Problem Set 2: Invariants, Monovariants, and Extrema
Math 149S

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1. There is a pile of pebbles on your desk. Pebbles are either black or white, and you may remove two pebbles and replace them by a *single* black pebble if they are the same color or a *single* white pebble if they are different colors. Prove that the final pebble will be the same regardless of the order in which you remove pebbles.

2. Find all real solutions of the system $(x+y)^3 = z, (y+z)^3 = x, (z+x)^3 = y$.

3. A group of people is split between several rooms. Each minute, a person leaves one room and enters another that had at least the same number of people immediately prior to his move. Prove that eventually, everyone will be in the same room.

4. Matt is competing in a rock-paper-scissors tournament, where every participant plays against every other participant exactly once. No game ends in a draw. Once the tournament is over, each player lists the names of all players that (1) were beaten by her or (2) were beaten by other players beaten by her. Prove that some player (probably not Matt) has a list containing the name of every other player.

5. Peng enjoys popcorn-flavored jellybeans, and he has a pile of 2009 of them on a table. Sometimes, Peng finds a pile with *at least* three jellybeans, eats one of the jellybeans from the pile, and arbitrarily splits up the remaining jellybeans into two smaller (but not necessarily equal-sized) piles. Peng hopes that if he repeatedly does this, someday the jellybeans will all be contained in one or more piles of three. Can his hope possibly be satisfied?

6. If $f : [0,1] \to \mathbb{R}$ is continuously differentiable and $\int_0^1 f(x)\, dx = 0$, prove that for any $\alpha \in (0,1)$,

$$\left| \int_0^\alpha f(x)\, dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|$$

7. A deck contains n cards labeled $1, 2, \ldots, n$. Starting with an arbitrary ordering of the cards, repeat the following operation: if the top card is labeled k, reverse the order of the top k cards. Prove that eventually the top card will be labeled 1.

8. A deck contains 50 cards. Specifically, it contains two cards labeled n for each $n = 1, 2, \ldots, 25$. There are 25 people sitting in a circle, and each of them holds two of the cards in this deck. Each minute every person passes the lower-numbered card of the two she is holding to the right. Prove that eventually someone has two cards of the same number.

9. Let $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$, and

$$a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|, \quad n \geq 4.$$

Determine $a_{14^{14}}$.

10. Bob writes a positive integer at each vertex of a hexagon. He then repeatedly performs the following operation: replace a number by the (non-negative) difference between the two numbers at adjacent vertices. If the starting values sum to $2009! + 1$, show that it must be possible for Bob to end with a hexagon that has 0 at every vertex.