Solutions to Problem Set 0: Math 149S

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Some terms or concepts here may be unfamiliar to you. We will cover them later, but in the meantime you are strongly encouraged to google them! If you have any questions about these solutions or notice any errors, please contact Matthew Rognlie at matthew.rognlie@duke.edu.

Problem 1. *Solve the recurrence (give an explicit formula for $f(n)$):*

$$f(n+1) = nf(n) + (n-1)f(n-1) + \ldots + f(1) + 1,\ \ f(0) = 1$$

Solution. We claim that $f(n) = n!$, which we prove by induction. First note that the problem provides $f(0) = 1$ as a base case. For the inductive step, we can rewrite the equation above as:

$$\begin{aligned} f(n+1) &= nf(n) + ((n-1)f(n-1) + \ldots + f(1) + 1) \\ &= (n+1)f(n) \end{aligned}$$

If we assume that $f(n) = n!$, it follows that $f(n+1) = (n+1)n! = (n+1)!$, completing the induction.

Problem 2. *How many ways can one pick four numbers from the first fifteen positive integers, such that among the four numbers any two differ by at least 2? (from 102 Combinatorial Problems, by Andreescu and Feng)*

First Solution. Let $a_1 < a_2 < a_3 < a_4$ be our four chosen numbers. Consider the numbers $(b_1, b_2, b_3, b_4) = (a_1, a_2 - 1, a_3 - 2, a_4 - 3)$. Then b_1, b_2, b_3, b_4 are four distinct numbers from the first twelve positive integers. Conversely, from any set $b_1 < b_2 < b_3 < b_4$ of distinct positive integers no greater than twelve, we can write the inverse map $(b_1, b_2, b_3, b_4) \mapsto (b_1, b_2 + 1, b_3 + 2, b_4 + 3)$ and obtain a set of four numbers no greater than fifteen among which any two differ by at least two. We have found a bijection between the sets of four numbers satisfying our conditions and the sets of four distinct numbers from the first twelve integers, and the answer is therefore $\binom{12}{4} = 495$.

Second Solution. For $m, n \in \mathbb{N}$, let $f(m, n)$ be the number of ways we can pick n nonconsecutive integers from the set $\{1, 2, \ldots, m\}$. We claim that in general, $f(m, n) = \binom{m-n+1}{n}$. Note that $f(1, 1) = 1$ and $f(1, n) = 0$ for any $n > 1$. These values are consistent with our claim, and we will use them as base cases for induction on m to prove our claim.

For the induction step, we will prove that $f(m, n) = f(m-1, n) + f(m-2, n-1)$. Given some choice of n nonconsecutive numbers from the first m integers, either (1) one of the numbers selected is m, or (2) all the numbers selected lie in $1, 2, \ldots, m-1$. In case 1, the number of possibilities is equal to the number of ways we can select $n-1$ nonconsecutive integers from the first $m-2$ integers, or $f(m-2, n-1)$. In case 2, the number of possibilities is equal to the number of ways we can select n nonconsecutive integers from the first $m-1$ integers, or $f(m-1,n)$. This verifies our assertion. Now, assuming that our claim $f(m',n') = \binom{m'-n'+1}{n'}$ holds for all values $1,\ldots,m-1$ of m':

$$\begin{aligned} f(m,n) &= \binom{m-n}{n} + \binom{m-n}{n-1} \\ &= \binom{m-n+1}{n} \end{aligned}$$

The last equality, which completes the induction, is Pascal's rule for binomial coefficients. (Credit to Daniel Vitek and Misha Lavrov)

Problem 3.

Let M be a set endowed with an operation $*$ satisfying the properties:

(a) there exists an element $e \in M$ such that $x * e = x$ for all $x \in M$

(b) $(x * y) * z = (z * x) * y$ for all $x, y, z \in M$

Show that the operation $*$ is both commutative and associative. (from Putnam and Beyond, by Gelca and Andreescu)

Solution. First, substitute $y = e$ into (b) to obtain $x * z = z * x$ for any $x, z \in M$, which establishes commutativity. Now, we may use commutativity to switch the order of the terms on the left side of (b), obtaining $z * (x * y) = (z * x) * y$ for all $x, y, z \in M$. This is associativity.

