# *The Pigeonhole Principle*

*Simple but immensely powerful*

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# The Basic Principle

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## The principle

If $m$ pigeons are in $n$ holes and $m > n$, then at least 2 pigeons are in the same hole. In fact, at least $\lceil \frac{m}{n} \rceil$ pigeons must be in the same hole.

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- Among three persons, two must be of the same gender.
- If there are 16 people and 5 possible grades, 4 people must have the same grade.
- Since nobody has more than $400,000$ hairs on their head, New York city must have at least twenty people with exactly the same number of hairs on their heads.

# Trivial Applications

## Example 1

Prove that in any group of six people there are either three mutual friends or three mutual strangers. (Assume that friendship is always reciprocated: two people are either mutual friends or mutual strangers.)

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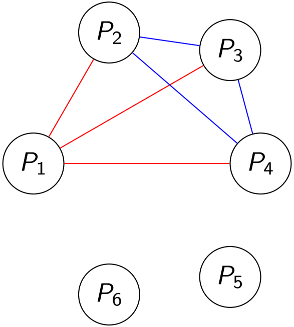
*Proof.*




Represent the people as nodes on a graph, and denote friendships using red edges and "stranger-ship" using blue edges. We have to show that there exists a monochromatic triangle. Consider the relationship of $P_1$ to the 5 others. By the pigeonhole principle, 3 of the others must have the same relationship to person 1. Without loss of generality, say $P_2$,$P_3$,$P_4$ are connected to $P_1$ by red edges. Consider the edges between $P_2$, $P_3$, and $P_4$. If any of them are red, then we have a red triangle. Otherwise we have a blue triangle. □

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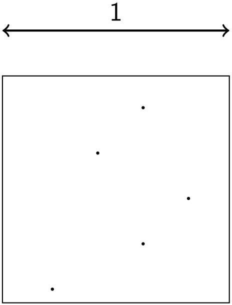
If we do this correctly, the proof should be slick. Otherwise, the problem may seem forbiddingly difficult.

When stuck, do not give up so easily! You learn and improve the most when you are stuck. Keep thinking of possible approaches, perhaps for a few hours, and you might be rewarded with an elegant solution. This is the ONLY way to learn mathematical problem solving.

# Geometric Example

## Example 2

Consider any five points $P_1, \cdots, P_5$ in the interior of a square $S$ of side length 1. Show that one can find two of the points at distance at most $\sqrt{2}/2$ apart.



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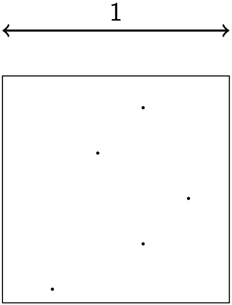


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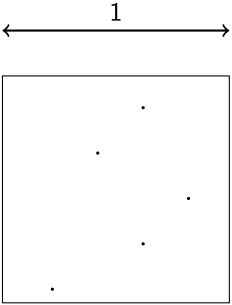


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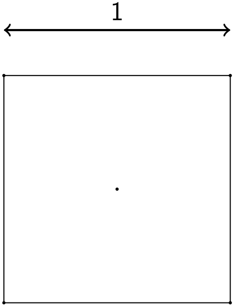


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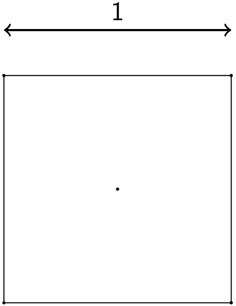


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When the points are in this configuration, the distances from the center point to each of the other points is exactly $\sqrt{2}/2$. Intuitively, this is as far as the points can be apart. So are we done? But this is not a proof!

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This class is as much about refining your proof-writing skills as training you to be better problem-solvers. Your proofs will be strictly examined, with perhaps an annoying amount of attention to details. A correct but badly written proof will not receive full marks!

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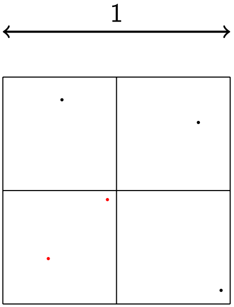


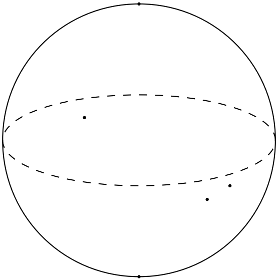

## Proof.

Consider partitioning the square into 4 subsquares as shown. By the pigeonhole principle, 2 of the points must be in one "sub-box." The distance between those two points must be less than the diameter of the sub-box: $\sqrt{2}/2$ (the length of the sub-box's diagonal). This proves the desired result. □

# More Careful "Hole" Construction

## Problem

*Suppose that 5 points lie on a sphere. Prove that there exists a closed semi-sphere (half a sphere including boundary), which contains 4 of the points.*



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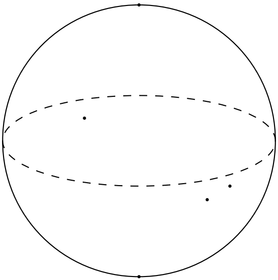

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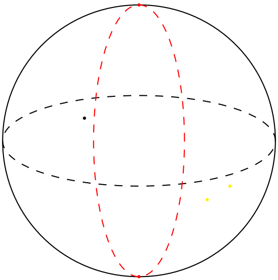

How would you construct the “holes”?

