same argument for $\dot{s}_2'' \geq \dot{s}_2'$, the optimal $\ddot{s}_2'' \geq \ddot{s}_2'$. Because the objective function is concave with respect to s. The optimal solutions to

$$s_2 = \underset{s: p_2(s,\xi) \leq p_1^*}{\arg\max}\ (1-\tau)sp_2(s,\xi) + \tau h(p_2(s,\xi))(I^* - s)$$

are $\max\{\dot{s}_2', d_2(p_1^*, \xi')\}$ and $\max\{\dot{s}_2'', d_2(p_1^*, \xi'')\}$ for ξ' and ξ'', respectively.

Because the objective is concave and $s_2'' < I''$ is the optimal solution to the original problem, it must be the case that $\max\{\dot{s}_2'', d_2(p_1^*, \xi'')\} = s_2''$. Because $d_2(p_1^*, \xi)$ is weakly increasing in ξ, $\max\{\dot{s}_2', d_2(p_1^*, \xi')\} \leq \max\{\dot{s}_2'', d_2(p_1^*, \xi'')\} = s_2'' < I'' \leq I'$ for $\xi' < \xi''$ so that the relaxed constraint is satisfied and $\max\{\dot{s}_2', d_2(p_1^*, \xi')\} (< I')$ is the unique optimal solution for ξ in the original formulation. Thus, we reach a contradiction. Therefore, if $r_2' = 0$ and $s_2' = I'$, donating excess inventory (i.e., $r_2'' = I'' - s_2''$, $s_2'' < I''$, and $p_2^* = p_2(s_2'', \xi'') \leq p_1^*$) is suboptimal at ξ''.

Therefore, if $r_2' = 0$ and $s_2' = I'$ under ξ', $r_2'' = 0$ and $s_2'' = I''$ for all $\xi'' \in (\xi', \breve{\xi}]$. Furthermore, the above analysis shows that within each of the three options, the firm's problem is concave and has a unique solution. Thus, the firm's optimal decision is continuous under each option. Given the tie-breaking rule, the firm's optimal decision is piecewise continuous with respect to ξ.

Now we show that at $\xi = \breve{\xi}$, $r_2^* = 0$ and $s_2^* = I^*$. Suppose that $\breve{\xi} < \bar{\xi}$. By the definition of $\breve{\xi}$ and continuity, $r_2^* = 0$, $s_2^* = I^*$, and $p_2^* = p_2(I^*, \xi) = p_1^*$ is optimal. Suppose that $\breve{\xi} = \bar{\xi}$, if $s_2^* < I^*$ at $\xi = \breve{\xi}$, $s_2^* < I^*$ for all $\xi \in [\underline{\xi}, \bar{\xi}]$. Let $\Delta = \inf\{I^* - s_2^* | \xi \in [\underline{\xi}, \bar{\xi}\}$. We have $\Delta > 0$ because $I^* - s_2^* > 0$ and $I^* - s_2^*$ is piecewise continuous with respect to ξ on the compact set. By increasing r_1^* by Δ and reducing either the donation quantity or the salvaging quantity by Δ, we (weakly) increase the firm's profit because the first period price is (weakly) higher than the second period price. Based on the tie-breaking, we reach a contradiction. Therefore, at $\xi = \breve{\xi}$, $r_2^* = 0$ and $s_2^* = I^*$. This proves (ii).

Define $\hat{\xi} = \inf\{\xi | s_2^* = I^*\}$. By continuity, at $\xi = \hat{\xi}$, $r_2^* = 0$ and $s_2^* = I^*$.

When $\xi < \hat{\xi}$, $s_2^* < I^*$ and the firm either salvages or donates the excess inventory. When salvaging is optimal, it must be the case that $p_2^* \leq c$; when donating is optimal, it must be the case that $p_2^* > c$. We have shown that when treating the problem as optimizing over the sales quantity, the problem is concave under either salvaging or donation. This implies that when treating the problem as optimization over the price, the problem is unimodal under either salvaging or donation. Therefore, $p_2^* = \arg\max_p (pD_2(p,\xi))$ under salvaging and $p_2^* = \min\{p_1^*, \arg\max_p ((1-\tau)pD_2(p,\xi) + \tau h(p)(I^* - D_2(p,\xi)))\}$ under donation. This proves (iii) and (iv). □

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Online Appendix

The online appendix contains proofs of the theorems in the paper.

Proofs for the Remaining Theoretical Results in Section 4

Proof of Theorem 4.2: Consider $\tau_1 < \tau_2$. For $i = 1, 2$, let $((p_t^i), q^i, (r_t^i)|\xi)$ be the firm's optimal price and quantity decisions at tax rate τ_i, and let $((s_t^i)|\xi)$ be the resulting sales quantity. Given the optimality of these decisions,

$$\begin{aligned} & \mathbb{E}_\xi \left[(1-\tau_1)(p_1^1 s_1^1 + p_2^1 s_2^1) + \tau_1(h(p_1^1) r_1^1 + h(p_2^1) r_2^1) - (1-\tau_1) c q^1 \mid \xi\right] \\ \geq\ & \mathbb{E}_\xi \left[(1-\tau_1)(p_1^2 s_1^2 + p_2^2 s_2^2) + \tau_1(h(p_1^2) r_1^2 + h(p_2^2) r_2^2) - (1-\tau_1) c q^2 \mid \xi\right] \end{aligned}$$

$$\begin{aligned} & \mathbb{E}_\xi \left[(1-\tau_2)(p_1^2 s_1^2 + p_2^2 s_2^2) + \tau_2(h(p_1^2) r_1^2 + h(p_2^2) r_2^2) - (1-\tau_1) c q^2 \mid \xi\right] \\ \geq\ & \mathbb{E}_\xi \left[(1-\tau_2)(p_1^1 s_1^1 + p_2^1 s_2^1) + \tau_2(h(p_1^1) r_1^1 + h(p_2^1) r_2^1) - (1-\tau_1) c q^1 \mid \xi\right] \end{aligned}$$

