Problem Set 0
Math 149S

August 27, 2009

The goal of this problem set is to provide us with information about your strengths, weaknesses, and preparation for Math 149S. We ask you to to work on it alone and turn in even incomplete solutions if you feel you've made substantial progress. We don't expect anyone to solve all the problems – just try as many as you can! If you take Math 149S, this will serve as your first graded problem set. Full credit in Math 149S requires 3 problems with a perfect score (10 out of 10) or more problems with partial points.

Please write neatly and staple your solutions, include your name and email address, and turn everything in by noon on Monday, August 31st in the box outside Professor Kraines's office (Math/Physics 211). Contact Matthew Rognlie at matthew.rognlie@duke.edu if you have any questions, and good luck!

1. Solve the recurrence (give an explicit formula for $f(n)$):

$$f(n+1) = nf(n) + (n-1)f(n-1) + \ldots + f(1) + 1,\ \ f(0) = 1$$

2. How many ways can one pick four numbers from the first fifteen positive integers, such that among the four numbers any two differ by at least 2?

3. Let M be a set endowed with an operation $*$ satisfying the properties:

   (a) there exists an element $e \in M$ such that $x * e = x$ for all $x \in M$

   (b) $(x * y) * z = (z * x) * y$ for all $x, y, z \in M$

   Show that the operation $*$ is both commutative and associative.

4. Prove that there is no triple of positive integers x, y, z satisfying:

$$x^2 + y^2 + z^2 = 2xyz$$

5. Let $n = 2^{21}3^9 5^{11}$. Find the number of positive integer divisors of n^2 that are less than, but do not divide, n.

6. Prove that every positive integer can be written as the sum of one or more integers of the form $2^s 3^t$, where s and t are nonnegative integers and no term in the sum is a multiple of another. (For example, $34 = 18 + 16$)

7. Find all polynomials whose coefficients are all equal to 1 or -1, and whose zeros are all real.

8. Is it possible for a countably infinite set to have an uncountable collection of distinct subsets among which the intersection of any two subsets is finite? If so, provide an example and prove its validity. If not, prove that it is impossible.