The Art of Counting

Bijections, Double Counting

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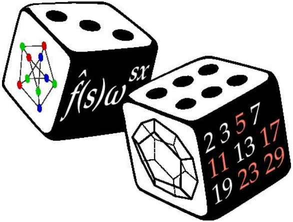
What we focus on in this talk?

Enumerative combinatorics is a huge branch of mathematics, involving many theorems and techniques, which we cannot hope to cover in one class



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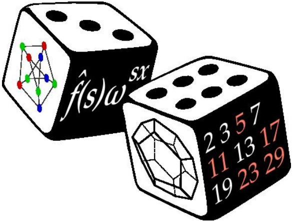

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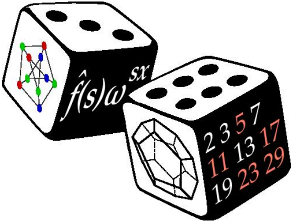

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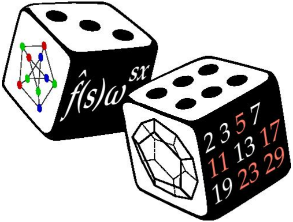

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- Straightforward, careful counting
- Bijection
- Counting in multiple ways

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- Comprehensive enumeration/case work
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Sum Rule:

If $A = A_1 \bigcup A_2 \bigcup \cdots \bigcup A_n$, $A_i \bigcap A_j = \emptyset$, then

$$|A| = |A_1| + |A_2| + \cdots + |A_n|$$

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Product Rule:
If $W = W_1 \times W_2 \times \cdots \times W_n$ (Cartesian set product), then

$$|W| = |W_1||W_2| \cdots |W_n|$$

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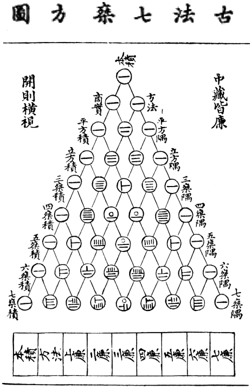
$$\binom{n}{k} = \frac{n(n-1)\cdots(n-m+1)}{m(m-1)\cdots 1} = \frac{n!}{(n-m)!m!}$$



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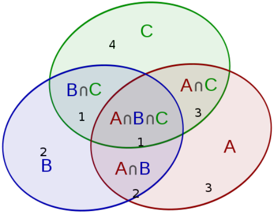
Example: suppose $|A| = 9$, $|B| = 6$, $|C| = 9$, $|A \bigcap B| = 3$, $|A \bigcap C| = 4$, $|B \bigcap C| = 2$, $|A \bigcap B \bigcap C| = 1$, what is $|A \bigcup B \bigcup C|$?



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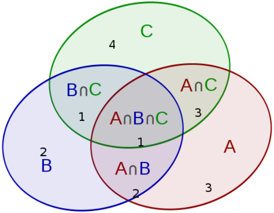

$$
\begin{aligned}
& |A \textstyle\bigcup B \bigcup C| \\
= \; & |A| + |B| + |C| - |A \textstyle\bigcap B| - |A \bigcap C| - |B \bigcap C| + |A \bigcap B \bigcap C| \\
= \; & 9 + 6 + 9 - 3 - 4 - 2 + 1 \\
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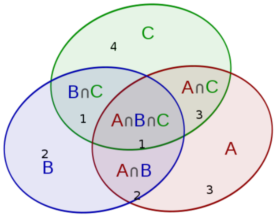

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In general,

$$|A_1 \bigcup \cdots \bigcup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \bigcap A_j| + \sum_{i<j<k} |A_i \bigcap A_j \bigcap A_k| \cdots$$

Putting the theory into practice

Example 1

[Derangements] Aat a Secret Santa party, there are n guests, who each brings a present. Once all presents are collected, they are permuted randomly, and redistributed to the guests. What is the probability no guest receives his/her own gift? What does this converge to as $n \to \infty$?