Summing up $(1-\tau_2)$ times the first inequality and $(1-\tau_1)$ times the second inequality, we have

$$(\tau_1 - \tau_1\tau_2)\mathsf{EATD}(\tau_1) + (\tau_2 - \tau_1\tau_2)\mathsf{EATD}(\tau_2) \geq (\tau_2 - \tau_1\tau_2)\mathsf{EATD}(\tau_1) + (\tau_1 - \tau_1\tau_2)\mathsf{EATD}(\tau_2)\ ,$$

which implies that $\mathsf{EATD}(\tau_1) \leq \mathsf{EATD}(\tau_2)$. □

Proof of Theorem 4.3: We consider a specific strategy under which the firm procures $q = \frac{M/(1+a) - F_c}{v_c}$ units of the product and donates all of them in the regular period with sufficiently high regular price (i.e., $p_1 \geq \frac{a+b}{b} c = \frac{(a+b)M}{(1+a)bq}$).

Under this strategy, the firm incurs a cost of $F_c + v_c q = M/(1+a)$ and claims an enhanced deduction of $(c + h(p_1))q = (c + \min\{ac, b(p_1 - c)\})q = (1+a)cq = M$. Therefore, the after-tax profit $U^*(\tau) \geq \tau M - M/(1+a) = \left(\tau - \frac{1}{1+a}\right) M$. □

Proof of Theorem 4.5: $s_i D_2(s_i, \xi) = s_i(\mathsf{WTP} - s_i)$, and $\frac{\partial (s_i D_2(s_i,\xi))}{\partial s_i} = \mathsf{WTP} - 2s_i$ for $i = 1, 2$. When $F_c \leq \hat{F}_c$, Theorem 4.4(I) shows that $\frac{\partial s_i D_2(s_i,\xi)}{\partial s_i}|_{s_i = s_i^*} = v_c$ for $i = 1, 2$. Therefore, $p_1^* = p_2^* = \frac{\mathsf{WTP} + v_c}{2}$, $s_1^* = s_2^* = \frac{\mathsf{WTP} - v_c}{2}$, $r_1^* = r_2^* = 0$, $q^* = s_1^* + s_2^*$, and the firm's optimal after-tax profit U^* is $(1-\tau)\left(\frac{(\mathsf{WTP} - v_c)^2}{2} - F_c\right)$.

By definition, $\bar{p}_1 = \mathsf{WTP}$. When $\mathsf{WTP} < \frac{a+b}{b} v_c$, $\hat{F} = \breve{F} = \bar{F} = \infty$ by Theorem 4.4. Now we focus on the case $\mathsf{WTP} \geq \frac{a+b}{b} v_c$ and derive the optimal solution when $F_c > \hat{F}$. First, we relax the markdown constraint. It is readily to verify that all the solutions identified satisfy the markdown constraint and this relaxation can be applied without loss of generality.

We derive the optimal solution for a given q (i.e., equivalent to a given c). By Theorem 4.4, when $F_c > \hat{F}$, $p_1^* \geq \frac{a+b}{b} c^*$ and $h(p_1^*) = ac^*$, and the firm's problem can be rewritten as the following convex optimization:

$$\begin{aligned} \max_{s_1, s_2} \quad & (1-\tau)\left(s_1(\mathsf{WTP} - s_1) + s_2(\mathsf{WTP} - s_2) - \frac{cF_c}{c - v_c}\right) + \tau ac\left(\frac{F_c}{c - v_c} - s_1 - s_2\right), \\ \text{s.t.} \quad & \frac{a+b}{b} c \leq p_1(s_1, \xi) \leq \bar{p}_1\ . \end{aligned}$$

Notice that $\frac{a+b}{b} \geq 1 > \frac{\tau}{1-\tau} a$, thus $\frac{2(a+b)}{b} - \frac{\tau}{1-\tau} a > \frac{a+b}{b}$. Denote $\check{c} \equiv \max\left\{v_c, \frac{\mathsf{WTP}}{\frac{2(a+b)}{b} - \frac{\tau a}{1-\tau}}\right\}$. We consider the potential c value when it belongs to $(v_c, \check{c}]$, $\left(\check{c}, \frac{b\mathsf{WTP}}{a+b}\right)$, or equals to $\frac{b\mathsf{WTP}}{a+b}$.

