Analysis

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What's in this talk?

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Next week Peng will cover problem-solving techniques in calculus, which is a branch of analysis; that lecture will focus much more on computation, while this will focus on theory.

Least upper bound property

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Note that in cases 1 and 3, the set contains its least upper bound, while in cases 2, 4, and 5 it does not.

Sup, inf, and all that

The least upper bound of a set S is denoted by $\sup S$. Similarly, there is also always a *greatest lower bound*, which we denote by $\inf S$. S sometimes, but not always, contains $\sup S$ or $\inf S$, although if S is finite we can be sure that it contains both.

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If S is a sequence $\{a_n\}_{n=1}^{\infty}$, then we also define:

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but we haven't defined limits yet!

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   Infinite limits are a little different: $\lim a_n = \infty$ if for any $y \in \mathbb{R}$ there is some N such that for all $n \geq N$, $a_n > y$. (In other words, for any y, there is some point past which the sequence is forever above y.) $\lim a_n = -\infty$ is defined similarly.

Infinite series

An infinite series

$$\sum_{n=1}^{\infty} b_n$$

converges if the limit of its partial sums

$$\lim_{k\to\infty} \sum_{n=1}^{k} b_n$$

converges as a sequence.

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1. We *could* just take some sequence $\{x_n\}$ with limit y and define $\lim_{x \to y} f(x)$ to be $\lim f(x_n)$. But how would we decide which sequence to pick? What if different sequences give different answers? For the limit to be well defined, we need to get the same answer no matter what sequence we choose. Thus we say that $\lim_{x \to y} f(x) = c$ if for *all* sequences $x_n \to y$, $f(x_n) \to c$.

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2. Here is another formal definition: we say that $\lim_{x \to y} f(x) = c$ if for every $\epsilon > 0$, we can find some $\delta > 0$ such that for all x such that $|x - y| < \delta$, $|f(x) - c| < \epsilon$. In other words, for any arbitrarily small ϵ, we can find some sufficiently small radius around y such that $f(x)$ is within ϵ of c for all x in that radius.

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If this limit exists, it will be the same as the limit of $f(x_n)$ for any sequence x_n with limit y.

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A function f that is continuous at x is **differentiable** at x if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is finite. If so, the limit is labeled $f'(x)$.

Open, Closed, Bounded, and Compact Sets

A subset $A \subset \mathbb{R}$ of the reals is **open** if for any point $x \in A$, we can find some $\delta > 0$ such that the set $B = \{y : |y - x| < \delta\}$ is a subset of A. In other words, A is open if for any point $x \in A$ we can find some "ball" around x contained within A.

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A subset $A \subset \mathbb{R}$ of the reals is **compact** if it is closed and bounded. Compactness is actually a more general property of topological spaces, defined in a more general way, but the **Heine-Borel Theorem** says that it is equivalent to being closed and bounded in real space.

Sequences

Squeeze Theorem

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$a_n \to a$ if and only if for any $\epsilon > 0$ we can find some N such that for all $m, n \geq N$, $|a_m - a_n| < \epsilon$.

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Sequential Compactness

Any sequence in a compact subset of the reals contains a convergent subsequence.

More on Sequences

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Cesaro-Stolz Theorem

Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers, where the y_n are positive, strictly increasing, and unbounded. If

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = L$$

then

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Cantor's Nested Intervals Theorem

If $I_1 \supset I_2 \supset \ldots$ is a decreasing sequence of closed intervals with lengths converging to zero, then $\cap_{n=1}^{\infty} I_n$ consists of one point.

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- The **geometric series** $\sum_{n=0}^{k} x^n$ has sum $\frac{1-x^k}{1-x}$. Taking $k \to \infty$, we see that this series converges if and only if $|x| < 1$, in which case the sum is $\frac{1}{1-x}$.

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- The **p-series** $\sum_{n=0}^{\infty} n^p$ converges for $p > 1$ (assuming p is positive). It notably *does not* converge when $p = 1$, where it is $1 + \frac{1}{2} + \frac{1}{3} + \dots$.