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$$\begin{array}{rcl} |A_1 \bigcup \cdots \bigcup A_n| & = & \sum_i |A_i| - \sum_{i<j} |A_i \bigcap A_j| \cdots \\ & = & (n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! \\ & = & n! \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i!} \end{array}$$

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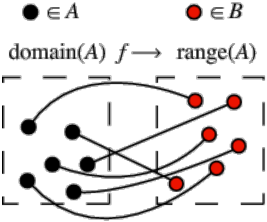
The probability is

$$\frac{|D|}{|U|} = \sum_{i=0}^{n} \frac{(-1)^i}{i!} \to \frac{1}{e}$$

Paradigm 2: Constructing a Bijection

A bijection is a one-to-one correspondence:



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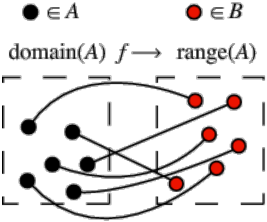

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Given sets A, B, a bijection f is $f : A \to B$ that is **one-to-one** (no two elements in A are mapped to the same in B) and **onto** (for every element in B, some element in A maps to it.)

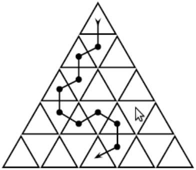
Equivalently, f is a bijection if there is an inverse map: $\exists g : B \to A$, s.t. $\forall a \in A$, $g(f(a)) = a$.

We frequently show that two sets are equal in size by constructing a bijection.

Simple Bijection

Example 2

(CMO 2005) Consider an equilateral triangle of side length n, which is divided into unit triangles, as shown. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated below for $n = 5$. Determine the value of $f(2005)$.



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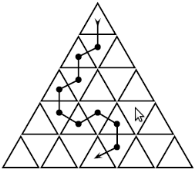


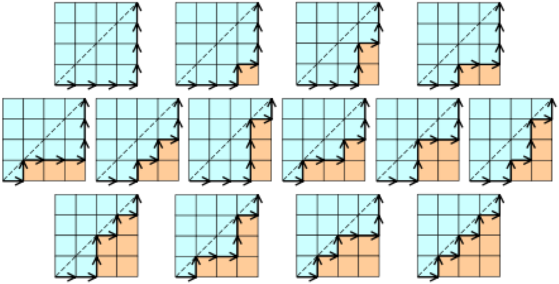
Solution.

We show that there is a bijective mapping between valid paths and ordered lists $(a_1, a_2, \cdots, a_n)$, where $1 \le a_i \le i$. Essentially a_i indicates where the path "exit" the ith row and enter the $i+1$th row. For any valid path, this ordered lists exists. For any ordered list, we can reconstruct the path uniquely. The number of such ordered lists is exactly $n!$, hence $f(2005) = 2005!$. □

More Involved Bijection

Example 3

(Catalan Numbers) In a $n \times n$ grid, we draw rectilinear paths from $(0,0)$ to (n,n), going only in positive x and y directions. How many such paths are there that stay below the line $y = x$?



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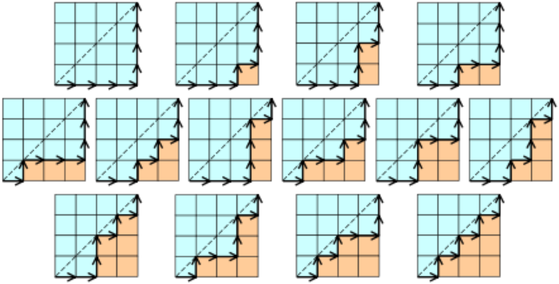

Answer: $C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$

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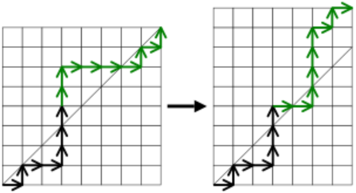

A path crosses $y = x$ iff it touches $y = x + 1$. Map f: take the first time the path touches $y = x + 1$ and reflect the following subpath across $y = x + 1$. Inverse map: app paths from $(0,0)$ to $(n-1, n+1)$ touch $y = x + 1$. Take the first touch, and reflect the following subpath across $y = x + 1$. The maps are inverses because the first touch is preserved by both maps.