When $c \in (v_c, \check{c}]$, the solution to the relaxed problem is $s_1 = s_2 = \frac{\mathsf{WTP} - \frac{\tau}{1-\tau} ac}{2}$, and $p_1 = p_2 = \frac{\mathsf{WTP} + \frac{\tau}{1-\tau} ac}{2} \geq \frac{a+b}{b} c$. Therefore, both $\frac{a+b}{b} c \leq p_1(s_1, \xi) \leq \bar{p}_1$ and the markdown constraint $p_1 \geq p_2$ are satisfied. The firm's after-tax profit is $(1-\tau)\frac{\left(\mathsf{WTP} - \frac{\tau}{1-\tau} ac\right)^2}{2} - (1 - \tau - \tau a) F_c \frac{c}{c - v_c}$.

When $c \in \left(\breve{c}, \frac{b\mathsf{WTP}}{a+b}\right]$, the solution to the relaxed problem violates the constraint $\frac{a+b}{b}c \le p_1(s_1,\xi)$. Therefore, at optimal, $p_1 = \frac{a+b}{b}c$, $s_1 = \mathsf{WTP} - \frac{a+b}{b}c$, $p_2 = \frac{\mathsf{WTP}+\frac{\tau}{1-\tau}ac}{2}$, and $s_2 = \frac{\mathsf{WTP}-\frac{\tau}{1-\tau}ac}{2}$. It is easy to verify that both $p_1(s_1,\xi) \le \bar{p}_1$ and the markdown constraint $p_1 \ge p_2$ are satisfied. The firm's after-tax profit is $\left((1-\tau)\frac{a+b}{b} - \tau a\right)c\left(\mathsf{WTP} - \frac{a+b}{b}c\right) + (1-\tau)\frac{\left(\mathsf{WTP}-\frac{\tau}{1-\tau}ac\right)^2}{4} - (1-\tau-\tau a)F_c\frac{c}{c-v_c}$.

Therefore, to find the optimal solution for a given F_c, we can search for the optimal c. When $F_c > \hat{F}$, Theorem 4.4 shows that $r_1^* > 0$. As a result, when the optimal $c \in \left(v_c, \frac{b\mathsf{WTP}}{a+b}\right)$, we can find the optimal solution using the first-order condition over the profit function.

When $c \in (v_c, \breve{c}]$, $-\tau a\left(\mathsf{WTP} - \frac{\tau}{1-\tau}ac\right) + \frac{(1-\tau-\tau a)F_c v_c}{(c-v_c)^2} = 0$ by the first-order condition (that is, $F_c = \frac{\tau a\left(\mathsf{WTP}-\frac{\tau}{1-\tau}ac\right)(c-v_c)^2}{(1-\tau-\tau a)v_c}$). Denote $F_1(x) \equiv \frac{\tau a\left(\mathsf{WTP}-\frac{\tau}{1-\tau}ax\right)(x-v_c)^2}{(1-\tau-\tau a)v_c}$. Now we show that $F_1(x)$ is a monotone increasing function on $(v_c, \breve{c}]$. Therefore, when $F_c \in (0, F_1(\breve{c})]$, there is a unique solution c^* to $F_c = F_1(c)$ on $(v_c, \breve{c}]$.

When $c > v_c$, the derivative $F_1'(c)$ has the same sign as $2\left(\mathsf{WTP} - \frac{\tau}{1-\tau}ac\right) - \frac{\tau}{1-\tau}a(c-v_c)$. The term would be non-negative if and only if $c \le \frac{2\mathsf{WTP}}{3\frac{\tau}{1-\tau}a} + \frac{1}{3}v_c$. It suffices to show that $\frac{\mathsf{WTP}}{\frac{2(a+b)}{b}-\frac{\tau a}{1-\tau}} \le \frac{2\mathsf{WTP}}{3\frac{\tau}{1-\tau}a} < \frac{2\mathsf{WTP}}{3\frac{\tau}{1-\tau}a} + \frac{1}{3}v_c$. Notice that $\frac{2\mathsf{WTP}}{3\frac{\tau}{1-\tau}a} \ge \frac{\mathsf{WTP}}{\frac{2(a+b)}{b}-\frac{\tau a}{1-\tau}} \Leftrightarrow \frac{4(a+b)}{b} - \frac{2\tau a}{1-\tau} \ge \frac{3\tau a}{1-\tau} \Leftarrow 4 + 4\frac{\tau}{1-\tau}a \ge 5\frac{\tau}{1-\tau}a$, which is true.

It is easy to verify that the solution c^* to $F_c = F_1(c)$ on $(v_c, \breve{c}]$ is a maximizer for the original profit maximization problem by the second-order condition. Furthermore, when $c \in (\breve{c}, \infty)$ solves $F_c = F_1(c)$ for $F_c \le (\breve{c})$, c is a minimizer.

