# Problem Set 4: Induction
# Math 149S

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1. Show that every positive integer may be expressed as the sum of distinct Fibonacci numbers.

2. Prove that
$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

3. If $0 \le x \le \pi$, prove for all $n \in \mathbb{N}$
$$|\sin nx| \le n \sin x$$

4. If $a_1, \ldots, a_n$ are nonnegative reals, show that:
$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \ge (1 + \sqrt[n]{a_1 a_2 \cdots a_n})^n$$

5. Let $\{u_n\}$ be a sequence defined by
$$u_1 = 1,\ u_2 = 2,\ u_{n+2} = 3u_{n+1} - u_n$$
Prove that
$$u_{n+2} + u_n \ge 2 + \frac{u_{n+1}^2}{u_n}$$
for all $n$.

6. Given some positive rational $u$, call $u+1$ and $\frac{u}{u+1}$ the *children* of $u$. Prove that every positive rational is the descendant of 1 in a unique way.

7. Evaluate
$$\int_0^\infty \left( x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \ldots \right) \cdot \left( 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots \right) dx$$

8. Prove that any $n \in \mathbb{N}$ can be represented as $\pm 1^2 \pm 2^2 \pm \ldots \pm m^2$ for some $m \in \mathbb{N}$ and some choice of signs.

9. Find a closed formula for the sequence $a_n$ defined by:
$$a_1 = 1, \quad a_{n+1} = \frac{1}{16}(1 + 4a_n + \sqrt{1 + 24a_n})$$