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Hence, $n = \frac{1}{2}\binom{15}{2} = 35$. $\square$

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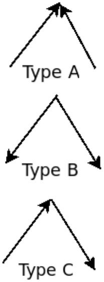
(CMO 2006) There are $2n + 1$ teams in a round-robin tournament, in which each team plays every other team exactly once, with no ties. We say that teams X, Y, Z form a *cycle triplet* if X beats Y, Y beats Z and Z beats X. Determine the maximum number of cyclic triplets possible.

Proof.

Count the # of the following types of "angles"



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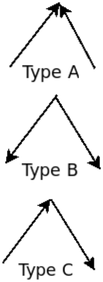
Each cyclic triangle has 3 type C angles, while each non-cyclic triagle has 1 angle of each time.



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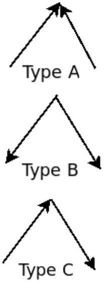

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\begin{aligned}
& (\#\text{ of non-cyclic triangles}) \\
= \;& \tfrac{1}{2}[(\#\text{ of type A angles}) + (\#\text{ of type B angles})] \\
= \;& \tfrac{1}{2}\textstyle\sum_{i=1}^{2n+1} \binom{a_i}{2} + \binom{2n-a_i}{2} \\
\geq \;& \tfrac{1}{2}\textstyle\sum_{i=1}^{2n+1} \binom{n}{2} + \binom{n}{2} \\
= \;& \frac{n(n-1)(2n+1)}{2}
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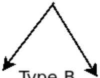

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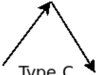

Type C

Each cyclic triangle has 3 type C angles, while each non-cyclic triagle has 1 angle of each time.

$$\begin{aligned}
& (\#\text{ of non-cyclic triangles}) \\
= \; & \tfrac{1}{2}[(\#\text{ of type A angles}) + (\#\text{ of type B angles})] \\
= \; & \tfrac{1}{2}\textstyle\sum_{i=1}^{2n+1} \binom{a_i}{2} + \binom{2n-a_i}{2} \\
\geq \; & \tfrac{1}{2}\textstyle\sum_{i=1}^{2n+1} \binom{n}{2} + \binom{n}{2} \\
= \; & \frac{n(n-1)(2n+1)}{2}
\end{aligned}$$

Hence, the # of cyclic triangles is at least $\binom{2n+1}{3} - \frac{n(n-1)(2n+1)}{2} = \frac{n(n+1)(2n+1)}{6}$.

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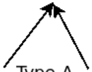
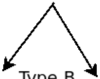
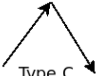

Type A


Type B


Type C

$$\begin{aligned}
& (\#\text{ of non-cyclic triangles}) \\
= \quad & \tfrac{1}{2}[(\#\text{ of type A angles}) + (\#\text{ of type B angles})] \\
= \quad & \tfrac{1}{2}\textstyle\sum_{i=1}^{2n+1} \binom{a_i}{2} + \binom{2n-a_i}{2} \\
\geq \quad & \tfrac{1}{2}\textstyle\sum_{i=1}^{2n+1} \binom{n}{2} + \binom{n}{2} \\
= \quad & \tfrac{n(n-1)(2n+1)}{2}
\end{aligned}$$

Hence, the # of cyclic triangles is at least $\binom{2n+1}{3} - \frac{n(n-1)(2n+1)}{2} = \frac{n(n+1)(2n+1)}{6}$.

To show this bound can be attained, label the vertices $1, 2, \cdots, 2n+1$ and put directed edge $i \to j$ iff $j - i (\mod 2n+1) \in \{1, 2, \cdots, n\}$. $\square$

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Enjoy problem set 3! All problems have nice solutions, so try not to brute force.