When $c \in \left(\breve{c}, \frac{b\mathsf{WTP}}{a+b}\right)$, $\left((1-\tau)\frac{a+b}{b} - \tau a\right)\left(\mathsf{WTP} - \frac{2(a+b)}{b}c\right) - \frac{1}{2}\tau a\left(\mathsf{WTP} - \frac{\tau}{1-\tau}ac\right) + \frac{(1-\tau-\tau a)F_c v_c}{(c-v_c)^2} = 0$ by the first-order condition (that is, $F_c = \frac{\left(\frac{1}{2}\tau a\left(\mathsf{WTP}-\frac{\tau}{1-\tau}ac\right) - \left((1-\tau)\frac{a+b}{b}-\tau a\right)\left(\mathsf{WTP}-\frac{2(a+b)}{b}c\right)\right)(c-v_c)^2}{(1-\tau-\tau a)v_c}$). Denote $F_2(x) \equiv \frac{\left(\frac{1}{2}\tau a\left(\mathsf{WTP}-\frac{\tau}{1-\tau}ax\right) - \left((1-\tau)\frac{a+b}{b}-\tau a\right)\left(\mathsf{WTP}-\frac{2(a+b)}{b}x\right)\right)(x-v_c)^2}{(1-\tau-\tau a)v_c}$. It is easy to verify that $F_1(\breve{c}) = F_2(\breve{c})$. Now we show that $F_2(x)$ is a monotone increasing function on $(\breve{c}, \infty)$. Therefore, when $F_c \in \left(F_2(\breve{c}), F_2\left(\frac{b\mathsf{WTP}}{a+b}\right)\right)$, there is a unique solution c^* to $F_c = F_2(c)$ on $\left(\breve{c}, \frac{b\mathsf{WTP}}{a+b}\right)$.

When $c > \breve{c} \ge v_c$, the derivative $F_2'(c)$ has the same sign as $\left(-\frac{1}{2}\tau a\frac{\tau a}{1-\tau} + \left((1-\tau)\frac{a+b}{b} - \tau a\right)\frac{2(a+b)}{b}\right)(c-v_c) + \left(\tau a\left(\mathsf{WTP} - \frac{\tau}{1-\tau}ac\right) - 2\left((1-\tau)\frac{a+b}{b} - \tau a\right)\left(\mathsf{WTP} - \frac{2(a+b)}{b}c\right)\right) = \left(-\frac{1}{2}\tau a\frac{\tau a}{1-\tau} + \left((1-\tau)\frac{a+b}{b} - \tau a\right)\frac{2(a+b)}{b}\right)(3c - v_c) - \left(2(1-\tau)\frac{a+b}{b} - 3\tau a\right)\mathsf{WTP}$.

The term would be non-negative if and only if $c \ge \frac{\left(2(1-\tau)\frac{a+b}{b}-3\tau a\right)\mathsf{WTP}}{3\left(-\frac{1}{2}\tau a\frac{\tau}{1-\tau}a + \left((1-\tau)\frac{a+b}{b}-\tau a\right)\frac{2(a+b)}{b}\right)} + \frac{1}{3}v_c$. Notice that $\breve{c} \ge \frac{2}{3}\frac{\mathsf{WTP}}{\frac{2(a+b)}{b}-\frac{\tau a}{1-\tau}} + \frac{1}{3}v_c$ and $\frac{2}{3}\frac{\mathsf{WTP}}{\frac{2(a+b)}{b}-\frac{\tau a}{1-\tau}} + \frac{1}{3}v_c \ge \frac{\left(2(1-\tau)\frac{a+b}{b}-3\tau a\right)\mathsf{WTP}}{3\left(-\frac{1}{2}\tau a\frac{\tau}{1-\tau}a + \left((1-\tau)\frac{a+b}{b}-\tau a\right)\frac{2(a+b)}{b}\right)} + \frac{1}{3}v_c \Leftrightarrow \frac{2}{\frac{2(a+b)}{b}-\frac{\tau a}{1-\tau}} \ge \frac{\left(2(1-\tau)\frac{a+b}{b}-3\tau a\right)}{\left(-\frac{1}{2}\tau a\frac{\tau}{1-\tau}a + \left((1-\tau)\frac{a+b}{b}-\tau a\right)\frac{2(a+b)}{b}\right)} \Leftrightarrow -\tau a\frac{\tau}{1-\tau}a + \left((1-\tau)\frac{a+b}{b} - \tau a\right)\frac{4(a+b)}{b} \ge (1-\tau)\frac{a+b}{b}\frac{4(a+b)}{b} - \tau a\frac{8(a+b)}{b} + 3\tau a\frac{\tau}{1-\tau}a \Leftrightarrow \frac{a+b}{b} \ge \frac{\tau}{1-\tau}a$, which is true. It is easy to verify that the solution c^* to $F_c = F_2(c)$ on $\left(\breve{c}, \frac{b\mathsf{WTP}}{a+b}\right)$ is a maximizer for the original profit maximization problem by the second-order condition.