*Proof.*

Consider the great circle through any two of the points. This partitions the sphere into two hemispheres. By the pigeonhole principle, 2 of the remaining 3 points must lie in one of the hemispheres. These two points, along with the original two points, lie in a closed semi-sphere. □

# Clever Construction of Pigeons and Holes

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*A chessmaster has 77 days to prepare for a tournament. He wants to play at least one game per day, but not more than 132 games. prove that there is a sequence of successive days on which he plays exactly 21 games.*

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## Problem

*A chessmaster has 77 days to prepare for a tournament. He wants to play at least one game per day, but not more than 132 games. prove that there is a sequence of successive days on which he plays exactly 21 games.*

## Proof.

Define $S_i$ $(1 \leq i \leq 77)$ as the total number games the chessmaster plays from day 1 up to day $i$. Because she plays at least one game a day, $1 \leq S_1 < S_2 < \cdots < S_{77} \leq 132$ (*i.e.* the $S_i$'s are distinct). Define $T_i = S_i + 21$. Note that the $T_i$'s are all distinct. Now, out of the $S_i$'s and $T_i$'s, there are $77 \times 2 = 154$ numbers, but these numbers can take at most $132 + 21 = 153$ possible values. By the pigeonhole principle, two of the numbers are equal. This implies that for some $i$, $j$, $S_j = T_i = S_i + 21$. Hence, the chessmaster plays exactly 21 games in the consecutive block from day $i + 1$ to day $j$. □

# One Commonly Used Idea

## Problem

*Let $\alpha$ be a positive real number and n be an arbitrary integer. Prove that there exists integer pairs $(h, k)$, with $1 < k \leq n$ such that*

$$|0 \leq k\alpha - h| < \frac{1}{n}$$

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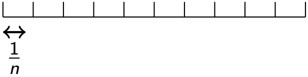


## Proof.

Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of $x$. It suffices to show that there exists integer $k \leq n$ such that $0 \leq \{\alpha\} < 1/n$. Partition $I = [0, 1)$ into $I_i = [i/n, (i+1)/n)$, for $0 \leq i \leq n-1$. Consider the numbers $\{\alpha\}, \{2\alpha\}, \cdots, \{n\alpha\}$. If any of them fall into $I_0$ or $I_{n-1}$, then we are done. Suppose on the contrary that none does, then by the pigeonhole principle, there exists $j < k$, $I_i$ such that both $\{j\alpha\} \in I_i$, $\{k\alpha\} \in I_i$. But $\{(k-j)\alpha\}$ must fall into $I_0$, or $I_{n-1}$, contradiction. This completes the proof. □

# Surprising Application of Pigeonhole

## Problem (Fermat)

*Show that every prime number of the form* $p \equiv 1 \pmod 4$ *can be written as a sum of squares of two integers.* (i.e. $p = a^2 + b^2$.)

# Surprising Application of Pigeonhole

## Problem (Fermat)

*Show that every prime number of the form $p \equiv 1(\mod 4)$ can be written as a sum of squares of two integers. (i.e. $p = a^2 + b^2$.)*

## Proof.

Let $p$ be such a prime, it is simple to show that $\exists x$ s.t. $x^2 \equiv -1(\mod p)$. (We will explain these techniques in a later lecture.)
It suffices to show that $\exists$ integer pair $(u, v)$, such that $u + xv \equiv 0(\mod p)$ and $|u|, |v| \leq \sqrt{p}$. The desired result would then follow because $u^2 + v^2 \equiv 0(\mod p)$, and $0 < u^2 + v^2 < 2p$.
Suppose on the contrary that such a pair does not exist, Consider all pairs $(u, v) \neq (0, 0)$, where $-\sqrt{p} \leq u, v \leq \sqrt{p}$. There are at least $(2\lfloor p \rfloor + 1)^2 - 1 \geq 4p$ such pairs. Consider all numbers of the form $u + vx$ (pigeons). Since there $p - 1$ possible residues mod $p$ (holes), by the pigeonhole principle, there must exist pairs $(u, v) \neq (u', v')$ such that $u + xv \equiv u' + xv'(\mod p)$. But $(|u - u'|, |v - v'|)$ is a pair satisfying the desired condition. Contradiction. □

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When you work on Problem set 1, you can collaborate, but you must write the proofs yourselfs. (No plagiarism!) We hope you enjoy the problems!