Math 149S Problem Set 1: Pigeonhole-Principle

Peng Shi

Sept. 1, 2009

Problems may not be sorted in level of difficulty. You are NOT expected to solve every problem. Each problem is worth 10 points (if the proof is completely correct and well-written). Points maybe subtracted from a correct solution if the proof is badly written. Depending on the amount of progress, partial points maybe awarded. To get an A you need at least 30 points.

Problem 1. *There are n persons present in a room. Prove that among them there are two persons who have the same number of acquaintances in the room.*

Problem 2. *Given any $n+2$ integers, show that there exist two of them whose sum, or else whose difference, is divisible by $2n$.*

Problem 3. *Show that among any nine distinct real numbers, there are two, say a and b, such that*

$$0 < \frac{a-b}{1+ab} < \sqrt{2}-1$$

Problem 4. *Suppose that there is a unit square, inside which lies a triangle. We know that the triangle lies strictly within the square, and that the triangle does not cover the center of the square. Prove that at least one side of the triangle has length less than 1.*

Problem 5. *Given ten distinct two-digit positive integers, prove that one can always choose two disjoint nonempty subsets, such that their elements have the same sum.*

Problem 6. *Suppose that there are 9 distinct lattice points in three dimensional Euclidean space. (A lattice point has all cordinates being integers.) Show that there is a lattice point on the interior of one of the line segments joining two of these points.*

Problem 7. *Suppose that we have two lists of positive integers $1 \leq a_1, \cdots, a_m \leq n$, $1 \leq b_1, \cdots, b_n \leq m$. Show that there are two consecutive "blocks" with the same sum:* i.e. *there are integers $p \leq q$ $r \leq s$ such that*

$$a_p + a_{p+1} + \cdots + a_q = b_r + b_{r+1} + \cdots + b_s$$

Problem 8 (Erdos and Szekeres)**.** *Consider a sequence of $k = (m-1)(n-1)+1$ numbers $a_1, a_2, \cdots, a_k$. Prove that the sequence either has an increasing subsequence of length m or a decreasing subsequence of length n. (A subsequence of length m denotes the sequence $a_{i_1}, a_{i_2}, \cdots a_{i_m}$, in which the indices satisfy $1 \leq i_1 < i_2 < \cdots < i_m \leq k$.)*

Problem 9. *Suppose that the squares of an $n \times n$ chessboard are labeled with numbers 1 though n^2 arbitrarily, using each number exactly once. Prove that there are two adjacent squares whose labels are at least n apart.*