Algebraic Puzzles

Introduction to Functional Equations

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- More advanced theory: recurrences, reduction to Cauchy's equation, etc.

Plug and check

Example

Let f be a real valued function which satisfies

- $\forall x, y \in \mathbb{R}$, $f(x+y) + f(x-y) = 2f(x)f(y)$
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If $f(0) = 0$, plug in $y = 0$, get $f \equiv 0$.
If $f(0) = 1$, plug in $x = y = \frac{x_0}{2}$, get $f(\frac{x_0}{2}) = 0$. Plug in $y = \frac{x_0}{2}$, we get $f(x + \frac{x_0}{2}) = -f(x - \frac{x_0}{2})$. So f reverses sign every x_0, and hence is periodic with period $2x_0$. $\square$

Using fixed points

Problem

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Solution.

Plug in $x = y = 1$, we get $f(f(1)) = f(1)$. Plug in $x = 1$ and $y = f(1)$, we get $f(1)^2 = f(f(f(1))) = f(1)$, so $f(1) = 1$ is a fixed point.

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Suppose $x > 1$ is a fixed point, then $xf(x) = x^2$ is also a fixed point. So x^{2^m} is a fixed point $\forall m \in \mathbb{N}$, contradicting $\lim_{x\to\infty} f(x) = 0$.

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If $x \in (0, 1)$ is a fixed point, then

$$1 = f(\frac{1}{x}x) = f(\frac{1}{x}f(x)) = xf(\frac{1}{x})$$

So $\frac{1}{x}$ is also a fixed point, contradicting the above.

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Hence 1 is the only fixed point, which implies $xf(x) \equiv 1$. $\square$

More Advanced Theory: Linear Recurrences

Recurrence relations are special forms of functional equations: *i.e.* Find $f : \mathbb{Z} \to \mathbb{Z}$ s.t. $f(n) = f(n-1) + f(n-2), f(0) = 0, f(1) = 1$.

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Consider the *characteristic equation* $p(x) = x^d - \sum_{i=1}^{d} a_i x^{d-i} = 0$. If $\{r_1, \cdots, r_d\}$ are the roots, then $f_j(n) = r_j^n$ satisfies the recurrence. If the r_j's are distinct, then we are done.

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Otherwise, if r_j has multiplicity m, then it satisfies $p'(x) = 0, \cdots, p^{(m-1)}(x) = 0$. Hence, $f_j'(n) = nr_j^{n-1}$, $f_j''(n) = n(n-1)r_j^{n-2}$, etc, also satisfy the recurrence. This provides us the basis.

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Example

Let $f(n) = f(n-1) + f(n-2)$, $f(0) = 0$, $f(1) = 1$. Prove that $\lim_{n\to\infty} \frac{f(n)}{f(n-1)} = \frac{1+\sqrt{5}}{2}$.

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Solution.

Let $\phi_1 = \frac{1+\sqrt{5}}{2}$, $\phi_2 = \frac{1-\sqrt{5}}{2}$. These are the roots of the characteristic equation $x^2 - x - 1 = 0$. Hence, $f_1(n) = \phi_1^n$ and $f_2(n) = \phi_2^n$ both satisfy the recurrence. We seek a solution of the form $f(n) = af_1(n) + bf_2(n)$.

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Which yields $a = 1/\sqrt{5}$, $b = -a$. So $f(n) = \frac{1}{\sqrt{5}}(\phi_1^n - \phi_2^n)$.

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□

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Theorem

If f satisfies Cauchy's equation and one of the following

- *is monotone*
- *is continuous*
- *is bounded from above in at least one interval $(p, p+s)$*

Then $f(x) \equiv cx$ for some $c \in \mathbb{R}$.

Example from the recent practice VTech

Example

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Proof.

Using simple arguments, one can show $f(x) = f(-x)$, and $f(x) > 0$ $\forall x \in \mathbb{R}$. Rewrite the equation as $\forall x, y \in \mathbb{R}^+$.

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Define $g = \ln(f(\sqrt{x}))$. g is continuous, and $g(1) = \ln 2$. The equation reduces to $\forall x, y \in \mathbb{R}^+$.

$$g(x) + g(y) = g(x+y)$$

So $g(x) \equiv \ln(2)x$. Hence, $\forall x \in \mathbb{R}^+$, $f(x) = 2^{x^2}$. By $f(x) = f(-x)$, $f(x) \equiv 2^{x^2}$. $\square$

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Thanks so much to all the volunteers for your help in Duke Math Meet! Good job everyone!