Inequalities Formula Sheet

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This is a partial list of commonly used inequalities. Please do not feel you have to memorize these; it is much better to be able to use one or two of them well than to know them all but not be able to use them in a problem.

1 Inequalities on convex functions

Theorem 1. *(Jensen) Let $f : I \to \mathbb{R}$ be a convex function. Then for any $x_1, \cdots, x_n \in I$ and any non-negative reals $w_1, \cdots, w_n$, $\sum_i w_i = 1$*

$$w_1 f(x_1) + \cdots + w_n f(x_n) \geq f(w_1 x_1 + \cdots + w_n x_n)$$

If f is concave, then the inequality is flipped.

Definition 1. *(Majorization) Sequence $x_1, \cdots, x_n$ is said to* majorize *sequence $y_1, \cdots, y_n$ if*

$$\begin{array}{rcl} x_1 & \geq & y_1 \\ x_1 + x_2 & \geq & y_1 + y_2 \\ \cdots & & \cdots \\ \sum_{i=1}^{n-1} x_i & \geq & \sum_{i=1}^{n-1} y_i \\ \sum_{i=1}^{n} x_i & = & \sum_{i=1}^{n} y_i \end{array}$$

Theorem 2. *(Majorization/Karamata) Let $f : I \to \mathbb{R}$ be a convex function and suppose that the sequence $x_1, \cdots, x_n$ majorizes the sequence $y_1, \cdots, y_n$. Then*

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n)$$

2 Inequalities generalizing AM-GM

Theorem 3. *(AM-GM-HM) If $x_1, \cdots, x_n$ are positive real numbers, then their arithmetic mean (AM) is larger than their geometric mean (GM), which is larger than their harmonic mean (HM).*

$$\frac{\sum_i x_i}{n} \geq \left(\prod_i x_i\right)^{1/n} \geq \frac{1}{\left(\sum_i \frac{1}{x_i}\right)/n}$$

with equality iff $x_1 = x_2 = \cdots = x_n$.

Definition 2. *(Weighted Power Means) Given positive reals $x_1, \cdots, x_n$, and positive weights $w_1, \cdots, w_n$ s.t. $\sum_i w_i = 1$, the* weighted r-th power mean *is*

$$M_w^r(x_1, \cdots, x_n) = (w_1 x_1^r + \cdots + w_n x_n^r)^{1/r}$$

Theorem 4. *(Weighted Power Means) If $r > s$, then*

$$M_w^r(x_1, \cdots, x_n) \geq M_w^s(x_1, \cdots, x_n)$$

with equality iff $x_1 = x_2 = \cdots = x_n$.

Theorem 5. *(Cauchy-Schwarz) For any real numbers $a_1, \cdots, a_n$, $b_1, \cdots, b_n$,*

$$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2$$

with equality iff the two sequences are proportional. (In vector notation, this is $\|\vec{a}\|\|\vec{b}\| \geq \vec{a} \cdot \vec{b}$.)

Theorem 6. *(Hölder) Let $a_1, \cdots, a_n; b_1, \cdots, b_n; z_1, \cdots, z_n$ be sequences of non-negative real numbers, and let $w_a, \cdots, w_z$ be positive real numbers whose sum is 1.*

$$(a_1 + \cdots + a_n)^{w_a}(b_1 + \cdots + b_n)^{w_b} \cdots (z_1 + \cdots + z_n)^{w_z} \geq a_1^{w_a} b_1^{w_b} \cdots z_1^{w_z} + \cdots + a_n^{w_a} b_n^{w_b} \cdots z_n^{w_z}$$

(This theorem generalizes Cauchy-Schwarz to many sequences.)

Theorem 7. *(Minkowski) Let $r > s$ be nonzero real numbers. Then for any positive real numbers a_{ij},*

$$\left(\sum_{j=1}^{m}\left(\sum_{i=1}^{m} a_{ij}^r\right)^{s/r}\right)^{1/s} \geq \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^s\right)^{r/s}\right)^{1/r}$$

(This theorem can be seen as the generalization of the triangles inequality to normed spaces.)

Theorem 8. *(Muirhead) Suppose the sequence $a_1, \cdots, a_n$ majorizes the sequence $b_1, \cdots, b_n$. Then for any positive reals $x_1, \cdots, x_n$,*

$$\sum_{sym} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \geq \sum_{sym} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$$

where the sums are taken over all permutations of the n variables.

3 Other Inequalities

Theorem 9. *(Rearrangement) Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be two nondecreasing sequences of real numbers. Then for any permutation π of $\{1, 2, \cdots, n\}$, we have*

$$\sum_i a_i b_i \geq \sum_i a_i b_{\pi(i)} \geq \sum_i a_i b_{n+1-i}$$

with equality on the left or right holding if and only if the sequences $\pi(1), \cdots, \pi(n)$ is decreasing or increasing respectively.

Theorem 10. *(Chebyshev) Let $a_1 \leq \cdots \leq a_n$; $b_1 \leq \cdots \leq b_n$ be two nondecreasing sequences of real numbers. Then*

$$\frac{\sum_i a_i b_i}{n} \geq \frac{\sum_i a_i}{n} \frac{\sum_i b_i}{n} \geq \frac{\sum_i a_i b_{n+1-i}}{n}$$

Theorem 11. *(Schur) Let a, b, c be nonnegative reals and $r > 0$. Then*

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0$$

with equality iff $a = b = c$ or some two of a, b, c are equal and the other is 0.

Theorem 12. *(Bernoulli) For all $r \geq 1$ and $x \geq -1$,*

$$(1+x)^r \geq 1 + xr$$