Problem 4. *Prove that there is no triple of positive integers x, y, z satisfying:*

$$x^2 + y^2 + z^2 = 2xyz$$

Solution. Suppose that such a triple (x, y, z) exists. Then let 2^k be the highest power of 2 that divides x, y, and z simultaneously, and write $x = 2^k a, y = 2^k b, z = 2^k c$. Our equation is now $2^{2k}(a^2 + b^2 + c^2) = 2^{3k+1}abc$, and dividing out the common factor of 2^{2k} we obtain:

$$a^2 + b^2 + c^2 = 2^{k+1}abc$$

where at least one of a, b, c is odd. Since the right side of the equation is even, the left side must be as well, and the only remaining possibility is that exactly two of a, b, c are odd. Without loss of generality, assume that a and b are odd, and c is even. Then a^2 and b^2 are equal to 1 mod 4 while c^2 is equal to 0 mod 4, implying that entire expression on the left equals 2 mod 4. Since c is even, however, the expression on the right must be divisible by 4. Thus equality cannot hold, and we have reached a contradiction. No such triple x, y, z can exist.

Problem 5. *Let $n = 2^{21}3^9 5^{11}$. Find the number of positive integer divisors of n^2 that are less than, but do not divide, n. (from 102 Combinatorial Problems, by Andreescu and Feng)*

Solution. More generally, suppose $n = p^s q^t r^u$, where p, q, r are distinct primes. Then $n^2 = p^{2s} q^{2t} r^{2u}$, and it therefore has

$$(2s+1)(2t+1)(2u+1)$$

factors. Moreover, the map $a \mapsto \frac{n^2}{a}$ provides a bijection between factors of n^2 less than n and factors greater than n. We can use this bijection to pair off all factors of n^2 except n, and the number of factors of n^2 less than n is therefore:

$$\frac{(2s+1)(2t+1)(2u+1)-1}{2}$$

Meanwhile, the number of factors of n, excluding n itself, is

$$(s+1)(t+1)(u+1)-1$$

Taking the difference, we find that the number of factors of n^2 less than n that are not themselves factors of n is:

$$\begin{aligned}
&\frac{(2s+1)(2t+1)(2u+1)-1}{2} - ((s+1)(t+1)(u+1)-1) \\
&= 4stu + 2(st+tu+su) + (s+t+u) - stu - (st+tu+su) - (s+t+u) \\
&= 3stu + st + tu + su
\end{aligned}$$

Plugging in the values $s = 21, t = 9, u = 11$, we obtain the answer: 6756.

Problem 6. *Prove that every positive integer can be written as the sum of one or more integers of the form $2^s 3^t$, where s and t are nonnegative integers and no term in the sum is a multiple of another. (For example, 34 = 18 + 16.) (Problem A1, 2005 Putnam contest)*

Solution. We use strong induction, with base case $1 = 2^0 3^0$. Say that all positive integers less than n can be represented in the desired way. If n is even, we can simply multiply the representation for $n/2$ by 2 to obtain a satisfactory representation for n. Otherwise, if n is odd, let 3^k be the largest power of 3 that is no greater than n, and find a representation $\frac{n-3^k}{2} = a_1 + \ldots + a_m$. We propose the following representation for n:

$$n = 3^k + 2a_1 + \ldots + 2a_m$$

Since $a_1 + \ldots + a_m$ is already a valid representation, none of the $2a_i$ terms divide each other, or 3^k. Additionally, since $2a_i \leq n - 3^k < 3^{k+1} - 3^k = 3^k \cdot 2 \Rightarrow a_i < 3^k$, we find that 3^k cannot divide any of the $2a_i$ terms either. Thus we have found a representation for n in all cases, completing the induction.

Problem 7. *Find all polynomials whose coefficients are all equal to 1 or -1, and whose zeros are all real. (from Putnam and Beyond, by Gelca and Andreescu)*

Solution. Let $p(x)$ be such a polynomial. If $p(x)$ does not contain a constant term, its terms must have a common divisor of the form x^k, and we can divide the polynomial by x^k to obtain another polynomial $q(x)$ satisfying our conditions and containing a constant term. From now on, we will deal only with such polynomials $q(x)$.

Write $q(x)$ as the product $\prod_{i=1}^{n}(x - x_i)$, where $x_1, \ldots, x_n$ are the roots of $q(x)$ (guaranteed by assumption to be real). Expanding, we obtain:

$$x^n - \left(\sum_{1 \leq i \leq n} x_i\right) x^{n-1} + \left(\sum_{1 \leq i < j \leq n} x_i x_j\right) x^{n-2} - \ldots \pm \left(\prod_{i=1}^{n} x_i\right)$$

Denoting the coefficient on x^k by a_k:

$$a_{n-1}^2 = (x_1 + \ldots + x_n)^2 = \sum_{1 \le i \le n} x_i^2 + 2 \sum_{1 \le i < j \le n} x_i x_j = \sum_{1 \le i \le n} x_i^2 + 2a_{n-2}$$

Imposing the assumption that all coefficients are 1, we find:

$$\begin{aligned} \sum_{1 \le i \le n} x_i^2 &= a_{n-1}^2 - 2a_{n-2} \\ &\le 1 + 2 \\ &= 3 \end{aligned}$$

Now, applying the AM-GM inequality:

$$\begin{aligned} &\frac{1}{n} \sum_{1 \le i \le n} x_i^2 \ge \sqrt[n]{x_1^2 x_2^2 \ldots x_n^2} \\ \iff &\frac{1}{n} \sum_{1 \le i \le n} x_i^2 \ge 1 \\ \iff &\sum_{1 \le i \le n} x_i^2 \ge n \end{aligned}$$

where the second line follows from the fact that $x_1^2 x_2^2 \ldots x_n^2 = a_0^2 = 1$.

