Inequalities

Basic tools and general techniques

Peng Shi

Department of Mathematics
Duke University

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2. Substitution
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4. Bash (not recommended)

Due to the nature of the topic, there will be quite a mess of expressions, so don't feel that you have to follow every line.

Basic tools

Theorem
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AM-GM If $x_1, \cdots, x_n$ *are positive real numbers, then*

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}$$

with equality iff $x_1 = x_2 = \cdots = x_n$.

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Example

$x^3 + y^3 + z^3 \geq 3xyz$

Basic tools

Theorem

(Cauchy-Schwarz) For any real numbers $a_1, \cdots, a_n$, $b_1, \cdots, b_n$,

$$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2$$

with equality iff the two sequences are proportional. ($\|\vec{a}\|\|\vec{b}\| \geq \vec{a} \cdot \vec{b}$.)

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with equality iff the two sequences are proportional. $(\|\vec{a}\|\|\vec{b}\| \geq \vec{a} \cdot \vec{b}.)$

Example

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}$$

because

$$(a+b+c+d)\left(\frac{1}{a} + \frac{1}{b} + \frac{2^2}{c} + \frac{4^2}{d}\right) \geq (1+1+2+4)^2$$

Basic tools

Theorem

(Jensen) Let f be a convex function. Then for any $x_1, \cdots, x_n \in I$ and any non-negative reals $w_1, \cdots, w_n$, $\sum_i w_i = 1$

$$w_1 f(x_1) + \cdots + w_n f(x_n) \geq f(w_1 x_1 + \cdots + w_n x_n)$$

If f is concave, then the inequality is flipped.

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Example

One proof of the AM-GM inequality uses the fact that $f(x) = \log(x)$ is concave, so

$$\frac{1}{b}(\log x_1 + \cdots + \log x_n) \leq \log \frac{x_1 + \cdots + x_n}{n}$$

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For other tools, see the formula sheet.

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Proof.

Let $\bar{x} = \frac{\sum_i x_i}{n}$. Say that $x_i < \bar{x}$ and $x_j > \bar{x}$. Consider replacing (x_i, x_j) by $(\bar{x}, 2\bar{x})$. Note that $x_i x_j \leq \bar{x}(2\bar{x} - x_i)$. Hence, this fixes LHS but increases RHS. So for fixed LHS, the RHS is maximized when $x_i = \bar{x}\ \forall i$. □

Connection to convexity

Usually smoothing is equivalent to arguing about a convex function: for fixed $\sum_i x_i$ and f convex, $\sum_i^n f(x_i)$ is minimized when all x_i's are equal.

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Sometimes the function is not convex, in which case the argument needs to be more intricate.

Another example

Example

If a, b, c, d, e are real numbers such that

$$\begin{aligned} a+b+c+d+e &= 8 \qquad (1) \\ a^2+b^2+c^2+d^2+e^2 &= 16 \qquad (2) \end{aligned}$$

What is the largest possible value of e?

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What is the largest possible value of e?

Solution.

Relax the second constraint to $a^2+b^2+c^2+d^2+e^2 \leq 16$ (2*). Call a 5-tuple valid if it satisfies (1) and (2*). We seek the valid 5-tuple with the largest possible e.

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$$\begin{aligned} 4k+e &= 8 \\ 4k^2+e &\leq 16 \end{aligned}$$

$$\implies \left(\frac{8-e}{4}\right)^2 = k^2 \leq \frac{16-e^2}{4}$$

$$\implies (5e-16)e \leq 0 \qquad \Rightarrow 0 \leq e \leq \frac{16}{5}$$

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Conversely, $(\frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{16}{5})$ satisfies the original equation, so the largest possible value is $\frac{16}{5}$. $\square$

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Commonly used subsitutions

- Simple manipluations
- Triangle related substitutions (cool)
- Homogenization (will show later)

Triangle related substitutions

- If a, b, c are sides of a triangle, then let $x = (b + c - a)/2$, $y = (a + c - b)/2$, $z = (a + b - c)/2$, so that $a = y + z$, $b = x + z$, $c = x + y$, and x, y, z are arbitrary positive real numbers.

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- If $a^2 + b^2 = 1$, let $a = \cos\theta$, $b = \sin\theta$.
- If $a + b + c = abc$, $a, b, c > 0$, let $a = \tan A$, $b = \tan B$, $c = \tan C$, where A, B, C are angles in a triangle.

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- If $a^2 + b^2 + c^2 + 2abc = 1$, let $a = \cos A$, $b = \cos B$, $c = \cos C$, where A, B, C are angles in a triangle.

Slick example

Example

For positive real numbers a, b, c with $a + b + c = abc$, show that

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2}$$

Proof.

