Linear Algebra

A wonderful little universe where everything seems to work...

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Cautionary Note

I will assume that you all have enough background in linear algebra to understand the terminology and basic ideas here—rehashing them all would take far too long, and it is not the purpose of this class. If you're having trouble, please:

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The goal of this (short) lecture will be to discuss a few useful ideas, facts, and examples that you might not have seen in linear algebra class.

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5. Swapping two rows or columns will invert the sign of the determinant.
6. Adding a multiple of one row/column to another row/column will not change the determinant.
7. Multiply a single row or column by a scalar c will multiply the determinant by c. Multiplying the entire matrix by c will multiply the determinant by c^n, where n is the dimension of the matrix.

Trace and other stuff

The **trace** is the sum of the diagonal elements of a matrix. We have:

1. $\text{Tr}(AB) = \text{Tr}(BA)$
2. $\text{Tr}(ABC) = \text{Tr}(BCA) = \text{Tr}(CAB)$
3. $\text{Tr}(PAP^{-1}) = \text{Tr}(A)$ (Trace is invariant to changes of basis.)
4. The trace is the sum of the eigenvalues of a matrix.

Note that since the trace is the sum of the eigenvalues of a matrix and the determinant is the product of the eigenvalues, the trace and determinant give us two coefficients of the characteristic polynomial:

$$P(\lambda) = \lambda^n - \text{Tr}(A)\lambda^{n-1} + \ldots + (-1)^n \det(A)$$

Some Awesome Theorems

Theorem (Spectral Mapping Theorem)

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct), and let $P(x)$ be a polynomial. Then the eigenvalues of $P(A)$ are $P(\lambda_1, \ldots, \lambda_n)$.

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Theorem (Perron-Frobenius Theorem)

Any square matrix with positive entries has a unique eigenvector with positive entries (up to a multiplication by a positive scalar), and the corresponding eigenvalue has multiplicity one and is strictly greater than the absolute value of any other eigenvalue.

Cayley-Hamilton Example

Problem

Let A and B be 2-by-2 matrices, each with determinant 1. Prove that:

$$tr(AB) - tr(A)tr(B) + tr(AB^{-1}) = 0$$

(From Putnam and Beyond)

Proof.

By Cayley-Hamilton, we must have:

$$B^2 - (\mathrm{tr}(B))B + \det(B) = 0$$

Multiplying on the left by AB^{-1}, we obtain:

$$AB - (\mathrm{tr}(B))A + AB^{-1} = 0$$

and then taking the trace we find

$$\mathrm{tr}(AB) - \mathrm{tr}(A)\mathrm{tr}(B) + \mathrm{tr}(AB^{-1}) = 0$$

A Putnam Problem

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Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2n elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Given a vector subspace V of $\mathbb{R}^n$, let $Z(V)$ denote the number of members of Z that lie in V. Let k be given, $0 \leq k \leq n$. Find the maximum, over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension k, of the number of points in $Z(V)$.

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Review: the *dimension* of a vector space equals the size of the largest set of linearly independent elements you can find in the vector space.

An awesome solution

Solution

Let V be a k-dimensional subspace. Form the matrix whose rows are the elements of $V \cap Z$. By construction, it has row rank of at most k. Therefore, it also has column rank of at most k; thus we can choose k coordinates such that each point of $Z \cap V$ is determined by k of these coordinates. Since each coordinate of a point in Z can only take two values, $V \cap Z$ can have at most 2^k elements. (proof from Catalin Zara)