Problem Set 6: Analysis
Math 149S

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1. Suppose f is defined for all real x, and suppose that $|f(x) - f(y)| \leq (x - y)^2$ for all real x and y. Prove that f is constant.

2. Let a and b be positive real numbers. Prove that
$$\lim_{n\to\infty} (a^n + b^n)^{1/n}$$
equals the larger of a and b. What happens if $a = b$?

3. Prove either the validity of ratio test or of the root test (your choice).

4. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of positive reals satisfying:
$$x_{n+1} \leq x_n + \frac{1}{n^2}, \quad \text{for all } n \geq 1$$
Prove that $\lim_{n\to\infty} x_n$ exists.

5. Let $f : [0, 1] \to [0, 1]$ be a continuous function such that $f(f(f(x))) = x$ for all $x \in [0, 1]$. Prove that $f(x) = x$.

6. Prove that the alternating series test is valid.

7. Let $f(x)$ be twice continuously differentiable (it has a continuous second derivative) in the interval $(0, \infty)$. If
$$\lim_{x\to\infty} (x^2 f''(x) + 4xf'(x) + 2f(x)) = 1$$
find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} xf'(x)$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with positive second derivative. Prove
$$f(x + f'(x)) \geq f(x)$$
for any $x \in \mathbb{R}$.

9. Given a sequence of positive real numbers with limit zero, show that every non-empty interval (a, b) contains a non-empty subinterval (c, d) that does not contain any numbers equal to a sum of 2010 distinct elements of the sequence.