Problem Set 9: Polynomials
Math 149S

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1. Solve the equation $z^8 + 4z^6 - 10z^4 + 4z^2 + 1 = 0$.

2. If $x < y < z$ are real numbers such that $x+y+z = 5$, $x^2+y^2+z^2 = 11$, and $x^3+y^3+z^3 = 26$, find y.

3. Solve the system

$$\begin{aligned} x + y + z &= 1 \\ xyz &= 1 \end{aligned}$$

   if x, y, z are complex numbers such that $|x| = |y| = |z| = 1$.

4. If a and b are integers, prove that the polynomial $(x-a)^2(x-b)^2 + 1$ is irreducible over the integers.

5. Prove that if $n \in \mathbb{N}$, the polynomial $P(x) = x^{2^n} + 1$ is irreducible over the integers.

6. Given that the polynomial $x^4 - 2x^2 + ax + b$ has four distinct real zeros, show that the absolute value of each zero is smaller than $\sqrt{3}$.

7. Find all polynomials P such that $P(x^2) = P(x)P(x-1)$.

8. Suppose $P(z)$ and $Q(z)$ are nonconstant polynomials with complex coefficients such that for any z, $P(z) = 0 \Leftrightarrow Q(z) = 0$ and $P(z) = 1 \Leftrightarrow Q(z) = 1$. Prove that the polynomials are equal.

9. Find all polynomials P such that $P(x^2 + x + 1) = P(x)P(x+1)$.

10. Let p be prime. Suppose $P(x)$ is a nonconstant polynomial such that for all integers $i = 0, 1, \ldots, p-1$, $P(i)$ equals 0 or 1. Prove that the degree of P is at least $p - 1$.