The analysis of the first-order conditions further reveals that when $F_c > \hat{F}$, if $F_c \in (0, F_1(\breve{c})]$, it would be suboptimal to choose $c > \breve{c}$; if $F_c \in \left(F_1(\breve{c}), F_2\left(\frac{b\mathsf{WTP}}{a+b}\right)\right)$, it would be suboptimal to choose either $c = \frac{b\mathsf{WTP}}{a+b}$ or $c \in (v_c, \breve{c})$; and if $F_c \ge F_2\left(\frac{b\mathsf{WTP}}{a+b}\right)$, it would be suboptimal to choose $c \in \left(v_c, \max\left\{v_c, \frac{b\mathsf{WTP}}{a+b}\right\}\right)$. Therefore, when $F_c > \hat{F}$, if $F_c \in (0, F_1(\breve{c})]$, $c^* \in (v_c, \breve{c}]$, $p_1^* = p_2^* = \frac{\mathsf{WTP}+\frac{\tau}{1-\tau}ac^*}{2}$, $s_1^* = s_2^* = \frac{\mathsf{WTP}-\frac{\tau}{1-\tau}ac^*}{2}$, $q^* = \frac{F_c}{c-v_c}$, $r_1^* = q^* - s_1^* - s_2^*$, and $r_2^* = 0$, where c^* is the unique solution to $\tau a\left(\mathsf{WTP} - \frac{\tau}{1-\tau}ac\right)(c-v_c)^2 - F_c(1-\tau-\tau a)v_c = 0$ on $(v_c, \breve{c}]$; if $F_c \in \left(F_1(\breve{c}), F_2\left(\frac{b\mathsf{WTP}}{a+b}\right)\right)$, $c^* \in \left(\breve{c}, \frac{b\mathsf{WTP}}{a+b}\right)$, $p_1^* = \frac{a+b}{b}c^*$, $p_2^* = \frac{\mathsf{WTP}+\frac{\tau}{1-\tau}ac^*}{2}$, $s_1^* = \mathsf{WTP} - \frac{a+b}{b}c^*$, $s_2^* = \frac{\mathsf{WTP}-\frac{\tau}{1-\tau}ac^*}{2}$, $q^* = \frac{F_c}{c^*-v_c}$, $r_1^* = q^* - s_1^* - s_2^*$, and $r_2^* = 0$, where c^* is the unique solution to $\frac{1}{2}\tau a\left(\mathsf{WTP} - \frac{\tau}{1-\tau}ac\right)(c-v_c)^2 - (1-$

$\tau)\frac{a+b}{b}-\tau a)\left(\mathsf{WTP}-\frac{2(a+b)}{b}c\right)(c-v_c)^2-F_c(1-\tau-\tau a)v_c=0$ on $\left(\check{c},\frac{b\mathsf{WTP}}{a+b}\right)$; and if $F_c\geq F_2(\frac{b\mathsf{WTP}}{a+b})$, $c^*=\frac{b\mathsf{WTP}}{a+b}$, $p_1^*=\frac{a+b}{b}c^*$, $p_2^*=\frac{\mathsf{WTP}+\frac{\tau}{1-\tau}ac^*}{2}$, $s_1^*=0$, $s_2^*=\frac{\mathsf{WTP}-\frac{\tau}{1-\tau}ac^*}{2}$, $q^*=\frac{F_c}{c^*-v_c}$, $r_1^*=q^*-s_1^*-s_2^*$, and $r_2^*=0$.

Furthermore, the claim that $\hat{F}=\infty$ if and only if $\tau\leq\frac{(a+b)v_c}{(a+b)v_c+ab\mathsf{WTP}}$ directly follows Theorem 4.4 and that $\bar{p}_1=\mathsf{WTP}$ under the linear demand case. The monotone property analyzed above implies that when $F_c>\hat{F}$, c^* is a continuous (weakly) increasing function of F_c. Given that p_1^* and p_2^* are continuous (weakly) increasing functions of c^*, when $F_c>\hat{F}$, p_1^* and p_2^* are also continuous (weakly) increasing functions of F_c. □

Proof of Theorem 4.6: When $\xi=\underline{\xi}=\overline{\xi}$, $s_1^*=D_1(p_1^*,\xi)$, $r_2^*=0$, $s_2^*=q-r_1^*-s_1^*$, and $p_2^*=p_2(s_2^*,\xi)$ by Theorem 4.4; similarly, $s_1^0=D_1(p_1^0,\xi)$, $s_2^0=q-s_1^0$, and $p_2^0=p_2(s_2^0,\xi)$. Therefore,

$$(q^0,p_1^0)=\underset{\substack{q\geq D_1(p_1,\xi)\\ p_1\geq p_2(q-D_1(p_1,\xi),\xi)}}{\arg\max}\; p_1D_1(p_1,\xi)+p_2(q-D_1(p_1,\xi),\xi)(q-D_1(p_1,\xi))-(F_c+v_cq).$$

$$(q^*,p_1^*,r_1^*)=\underset{\substack{q\geq D_1(p_1,\xi)+r_1\\ p_1\geq p_2(q-r_1-D_1(p_1,\xi),\xi)\\ r_1\geq 0}}{\arg\max}\; p_1D_1(p_1,\xi)+p_2(q-r_1-D_1(p_1,\xi),\xi)(q-r_1-D_1(p_1,\xi))-(F_c+v_cq)+\frac{\tau}{1-\tau}h(p_1)r_1.$$

We will prove by contradiction. Suppose that $p_1^*<p_1^0$. Let

$$(q^\#,p_1^\#)=\underset{\substack{q\geq D_1(p_1,\xi)\\ p_1=p_1^*\geq p_2(q-D_1(p_1,\xi),\xi)}}{\arg\max}\; p_1D_1(p_1,\xi)+p_2(q-D_1(p_1,\xi),\xi)(q-D_1(p_1,\xi))-(F_c+v_cq).$$

That is, $(q^\#,p_1^\#)$ maximizes the profit without the enhanced tax deduction when the first period price $p_1^\#$ is fixed at p_1^*. Let $p_2^\#=p_2(q^\#-D_1(p_1^\#,\xi),\xi)$ and $U^\#$ and $(s_1^\#,s_2^\#)$ be the associated profit and sales quantities, respectively. Because the profit maximization problem without the enhanced tax deduction is concave and has a unique solution, $U^0>U^\#$, $p_2^\#\leq p_1^\#=p_1^*<p_1^0$, and $s_1^0=D_1(p_1^0,\xi)<D_1(p_1^*,\xi)=s_1^*=s_1^\#$.

