Math 149S Problem Set 10: Number Theory

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Problems may not be sorted in level of difficulty. You are NOT expected to solve every problem. Each problem is worth 10 points (if the proof is completely correct and well-written). Points maybe subtracted from a correct solution if the proof is badly written. Depending on the amount of progress, partial points maybe awarded. To get an A you need at least 30 points.

Problem 1. *Let x and y be integers. Prove that $2x + 3y$ is divisible by 17 if and only if $9x + 5y$ is divisible by 17.*

Problem 2. *Find the last two digits of $7^{7^{7^{7^{7^{7^{7}}}}}}$. (The tower has seven 7's).*

Problem 3. *Prove that the only solution in rational numbers of the equation*

$$x^3 + 3y^3 + 9z^3 - 9xyz = 0$$

is $x = y = z = 0$.

Problem 4. *Show that the harmonic number,*

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

is not an integer for any $n > 1$.

Problem 5. *Suppose that $a_1, a_2, \cdots, a_{2n}$ are distinct integers such that the equation*

$$(x - a_1)(x - a_2)\cdots(x - a_{2n}) = (-1)^n (n!)^2$$

has an integer solution r. Show that $r = \frac{a_1 + a_2 + \cdots + a_{2n}}{2n}$

Problem 6. *Show that if n is an integer greater than 1, then n does not divide $2^n - 1$.*

Problem 7. *For all positive integers n, let $T_n = 2^{2^n} + 1$. Show that if $m \neq n$, then T_m and T_n are relatively prime.*

Problem 8. *Find all positive integer solutions to*

$$3^x + 4^y = 5^z$$

Problem 9. *If p is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum*

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.

Problem 10. *Show that for $n \geq 2$,*

$$\overbrace{2^{2^{\cdots^{2}}}}^{n \text{ terms}} \equiv \overbrace{2^{2^{\cdots^{2}}}}^{n-1 \text{ terms}} \pmod{n}$$