# Math 149s: Analysis Cheat Sheet

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## 1 Definitions

If $S$ is some set of real numbers:

1. $\sup S$ is the **least upper bound** of $S$.
2. $\inf S$ is the **greatest lower bound** of $S$.

$S$ may or may not contain its sup or inf; if it does, we say that the sup is its *maximum* and the inf is its *minimum*.

For a sequence $\{a_n\}_{n=1}^{\infty}$, we also define:

1. $\limsup a_n = \inf_k \sup_n \{a_n\}_{n=k}^{\infty}$
2. $\liminf a_n = \sup_k \inf_n \{a_n\}_{n=k}^{\infty}$

We can define the **limit** $\lim a_n$ of a sequence in two equivalent ways:

1. The limit is defined if the lim inf and lim sup of the sequence exist and have the same value, in which case $\lim a_n = \liminf a_n = \limsup a_n$.
2. $\lim a_n = c$ if for any $\epsilon > 0$, we can find some $N$ such that for all $n \geq N$, $|a_n - c| < \epsilon$. $\lim a_n = \infty$ if for any $y \in \mathbb{R}$ there is $N$ such that for all $n \geq N$, $a_n > y$.

We say that an **infinite series** $\sum_{n=1}^{\infty} b_n$ **converges** if the limit of its partial sums $\lim_{k\to\infty} \sum_{n=1}^{k} b_n$ converges as a sequence.

There are also two equivalent notions of the **limit of a function** $f(x)$ as $x \to y$:

1. $\lim_{x\to y} f(x) = c$ if for all sequences $x_n \to y$, $f(x_n) \to c$.
2. $\lim_{x\to y} f(x) = c$ if for every $\epsilon > 0$, we can find some $\delta > 0$ such that for all $x$ such that $|x - y| < \delta$, $|f(x) - c| < \epsilon$.

A function $f$ is **continuous** at point $y$ if $\lim_{x\to y} f(x) = f(y)$. Using our two definitions of limits, we can write this as:

1. $f$ is continuous at $y$ if for any sequence $x_n \to y$, $f(x_n) \to f(y)$.
2. $f$ is continuous at $y$ if for any $\epsilon > 0$, we can find some $\delta > 0$ such that for all $x$ such that $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$.

A function $f$ that is continuous at $x$ is **differentiable** at $x$ if the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists and is finite. If so, the limit is labeled $f'(x)$.

A subset $A \subset \mathbb{R}$ is:

1. **Open** if for any point $x \in A$, we can find some $\delta > 0$ such that the set $B = \{y : |y-x| < \delta\}$ is a subset of $A$.
2. **Closed** if for any sequence $x_n \to x$, where all $x_n \in A$, $x \in A$ as well.
3. **Bounded** if $\sup_{x,y\in\mathbb{R}} |x-y| < \infty$.
4. **Compact** if it is closed and bounded.

The complement of an open set is closed, and vice versa.

# 2 Facts

Some facts about sequences include:

1. **Squeeze Theorem:**
   (a) If $a_n \le c_n \le b_n$ for all $n$, $a_n \to L$ and $b_n \to L$, then $c_n \to L$ as well.
   (b) If $a_n \le b_n$ for all $n$ and $a_n \to \infty$, then $b_n \to \infty$ as well.
2. **Cauchy Criterion:** $a_n \to a$ if and only if for any $\epsilon > 0$ we can find some $N$ such that for all $m, n \ge N$, $|a_m - a_n| < \epsilon$.
3. **Weierstrass Theorem:** A monotonic bounded sequence converges.
4. **Sequential Compactness:** A compact subset of the reals is also sequentially compact, meaning that any sequence in it contains a convergent subsequence.
5. **Cezaro-Stolz Theorem:** Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers, where the $y_n$ are positive, strictly increasing, and unbounded. If $\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = L$ then $\lim \frac{x_n}{y_n}$ exists and is equal to $L$.
6. **Cantor's Nested Intervals Theorem**: If $I_1 \supset I_2 \supset \dots$ is a decreasing sequence of closed intervals with lengths converging to zero, then $\cap_{n=1}^{\infty} I_n$ consists of one point.

Two types of **series** are especially important:

1. The **geometric series** $\sum_{n=0}^{k} x_n$ has sum $\frac{1-x^k}{1-x}$. Taking $k \to \infty$, the series converges iff $|x| < 1$, in which case the sum if $\frac{1}{1-x}$.
2. The **p-series** $\sum_{n=0}^{\infty} n^p$ converges for $p > 1$ (assuming $p$ is positive).