We consider the two possibilities:

$p_2^\#<p_1^\#$: In this case, by Theorem 4.4, $s_2^0=s_2^\#=\arg\max_s(sp_2(s,\xi)-v_cs)$ as the constraints are not binding; thus, $p_2^\#=p_2^0$. Because $U^0>U^\#$, $p_1^0s_1^0-v_cs_1^0>p_1^\#s_1^\#-v_cs_1^\#=p_1^*s_1^*-v_cs_1^*$.

Now we show that (q^*,p_1^0,p_2^*) is feasible and provides a profit higher than U^* with the enhanced tax deduction. (q^*,p_1^0,p_2^*) is feasible because $p_1^0>p_1^*\geq p_2^*$. Under (q^*,p_1^0,p_2^*), the donation quantity in the first period is $q^*-s_1^0-s_2^*=r_1^*+s_1^*-s_1^0>r_1^*$. Furthermore, $p_1^*<p_1^0$, implies that $h(p_1^0)r_1^*\geq h(p_1^*)r_1^*$ and Theorem 4.4 shows that $\frac{\tau}{1-\tau}h(p_1^0)\geq\frac{\tau}{1-\tau}h(p_1^*)>v_c$. Together with $p_1^0s_1^0-v_cs_1^0>p_1^*s_1^*-v_cs_1^*$, the profit under (q^*,p_1^0,p_2^*) is greater than U^*. Thus, we reach a contradiction.

$p_2^\#=p_1^\#$: In this case, by Theorem 4.4, $s_2^\#=D_2(p_1^\#,\xi)=\underset{s\geq D_2(p_1^\#,\xi)}{\arg\max}(sp_2(s,\xi)-v_cs)$ as the constraints is binding. Recall that $(sp_2(s,\xi)-v_cs)$ is concave and $p_1^*=p_1^\#$. Because $D_2(p_1^\#,\xi)=D_2(p_1^*,\xi)$ and $\frac{\tau}{1-\tau}h(p_1^*)>v_c$ by Theorem 4.4, $s_2^*=\underset{s\geq D_2(p_1^*,\xi)}{\arg\max}\left(sp_2(s,\xi)-\frac{\tau}{1-\tau}h(p_1^*)s\right)=s_2^\#$ as the constraint must also be binding. Furthermore, $p_1^0>p_1^*=p_1^\#\Leftrightarrow D_2(p_1^\#,\xi)>D_2(p_1^0,\xi)$, $s_2^0=\underset{s\geq D_2(p_1^0,\xi)}{\arg\max}(sp_2(s,\xi)-v_cs)\leq s_2^\#=s_2^*$ and $p_2^*=p_2^\#\geq p_2^0$. Because $U^0>U^\#$, $p_1^0s_1^0+p_2^0s_2^0-v_c(s_1^0+s_2^0)>p_1^\#s_1^\#+p_2^\#s_2^\#-v_c(s_1^\#+s_2^\#)=p_1^*s_1^*+p_2^*s_2^*-v_c(s_1^*+s_2^*)$.

Now we show that (q^*,p_1^0,p_2^0) is feasible and provides a profit higher than U^* with the enhanced tax deduction. (q^*,p_1^0,p_2^0) is feasible because $p_1^0>p_2^0$. Under (q^*,p_1^0,p_2^0), the donation quantity in the first period is $q^*-s_1^0-s_2^0=r_1^*+s_1^*+s_2^*-(s_1^0+s_2^0)>r_1^*$. The profit under (q^*,p_1^0,p_2^0) is higher than U^* because $p_1^0s_1^0+p_2^0s_2^0-v_c(s_1^0+s_2^0)>p_1^*s_1^*+p_2^*s_2^*-v_c(s_1^*+s_2^*)$, $h(p_1^0)r_1^*\geq h(p_1^*)r_1^*$ due to $p_1^*<p_1^0$, and $\frac{\tau}{1-\tau}h(p_1^0)\geq\frac{\tau}{1-\tau}h(p_1^*)>v_c$ by Theorem 4.4. Thus, we reach a contradiction. □

Proof of Theorem 4.9: Statements I) and II) parts (i) and (ii) directly follow from Theorem 4.8.

When it is optimal for the firm to choose salvaging, the optimal sales quantity $s_2 = \arg\max sp_2(s,\xi)$, which is a concave function and has a unique solution. When either $D_2(p,\xi) = D(p)$ or $D_2(p,\xi) = \xi D(p)$, the optimal price corresponds to $\hat{p}$, the unique price that maximizes $pD(p)$. $\hat{p} \le c$ follows from Theorem 4.8. This proves the first part of the theorem.

