Math 149S Problem Set 5: Inequalities

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Problems may not be sorted in level of difficulty. You are NOT expected to solve every problem. Each problem is worth 10 points (if the proof is completely correct and well-written). Points maybe subtracted from a correct solution if the proof is badly written. Depending on the amount of progress, partial points maybe awarded. To get an A you need at least 30 points.

Problem 1. *Given real numbers $x_1, \cdots, x_n$, what is the minimum value of*

$$|x - x_1| + \cdots + |x - x_n| \text{ ?}$$

Problem 2. *Prove that if a, b, c are the side lengths of a triangle, then*

$$abc \geq (a + b - c)(a + c - b)(b + c - a)$$

Problem 3. *Suppose that x, y, z are positive real numbers such that $x + y + z + 2\sqrt{xyz} = 1$. Prove that*

$$\sqrt{1-x} + \sqrt{1-y} + \sqrt{1-z} \leq \frac{3\sqrt{3}}{2}$$

Problem 4. *Prove that, for positive reals,*

$$\sqrt{ab} + \sqrt{cd} \leq \sqrt{(a + d)(b + c)}$$

Problem 5. *Let $a_0, \cdots, a_n$ be numbers in the interval $(0, \pi/2)$ such that*

$$\tan(a_0 - \pi/4) + \tan(a_1 - \pi/4) + \cdots + \tan(a_n - \pi/4) \geq n - 1$$

Prove that $\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}$.

Problem 6. *If a, b, c, d are positive reals, then prove that*

$$1 < \frac{a}{d + a + b} + \frac{b}{b + c + a} + \frac{c}{b + c + d} + \frac{d}{c + d + a} < 2$$

Problem 7. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b + c)} + \frac{1}{b^3(c + a)} + \frac{1}{c^3(a + b)} \geq \frac{3}{2}$$

Problem 8. *Prove that*

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 9(ab + bc + ac)$$

for all real numbers $a, b, c > 0$.

Problem 9. *For any $x, y, z > 0$ with $xyz = 1$, prove that*

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+x)(1+z)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$$

Problem 10. *Let $n \geq 2$ be a positive integer and $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ be a sequence of $2n$ positive reals. Suppose $z_2, z_3, \cdots, z_{2n}$ is such that $z_{i+j}^2 \geq x_i y_j$ for all $i, j \in \{1, \cdots, n\}$. Let $M = \max z_2, z_3, \cdots, z_{2n}$. Prove that*

$$\left(\frac{M + z_2 + z_3 + \cdots + z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right)$$