You will often apply the **comparison test**, which states that if $a_n, b_n \ge 0$, $a_n \le b_n$ for all $n$ and $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges as well. If $\sum_{n=0}^{\infty} a_n$ diverges, then so does $\sum_{n=0}^{\infty} b_n$.

The series $\sum_{n=0}^{\infty} a_n$ **converges absolutely** if $\sum_{n=0}^{\infty} |a_n| < \infty$; absolute convergence implies normal convergence. Rearranging the terms of a convergent series is only guaranteed to leave the sum the same if the series converges absolutely. Some tests for absolute convergence (and convergence more generally) include:

1. **Ratio Test.** Letting $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, we can conclude that the series converges if $L < 1$ and diverges if $L > 1$; the common case $L = 1$ is ambiguous.
2. **Root Test.** Letting $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$, we can conclude that the series converges if $L < 1$ and diverges if $L > 1$; again, the $L = 1$ case is ambiguous.
3. **Integral Test.** If $|a_n|$ is monotone decreasing, and $|a_n| = f(n)$, where $f$ is some monotone decreasing continuous function on the interval $[0, \infty)$, then $\sum_{n=0}^{\infty} |a_n|$ converges if and only if the integral $\int_0^\infty f(n)$ is finite.

According to the **Alternating Series Test**, a series $\sum_{n=0}^{\infty}(-1)^n a_n$ where the $a_n$ are positive and decreasing will converge.

The **Squeeze Theorem for Functions** states that if $f, g, h$ are functions defined on some interval $I$ such that $g(x) \le f(x) \le h(x)$ for all $x \in I$, then if $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$.

Useful facts about **continuous functions** include:

1. **Intermediate Value Theorem.** If $f$ is continuous on the interval $[a, b]$, for any $\gamma$ between $f(a)$ and $f(b)$ there exists $c \in [a, b]$ such that $f(c) = \gamma$.
2. **Extreme Value Theorem.** If $f$ is continuous on the interval $[a, b]$, then $f$ attains an absolute maximum value $f(c)$ and an absolute minimum $f(d)$ at some numbers $c$ and $d$ in $[a, b]$.
3. If $f$ is continuous and $A$ is open, then the inverse image $f^{-1}(A)$ is also open.

Useful facts about **differentiable functions** include:

1. **Mean Value Theorem.** If $f$ is continuous on $[a, b]$ and is differentiable on $(a, b)$, then there exists some $c \in (a, b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.
2. **L'Hopital's Rule.** Let $f, g$ be differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$
Similarly, if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, we also have $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.
3. **Increasing and Decreasing Functions.** Let $f$ be a function on some interval $[a, b]$. If $f'(x) > 0$ for all $x \in (a, b)$, $f$ must be strictly increasing on $[a, b]$. If $f'(x) \ge 0$ for all $x \in (a, b)$, then $f$ is nondecreasing on $[a, b]$. The opposite holds for negative first derivatives.
4. **Convexity** If $f''(x) \ge 0$ for all $x \in [a, b]$, then $f$ is **convex** on that interval and we have for any $\alpha \in [0, 1]$, $a \le c \le d \le b$:
$$f(\alpha c + (1-\alpha)d) \le \alpha f(c) + (1-\alpha) f(d)$$
We can then apply Jensen's inequality, which states that for any nonnegative $w_1, \ldots, w_n$, $\sum_i w_i = 1$, and $x_1, \ldots, x_n \in [a, b]$:
$$f(w_1 x_1 + \ldots + w_n x_n) \le w_1 f(x_1) + \ldots + w_n f(x_n)$$
The opposite inequalities hold if $f''(x) \le 0$.

5. **Extrema** If $f$ is differentiable on $(a, b)$, then the maximum and minimum of $f$ on $[a, b]$ either lie at the endpoints $a$ and $b$ or satisfy $f'(x) = 0$. If $f''(x) < 0$ as well, then $x$ is a maximum; if $f''(x) > 0$, then $x$ is a minimum.

6. **Fundamental Theorem of Calculus.** Let $f$ be a continuous real-valued function on some interval $I \subset \mathbb{R}$ and let $a \in I$. If $F(x) = \int_a^x f(t)\, dt$ for all $x \in I$, then $F$ has a continuous first derivative equal to $f$.