To prove the last part, we consider two demand states $\xi' < \xi''$. Let the optimal price, donation quantity, and sales in the second period be (p_2', r_2', s_2') and (p_2'', r_2'', s_2'') for these two states, respectively. Let the available inventory at beginning of second period be I' and I'' for these two states, respectively.

Suppose that $\xi' < \xi''$ and $r_2'' > 0$, we show that $r_2' > 0$ as well. Because $r_2'' > 0$, donating is preferred over salvage at ξ''. When $D_2(p,\xi) = \xi D(p)$, this implies $(1-\tau)\xi'' p_2'' D(p_2'') + \tau h(p_2'')(I'' - \xi'' D(p_2'')) > (1-\tau)\xi'' \hat{p} D(\hat{p})$.

At ξ', the profit of donating excess inventory at price p_2'' is $(1-\tau)\xi' p_2'' D(p_2'') + \tau h(p_2'')(I' - \xi' D(p_2'')) \ge \frac{\xi'}{\xi''}((1-\tau)\xi'' p_2'' D(p_2'') + \tau h(p_2'')(I'' - \xi'' D(p_2''))) > (1-\tau)\xi' \hat{p} D(\hat{p})$, which is the profit under salvaging at ξ'. The first inequality holds because $I' \ge I''$ and $\xi' < \xi''$. Therefore, if donating is optimal at ξ'', donating is optimal at $\xi' < \xi''$ when $D_2(p,\xi) = \xi D(p)$. When $D_2(p,\xi) = D(p)$, the same conclusion can be established using a similar argument.

Because the firm's profit is continuous under either donating or salvaging and the tie-breaking favors salvaging, there exists $\check{\xi}$ such that when $\underline{\xi} \le \xi < \check{\xi}$, the firm chooses donation. That is, $r_2^* > 0$, $s_2^* = \max\{D_2(p_1^*, \xi), \arg\max_s((1-\tau)sp_2(s,\xi) + \tau h(p_2(s,\xi))(I^* - s)\}$, and $p_2^* = p_2(s_2^*, \xi) > c$ by Theorem 4.8.

Now we show that p_2^* and r_2^* are decreasing in ξ when $\underline{\xi} \le \xi < \check{\xi}$. We will prove by contradiction. Suppose that the optimal solutions $p_2' < p_2''$ under $\xi' < \xi''$. Because of the continuity of the profit function and the uniqueness of the optimal solution under donating, p_2 can be viewed as a continuous function of ξ. Because $I'' - D_2(p_2'', \xi) > 0$, without loss of generality, we can assume that $I'' - D_2(p_2', \xi) > 0$. Also, $I' - D_2(p_2'', \xi) > 0$ follows from $I' > I''$. Furthermore, when $D_2(p,\xi) = \xi D(p)$, the optimality of p_2' and p_2'' implies that

$$(1-\tau)\xi'' p_2'' D(p_2'') + \tau h(p_2'')(I'' - \xi'' D(p_2'')) > (1-\tau)\xi'' p_2' D(p_2') + \tau h(p_2')(I'' - \xi'' D(p_2')) \tag{EC.1}$$

and

$$(1-\tau)\xi' p_2' D(p_2') + \tau h(p_2')(I' - \xi' D(p_2')) > (1-\tau)\xi' p_2'' D(p_2'') + \tau h(p_2'')(I' - \xi' D(p_2'')). \tag{EC.2}$$

Note that $p_2' < p_2''$ implies that $h(p_2') \le h(p_2'')$. Therefore,

$$\begin{aligned}(1-\tau)(p_2' D(p_2') - p_2'' D(p_2'')) &> \tfrac{\tau I'}{\xi'}(h(p_2'') - h(p_2')) + \tau(h(p_2')D(p_2') - h(p_2'')D(p_2''))\\ &\ge \tfrac{\tau I''}{\xi''}(h(p_2'') - h(p_2')) + \tau(h(p_2')D(p_2') - h(p_2'')D(p_2'')) > (1-\tau)(p_2' D(p_2') - p_2'' D(p_2'')),\end{aligned}$$

where the first inequality follows from (EC.2), the second inequality follows from $I' > I''$ and $\xi' < \xi''$, and the last inequality follows from (EC.1). Thus we reach a contradiction. When $D_2(p,\xi) = D(p)$, we can reach a contradiction by a similar argument.

Therefore, when $\underline{\xi} \le \xi < \check{\xi}$, p_2^* is (weakly) decreasing in ξ; $s_2^* = D(p_2^*, \xi)$ is (weakly) increasing in ξ; and $r_2^* = I^* - s_2^*$ is (weakly) decreasing in ξ. □

Proof of Lemma 1: We first show that when $p_1' < p_1''$, $U_2(x|p_1', \xi) - U_2(x|p_1', \xi, r_2 = 0) \le U_2(x|p_1'', \xi) - U_2(x|p_1'', \xi, r_2 = 0)$. Suppose that the second period price p_2' is part of the optimal solution to $U_2(x|p_1', \xi, r_2 = 0)$. We consider two possibilities:

$p'_2 = p'_1$: In this case, all x units of inventory are sold at $p'_2 = p'_1$ under the optimal solution to $U_2(x|p'_1, \xi, r_2 = 0)$. Therefore, $U_2(x|p_1, \xi) = p_1 x = U_2(x|p'_1, \xi, r_2 = 0)$, and $U_2(x|p'_1, \xi) - U_2(x|p'_1, \xi, r_2 = 0) = 0 \leq U_2(x|p''_1, \xi) - U_2(x|p''_1, \xi, r_2 = 0)$.

