Induction

When nothing else seems to work...

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A Seemingly Intractable Problem

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Let $0 < a_1 < a_2 < \ldots < a_n$*, and let* $e_i = \pm 1$*. Prove that* $\sum_{i=1}^{n} e_i a_i$ *assumes at least* $\binom{n+1}{2}$ *distinct values as the* e_i *range over all* 2^n *possible choices of sign. (from Larson's Problem Solving Through Problems)*

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How can we possibly prove this? It looks really difficult! It's hard to imagine showing directly that there are at least $\binom{n+1}{2}$ values.

Making Life Easier

If we're trying to prove the statement for n, let's give ourselves more information and say that the statement is *already* true for $n-1$. In other words, for $0 < a_1 < a_2 < \ldots < a_{n-1}$, $\sum_{i=1}^{n-1} e_i a_i$ assumes at least $\binom{n}{2}$ distinct values.

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Now suppose some additional element a_n is given, with $a_n > a_{n-1}$. We already know that there are at least $\binom{n}{2}$ values of $\sum_{i=1}^{n-1} e_i a_i$. If we add $e_n = -1$, we will hence obtain $\binom{n}{2}$ distinct values of $\sum_{i=1}^{n} e_i a_i$. These values are all less than $S = \sum_{i=1}^{n} a_i$. Now consider the values:

$$S + a_n, S + (a_n - a_{n-1}), S + (a_n - a_{n-2}), \ldots, S + (a_n - a_1)$$

Since $0 < a_1 < a_2 < \ldots, a_n$, these n values are all distinct, and they are all also *greater* than S. We have now found a total of $\binom{n}{2} + n = \binom{n+1}{2}$ distinct values, as desired.

Turning This into a Proof

We showed that if the statement is true for n, it's also true for $n+1$. As long as we can show that the statement is true for $n = 1$, we can use this fact to prove it for *any* higher n: if it holds for $n = 1$, it holds for $n = 2$, and then it holds for $n = 3$, and so on.

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We are really using the *principle of mathematical induction*.

Principle of Mathematical Induction

The principle of mathematical induction states that if $P(n)$ is a proposition about natural numbers $n \geq a$, and if:

1. $P(a)$ is true, and
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This is actually a limited type of induction called "weak induction," although it's usually called "induction" when there is no ambiguity. We'll talk about "strong" induction later.

Another Example of Weak Induction

Problem

Let $x_0 = 1$ *and for* $n \geq 0$, *let* $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5} \rfloor$. *In particular,* $x_1 = 5, x_2 = 26, x_3 = 136, x_4 = 712$. *Find a closed form expression for* x_{2007}. *(Putnam B3, 2007)*

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Proof.

Based on the first few values, we guess that $x_n = 2^{n-1} F_{2n+3}$, where F_k is the kth Fibonacci number. Using the formula for Fibonacci numbers, this is equivalent to $x_n = \frac{2^{n-1}}{\sqrt{5}}(\alpha^{2n+3} - \alpha^{-(2n+3)})$, where $\alpha = \frac{1+\sqrt{5}}{2}$.

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We use induction. The base case $x_0 = 1$ is true, and it suffices to show that our formula for x_n satisfies the recursion $x_{n+1} = 3xn + \lfloor x_n\sqrt{5} \rfloor$. Using $\alpha^2 = \alpha + 1 = \frac{3+\sqrt{5}}{2}$, we find:

$$\begin{aligned} x_{n-1} - (3+\sqrt{5})x_n &= \frac{2^{n-1}}{\sqrt{5}}(2(\alpha^{2n+5} - \alpha^{-(2n+5)}) \\ &\qquad - (3+\sqrt{5})(\alpha^{2n+3} - \alpha^{-(2n+3)})) \\ &= 2^n\alpha^{-(2n+3)} \end{aligned}$$

Proof, continued

The recurrence we are trying to satisfy is $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5} \rfloor$, and we have shown:

$$x_{n-1} - (3+\sqrt{5})x_n = 2^n \alpha^{-(2n+3)}$$

We can rewrite this:

$$x_{n-1} - (3+\sqrt{5})x_n = \left(\frac{1-\sqrt{5}}{2}\right)^3 (3-\sqrt{5})^n$$

which is strictly between -1 and 0 for all n. The recursion follows.

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We conclude

$$x_{2007} = \frac{2^{2006}}{\sqrt{5}}(\alpha^{3997} - \alpha^{-3997})$$

What is Strong Induction?