Combining results, we find that $n \le \sum_{1 \le i \le n} x_i^2 \le 3$, and therefore $n \le 3$. We can now proceed case-by-case, imposing a positive leading term for simplicity (multiplication by -1 preserves the desired properties and can be used to obtain the other solutions):

Degree 1: The only candidates are $x+1$ and $x-1$, both of which have a single real solution.

Degree 2: Candidates include x^2-1, x^2+1, x^2+x+1, x^2+x-1, x^2-x+1, and x^2-x-1. We must have $a_1^2 \ge 4a_2a_0$ for the solutions of a quadratic to be real, and this leaves us with x^2-1, x^2+x-1, and x^2-x-1.

Degree 3: Here we have equality in the AM-GM inequality we applied earlier, which implies that all roots x_i must have the same absolute value. Since the product of these roots has absolute value 1, they must each have absolute value 1. There are thus 4 possibilities: (1) all roots are 1, (2) two roots are 1 are one is -1, (3) one root is 1 and two are -1, and (4) all roots are -1. Multiplying out the product $\prod_i x_i$ reveals that only the middle two possibilities produce a polynomial where all coefficients are 1 or -1, and the resulting polynomials are x^3+x^2-x-1 and x^3-x^2-x+1.

We have classified all polynomials satisfying the conditions of the problem up to a factor of $\pm x^k$.

Problem 8. *Is it possible for a countably infinite set to have an uncountable collection of distinct subsets among which the intersection of any two subsets is finite? If so, provide an example and prove its validity. If not, prove that it is impossible. (Problem B4, 1989 Putnam)*

First Solution. Since the rationals $\mathbb{Q}$ are dense in the reals $\mathbb{R}$, for any $x \in \mathbb{R}$ we can find a sequence $\{a_n\}$ of rationals such that $a_n \to x$. Let this sequence be the countable set corresponding to each real; over all reals, this produces an uncountable collection of countable sets.

Furthermore, the sets corresponding to any two reals can only have finite intersection. To prove this, we consider $x, y \in \mathbb{R}$ and rational sequences $a_n \to x, b_n \to y$. Set $\epsilon = \frac{x-y}{2}$. Then there must be some N_1, N_2 such that for all $n \geq N_1$, $|x - a_n| < \epsilon$, and for all $n \geq N_2$, $|y - b_n| < \epsilon$. Let $N_3 = \max(N_1, N_2)$. Then the sets $(a_{N_3}, a_{N_3+1}, \ldots)$ and $(b_{N_3}, b_{N_3+1}, \ldots)$ are disjoint, implying that $\{a_n\}$ and $\{b_n\}$ only have finite intersection, as desired.

Second Solution. To each $x \in \mathbb{R}$, associate the set of its successive decimal approximations. (For instance, to $1/3$ associate $0, 0.3, 0.33, \ldots$) Since decimal approximations are rational, these sets are subsets of $\mathbb{Q}$, which is countable. Moreove, if x and y are different real numbers, eventually their decimal approximations must diverge: for some N, all approximations with more than N digits will be different from each other. This implies that each pair of subsets must have finite intersection, as desired. (Note that this solution is really just a specific implementation of the first solution. Credit to Misha Lavrov.)

Third Solution. Consider the countably infinite set $\mathbb{Z} \times \mathbb{Z}$ of lattice points on the plane. To every $x \in \mathbb{R}$ associate the following subset of $\mathbb{Z} \times \mathbb{Z}$:

$$\{(n, \lfloor xn \rfloor) \ : \ n \in \mathbb{Z}\}$$

Now consider the subsets corresponding to any two distinct reals x, y. If some element (n, m) is in both subsets, we must have $\lfloor xn \rfloor = \lfloor yn \rfloor$. For all n greater in absolute value than $\frac{1}{|x-y|}$, however, $|(x - y)n| > 1$, and $\lfloor xn \rfloor = \lfloor yn \rfloor$ is impossible. Thus these two subsets can only intersect in finitely many elements. (Credit to Misha Lavrov)