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- The **p-series** $\sum_{n=0}^{\infty} n^p$ converges for $p > 1$ (assuming p is positive). It notably *does not* converge when $p = 1$, where it is $1 + \frac{1}{2} + \frac{1}{3} + \dots$.

One of the best ways to determine whether a series converges is to use an adaptation of the squeeze theorem for sequences called the **comparison test**. If each term of a series with positive terms is less than the corresponding term of another series that we know to converge, we can conclude that it converges. Alternatively, if each term of a series with positive terms is greater than the corresponding term of another series we know to diverge, we can conclude that it diverges.

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- **Ratio Test.** Letting $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, we can conclude that the series converges if $L < 1$ and diverges if $L > 1$; the common case $L = 1$ is ambiguous.

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- **Root Test.** Letting $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$, we can conclude that the series converges if $L < 1$ and diverges if $L > 1$; again, the $L = 1$ case is ambiguous.

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- **Root Test.** Letting $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$, we can conclude that the series converges if $L < 1$ and diverges if $L > 1$; again, the $L = 1$ case is ambiguous.
- **Integral Test.** If $|a_n|$ is monotone decreasing, and $|a_n| = f(n)$, where f is some monotone decreasing continuous function on the interval $[0, \infty)$, then $\sum_{n=0}^{\infty} |a_n|$ converges if and only if the integral $\int_0^{\infty} f(n)$ is finite.

Alternating Series

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Important fact: rearranging the terms in an alternating series, or indeed *any* series, will only keep the sum the same *if* the series converges absolutely. Otherwise, rearranging the terms may change the sum, or even cause a previously convergent series to diverge.

Functions and Continuity

Squeeze Theorem for Functions

If f, g, h are functions defined on some interval I such that $g(x) \leq f(x) \leq h(x)$ for all $x \in I$, then if $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$.

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If f is continuous on the interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum $f(d)$ at some numbers c and d in $[a, b]$.

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Some Basic Topology

If f is continuous and A is open, then the inverse image $f^{-1}(A)$ is also open.

Differentiable Functions

Mean Value Theorem

If f is continuous on $[a, b]$ and is differentiable on (a, b), then there exists some $c \in (a, b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

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L'Hopital's Rule

Let f, g be differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Similarly, if $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, we also have $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

More on Differentiable Functions

Increasing and decreasing functions

Let f be a function on some interval $[a, b]$. If $f'(x) > 0$ for all $x \in (a, b)$, f must be strictly increasing on $[a, b]$. If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is nondecreasing on $[a, b]$. The opposite holds for negative first derivatives.

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Convexity

If $f''(x) \geq 0$ for all $x \in [a, b]$, then f is **convex** on that interval and we have for any $\alpha \in [0, 1]$, $a \leq c \leq d \leq b$:

$$f(\alpha c + (1 - \alpha)d) \leq \alpha f(c) + (1 - \alpha)f(d)$$

We can then apply Jensen's inequality, which states that for any nonnegative $w_1, \ldots, w_n$, $\sum_i w_i = 1$, and $x_1, \ldots, x_n \in [a, b]$:

$$w_1 f(x_1) + \ldots + w_n f(x_n) \geq f(w_1 x_1 + \ldots + w_n x_n)$$

The opposite inequalities hold if $f''(x) \leq 0$.

Even More Differentiable Function Stuff

Extrema

If f is differentiable on (a, b), then the maximum and minimum of f on $[a, b]$ either lie at the endpoints a and b or satisfy $f'(x) = 0$. If $f''(x) < 0$ as well, then x is a maximum; if $f''(x) > 0$, then x is a minimum.

Fundamental Theorem of Calculus

Let f be a continuous real-valued function on some interval $I \subset \mathbb{R}$ and let $a \in I$. If $F(x) = \int_a^x f(t)\, dt$ for all $x \in I$, then F has a continuous first derivative equal to f.