Until now, we've declined to use some potentially valuable information. We've assumed that a statement is true for $n-1$ and proven that it's also true for n, but why not assume that the statement is true for *all* values $a, a+1, \ldots, n-1$?

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The *strong form* of the principle of mathematical induction states that if $P(n)$ is a proposition about natural numbers $n \geq a$, and:

1. $P(a)$ is true, and
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An aside: this is equivalent to the original principle of mathematical induction. If we define a new proposition $R(n)$ as true when $P(a), P(a+1), \ldots, P(n)$ are all true, then the original principle of mathematical induction on $R(n)$ gives us the strong form on $P(n)$.

Strong Induction: An Example

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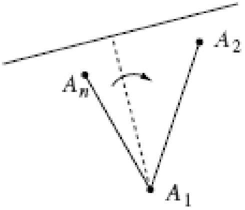
We use strong induction on the number of vertices. Our base case is the triangle, where there is nothing to prove. Assume that the property holds for all polygons with fewer than n vertices. We want to prove it for a polygon with n vertices. Any interior diagonal in this polygon will complete the inductive step, because it will divide the polygon into two polygons with fewer vertices. $\square$

Proof, continued

Since the sum of all n angles is $(n-2)\pi$, some angle must be less than π. Let the polygon be $A_1 A_2 \ldots A_n$, where is $\angle A_n A_1 A_2$ is our chosen interior angle. Rotate the ray $A_1 A_n$ toward $A_1 A_2$ inside the angle, and consider the point in the intersection of the ray and the polygon that is closest to A_1. If this is ever a vertex, we have obtained an interior diagonal. If not, $A_2 A_n$ is an interior diagonal.



An Alternative Approach

Say we want to prove the classic A.M.-G.M. inequality:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$

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We focus on a special case of the inequality, where $n = 2^k$. Why not attempt induction on k? Certainly the inequality holds for the base case $k = 0$.

Induction Step for the Special Case

Lemma

If the A.M.-G.M. inequality holds for $n = 2^{k-1}$, it holds for $n = 2^k$ as well.

Proof.

$$
\begin{aligned}
\frac{x_1 + x_2 + \cdots + x_{2^k}}{2^k} &= \frac{\frac{x_1+x_2+\cdots+x_{2^{k-1}}}{2^{k-1}} + \frac{x_{2^{k-1}+1}+x_{2^{k-1}+2}+\cdots+x_{2^k}}{2^{k-1}}}{2} \\
&\geq \frac{\sqrt[2^{k-1}]{x_1 x_2 \cdots x_{2^{k-1}}} + \sqrt[2^{k-1}]{x_{2^{k-1}+1} x_{2^{k-1}+2} \cdots x_{2^k}}}{2} \\
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&= \sqrt[2^k]{x_1 x_2 \cdots x_{2^k}}
\end{aligned}
$$

□

Downward Induction Step

Lemma

If the A.M.-G.M. inequality holds for n, it holds for $n-1$ as well.

Proof.

We know that $\frac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$. Now for any suitable $x_1, \ldots, x_{n-1}$, let $x_n = \frac{x_1+x_2+\cdots+x_{n-1}}{n-1}$. Then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}$$

and

$$\begin{aligned} \sqrt[n]{x_1 x_2 \cdots x_n} &= \sqrt[n]{x_1 x_2 \cdots \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}} \\ &\geq \sqrt[n]{x_1 x_2 \cdots \sqrt[n-1]{x_1 x_2 \cdots x_{n-1}}} \\ &= \sqrt[n-1]{x_1 x_2 \cdots x_{n-1}} \end{aligned}$$

$\square$

Finished Proof

We have shown by induction that the A.M.-G.M. inequality holds for all $n = 2^k$, and that if it holds for some n, it also holds for $n - 1$ and thus all $m \leq n$. Combining our results, we see that the A.M.-G.M. inequality holds for any positive m such that $m \leq 2^k$ for some k. We can pick k high enough to satisfy this condition for any m, and thus the A.M.-G.M. inequality is true in general.

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That was a strange induction! Note that we can also use the same basic results a little differently: if $n \to 2n$ and $n \to n - 1$, then we can immediately make a strong induction argument without talking about powers of 2 at all. The lesson of the approach we used here, however, is that we can induct over exponents, not just the numbers themselves.

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- Whenever you're stuck.

Remember, induction is **free**! If you need to prove some statement P for all $n \in \mathbb{N}$, and you can find a base case, when you're trying to prove $P(n)$ it can't hurt to think about the implications of P being true for all $1 \ldots n-1$.