Math 149S Problem Set 7: Integration

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Problems may not be sorted in level of difficulty. You are NOT expected to solve every problem. Each problem is worth 10 points (if the proof is completely correct and well-written). Points maybe subtracted from a correct solution if the proof is badly written. Depending on the amount of progress, partial points maybe awarded.

This week, you need to also submit original problems for the Duke Math Meet. YOU NEED TO SUBMIT AT LEAST 2 Problems, but please submit as many as you can to help us run the Math meet. The problems can be of varying levels of difficulties. Possible topics include Euclidean geometry, counting, probability, basic number theory, sequences, etc (anything is fine but stay away from college topics such as linear algebra and advanced calculus). Try to be as creative as possible and come up with interesting, original problems. This means that you can take an existing problem and simply change the numbers. Each submitted problem is worth 7 points. For this week, you can submit fewer solutions and substitute with problems.

Problem 1. *Evaluate $\int_0^x \frac{d\theta}{2+\tan\theta}$ where $0 \le x \le \frac{\pi}{2}$. Use your result to show that*

$$\int_0^{\pi/4} \frac{d\theta}{2+\tan\theta} = \frac{\pi + \ln(9/8)}{10}$$

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Problem 2. *Evaluate*

$$\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}} dy dx$$

where a and b are positive.

Problem 3. *Define $C(\alpha)$ to be the coefficient of x^{2009} in the power series about $x = 0$ of $(1+x)^\alpha$. Evaluate*

$$\int_0^1 \left(C(-y-1) \sum_{k=1}^{2009} \frac{1}{y+k} \right) dy$$

Problem 4. *Compute*

$$\int_0^1 ((e-1)\sqrt{\ln(1+ex-x)} + e^{x^2})dx$$

Problem 5. *Let T be a solid tetrahedron whose edges all have length 1. Determine the volume of the region consisting of points which are at distance at most 1 from some point in T.*

Problem 6. *We can define the improper integral $\int_0^1 \int_0^1 \frac{1}{1-xy} dx dy$ as the limit of the integral over the region $[0,t] \times [0,t]$ as $t \to 1^-$. Show that*

$$\int_0^1 \int_0^1 \frac{1}{1-xy} dx dy = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Use this fact to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Problem 7. *Evaluate*

$$\int_0^1 \int_{\sqrt{y-y^2}}^{\sqrt{1-y^2}} x e^{(x^4+2x^2y^2+y^4)} dx dy$$

Problem 8. *Evaluate*

$$\int_0^1 \frac{\ln(x+1)}{x^2+1} dx$$

Problem 9. *Find the maximum value of*

$$\int_0^y \sqrt{x^4 + (y-y^2)^2} dx$$

for $0 \leq y \leq 1$.