Math 149S Problem Set 3: Counting[*]

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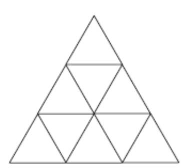
Problems may not be sorted in level of difficulty. You are NOT expected to solve every problem. Each problem is worth 10 points (if the proof is completely correct and well-written). Points maybe subtracted from a correct solution if the proof is badly written. Depending on the amount of progress, partial points maybe awarded. There are 12 problems for a maximum of 120 points. To get an A you need at least 30 points.

Problem 1. *In a certain community, each member belongs to exactly three committees, and each committee has exactly three members. Prove that the number of members equals to the number of committes.*

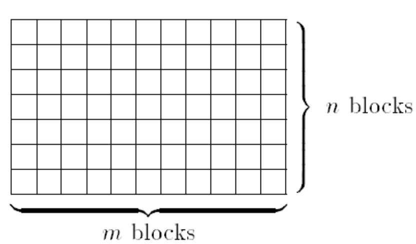
Problem 2. *Is there a polyhedron with an odd number of faces, each face having an odd number of edges?*

Problem 3. *Given $S = \{1, 2, \cdots, n\}$. How many unordered pairs $\{A, B\}$ are there where A, B are non-empty subsets of S with $A \bigcap B = \emptyset$?*

Problem 4. *In the following figure, the side lengths of the large equilateral triangle is 3 (each small triangle has side length 1) and $f(3)$, the number of parallelograms bounded by the grid sides, is 15. Find a closed-form formula[1] for $f(n)$, the number of parallelograms in a triangle of side length n.*



Problem 5. *A rectangular city has exactly $m \times n$ blocks. (Hence, there are $m + 1$ parallel streets in one direction and $n+1$ parallel streets in an orthogonal direction; see diagram.) A woman lives in the southwest corner of the city and she works in the northest corner. She walks to work each day, following the streets, and on any given trip, she makes sure that her path does not include any intersection twice. Show that the number of different paths she can take to work is* at most 2^{mn}.



[*]Although other types of solutions are acceptable, this problem set focuses on bijections and counting in multiple ways, so you can be sure there is a nice, relatively short solution to every problem.

[1]A formula is closed-form if it involves no summations.

Problem 6. *A person has a coat of area* 1 *composed of five possibly overlapping patches. The area of each patch is* $\geq 1/2$*. Prove that there are two patches whose overlap has area* $\geq 1/5$*.*

Problem 7. *Two hundred students participated in a math contest. The had six problems to solve. Each problem was correctly solved by at least* 120 *participants. Prove that there must be two participants such that every problem was solved by at least one of these students.*

Problem 8. *Let* $p_n(k)$ *denote the number of permutations of* $\{1, 2, \cdots, n\}$ *with exactly* k *fix points. Prove that*

$$\sum_{k=1}^{n} k p_n(k) = n!$$

Problem 9. *Given positive integer* n *and a divisors* d *(*$d|n$*), let set* S *contain all ordered* n*-tuples* $(x_1, x_2, \cdots, x_n)$ *of integers such that* $0 \leq x_1 \leq \cdots x_n \leq n$ *and* $d|x_1 + x_2 + \cdots + x_n$*. Prove that exactly half of the elements in* S *satisfy the property* $x_n = n$*.*

Problem 10. *A competition has* $a > 0$ *contestants and* b *judges, where* $b \geq 3$ *is odd. Each judge rates each contestant as either "pass" or "fail." Define* $d = \lceil \frac{a(b-1)}{2b} \rceil$*. Prove that there exists two judges whose ratings agree for at least* d *contestants.*

Problem 11. *In a certain math competition,* 6 *problems were posed to the participants. We know the following: every two of these problems were solved by more than* $\frac{2}{5}$ *of the contestants; no contestant solved all the* 6 *problems. Show that there are at least* 2 *contestants who solved exactly* 5 *problems each.*

Problem 12. *There are* 10001 *students at a university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of* k *societies. Suppose the following conditions hold:*

1. *Each pair of students is in exactly one club.*
2. *For each student and each society, the student is in exactly one club of the society.*
3. *Each club has an odd number of students. In addition, a club with* $2m + 1$ *students (*m *is a positive integer) is in exactly* m *societies.*

Find all possible values of k*.*