$p'_2 < p'_1$: In this case, the constraint $p'_2 < p'_1$ is not binding, because $U_2(x|p'_1, \xi, r_2 = 0)$ can be formulated as a convex optimization problem of the sales quantity. When $p'_2 < p'_1 < p''_1$, the optimal solution to $U_2(x|p'_1, \xi, r_2 = 0)$ is the optimal solution to $U_2(x|p''_1, \xi, r_2 = 0)$. Therefore, $U_2(x|p'_1, \xi) - U_2(x|p'_1, \xi, r_2 = 0) \leq U_2(x|p''_1, \xi) - U_2(x|p''_1, \xi, r_2 = 0)$ because $U_2(x|p'_1, \xi) \leq U_2(x|p''_1, \xi)$ when $p'_1 < p''_1$.

Now we show that when $x' < x''$, $U_2(x'|p_1, \xi) - U_2(x'|p_1, \xi, r_2 = 0) \leq U_2(x''|p_1, \xi) - U_2(x''|p_1, \xi, r_2 = 0)$. Suppose that the second period donation r'_2 is part of the optimal solution to $U_2(x'|p_1, \xi)$. We consider two possibilities:

$r'_2 = 0$: In this case, $U_2(x'|p_1, \xi) - U_2(x'|p_1, \xi, r_2 = 0) = 0 \leq U_2(x''|p_1, \xi) - U_2(x''|p_1, \xi, r_2 = 0)$.

$r'_2 > 0$: In this case, it suffices to show that $\tau h(p'_2) \geq U'_2(x'|p_1, \xi, r_2 = 0)$ because $U_2(x''|p_1, \xi) - U_2(x'|p_1, \xi) \geq h(p'_2)(x'' - x')$ and $U_2(x''|p_1, \xi, r_2 = 0) - U_2(x'|p_1, \xi, r_2 = 0) \leq U'_2(x'|p_1, \xi, r_2 = 0)(x'' - x')$ due to the concavity of $U_2(x|p_1, \xi, r_2 = 0)$.

Notice that $\tau h(p'_2) r'_2 > U_2(x'|p_1, \xi, r_2 = 0) - U_2(x' - r'_2|p_1, \xi, r_2 = 0) = U'(x'''|p_1, \xi, r_2 = 0) r'_2$, where $x''' \in (x' - r'_2, x')$. Therefore, $\tau h(p'_2) > U'(x'''|p_1, \xi, r_2 = 0) > U'(x'|p_1, \xi, r_2 = 0)$ by the concavity of $U_2(x|p_1, \xi, r_2 = 0)$. □

Proof of Theorem 4.10: Notice that $r_1 = 0$ at optimal due to $F_c = 0$ by Theorem 4.7. We prove the result by contradiction. Suppose that $p_1^* < p_1^0$ and $q^* < q^0$, we show that by procuring q^0 and setting the first period price as p_1^0, the firm achieves a profit higher than U^* with the enhanced tax deduction.

$$
\begin{aligned}
U^* &= U_1(q^*) - (1-\tau) v_c q^* = (1-\tau)(-v_c q^* + p_1^* E_\xi[\min\{D_1(p_1^*, \xi), q^*\}]) + E_\xi[U_2((q^* - D_1(p_1^*, \xi))^+ | p_1^*, \xi)] \\
&= (1-\tau)(-v_c q^* + p_1^* E_\xi[\min\{D_1(p_1^*, \xi), q^*\}]) + E_\xi[U_2((q^* - D_1(p_1^*, \xi))^+ | p_1^*, \xi, r_2 = 0)] \\
&+ E_\xi[U_2((q^* - D_1(p_1^*, \xi))^+ | p_1^*, \xi) - U_2((q^* - D_1(p_1^*, \xi))^+ | p_1^*, \xi, r_2 = 0)] \\
&< (1-\tau)(-v_c q^0 + p_1^* E_\xi[\min\{D_1(p_1^0, \xi), q^0\}]) + E_\xi[U_2((q^0 - D_1(p_1^0, \xi))^+ | p_1^0, \xi, r_2 = 0)] \\
&+ E_\xi[U_2((q^0 - D_1(p_1^*, \xi))^+ | p_1^0, \xi) - U_2((q^0 - D_1(p_1^0, \xi))^+ | p_1^0, \xi, r_2 = 0)] \\
&= (1-\tau)(-v_c q^0 + p_1^0 E_\xi[\min\{D_1(p_1^0, \xi), q^0\}]) + E_\xi[U_2((q^0 - D_1(p_1^0, \xi))^+ | p_1^0, \xi)] \\
&\leq U_1(q^0) - (1-\tau) v_c q^0
\end{aligned}
$$

The first inequality is due to the optimality of (q^0, p_1^0), Lemma 1, and $(q^0 - D_1(p_1^0, \xi))^+ \geq (q^* - D_1(p_1^*, \xi))^+$ when $p_1^* < p_1^0$ and $q^* < q^0$. The second inequality is due to the definition of U_1. Therefore, we reach a contradiction and we conclude the proof. □