WLOG, let $a = \tan A$, $b = \tan B$, $c = \tan C$, where A, B, C are angles in a triangle. The inequality is equivalent to

$$\cos A + \cos B + \cos C \leq \frac{3}{2}$$

But $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$, so inequality follows from $f(x) = \log \sin x$ being concave, where $0 \leq x \leq \frac{\pi}{2}$. □

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Can only learn this through experience.

How did they come up with that?

Example

(IMO 2005 P3) Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0$$

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Solution.

By Cauchy-Schwarz

$$(x^5 + y^2 + z^2)(\frac{1}{x} + y^2 + z^2) \geq (x^2 + y^2 + z^2)^2$$

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$$\implies \frac{yz - x^2}{x^2 + y^2 + z^2} + 1 \geq \frac{x^2 - x^5}{x^5 + y^2 + z^2} + 1$$

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$$\implies \frac{yz - x^2}{x^2 + y^2 + z^2} + 1 \geq \frac{x^2 - x^5}{x^5 + y^2 + z^2} + 1$$

$$\implies LHS \geq \frac{x^2 + y^2 + z^2 - xy - yz - xz}{x^2 + y^2 + z^2} \geq 0$$

□

Idea 4: Bash (for symmetric inequalities)

Not really an idea, but what you do when you run out of ideas and feel like an algebraic workout.

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Preliminaries:

- Symmetric notation: *i.e.*

$$(2,1,1) = \sum_{sym} x^2yz = x^2yz + x^2zy + y^2xz + y^2zx + z^2xy + z^2yx$$

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- Homogenization: *i.e.* If $xyz = 1$, then

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} = \frac{x^5 - x^3yz}{x^5 + xy^3z + xyz^3}$$

Equivalently one can subsitute $x = \frac{bc}{a}, y = \frac{ac}{b}, z = \frac{ab}{c}$.

Ultimate Bash Toolbox

Definition

(Majorization) Sequence $x_1, \cdots, x_n$ is said to *majorize* sequence $y_1, \cdots, y_n$ if

$$\begin{array}{rcl} x_1 & \geq & y_1 \\ x_1 + x_2 & \geq & y_1 + y_2 \\ \ldots & & \ldots \\ \sum_{i=1}^{n-1} x_i & \geq & \sum_{i=1}^{n-1} y_i \\ \sum_{i=1}^{n} x_i & = & \sum_{i=1}^{n} y_i \end{array}$$

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Theorem

(Muirhead) Suppose the sequence $a_1, \cdots, a_n$ majorizes the sequence $b_1, \cdots, b_n$. Then for any positive reals $x_1, \cdots, x_n$,

$$\sum_{sym} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \geq \sum_{sym} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$$

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where the sums are taken over all permutations of the n variables.

Example

$\sum_{sym} x^4 y \geq \sum_{sym} x^3 yz$ (In simplified notation, $(4, 1, 0) \geq (3, 1, 1)$)

Ultimate Bash Toolbox

Theorem

(Schur) Let a, b, c be nonnegative reals and $r > 0$. Then

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0$$

with equality iff $a = b = c$ or some two of a, b, c are equal and the other is 0. ($\sum_{sym} a^{r+2} + \sum_{sym} a^r bc \geq 2\sum_{sym} a^{r+1}b$)

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Example

$$\sum_{sym} x^3 + \sum_{sym} xyz \geq 2\sum_{sym} x^2 y$$

Equivalently

$$2(x^3 + y^3 + z^3 + 3xyz) \geq 2(x^2(y+z) + y^2(x+z) + z^2(x+y))$$

The Idiot's Guide to Symmetric Inequalities

1. Homogenize
2. Multiply out all denomiators, expand, and rewrite using symmetric notation.
3. Apply AM-GM, Muirhead, and Schurs.

Example of Bash

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Solution.

Homogenizing and rearranging, it suffices to show

$$3 \geq (x^3yz + xy^3z + xyz^3)\left(\frac{1}{x^5 + xy^3z + xyz^3} + \frac{1}{x^3yz + y^5 + xyz^3} + \frac{1}{x^3yz + xy^3z + z^5}\right)$$

Multiply out and using symmetric notation, this is equivalent to

$$\sum_{sym} x^1 0yz + 4\sum_{sym} x^7y^5 + \sum_{sym} x^6y^3z^3 \geq 2\sum_{sym} x^6y^5z + \sum_{sym} x^8y^2z^2 + \sum_{sym} x^5y^5z^2 + \sum_{sym} x^6y^4z^2$$

Which follows from

$$\begin{aligned}
(10,1,1) + (6,3,3) &\geq 2(8,2,2) \\
(8,2,2) &\geq (6,4,2) \\
2(7,5,0) &\geq 2(6,5,1) \\
(7,5,0) &\geq (5,5,2) \\
(7,5,0) &\geq (6,4,2)
\end{aligned}$$

□