History dependence
Appendix to "Investment and liquidation in renegotiation-proof contracts with moral hazard"

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In these pages I extend the model to allow for history dependence. I still assume that the shock takes values in the set $\eta \in [\underline{\eta}, \bar{\eta}]$ and is observed only by the entrepreneur. However, the probability density depends on a variable z that follows a finite state Markov chain with transition probabilities $\Gamma(z/z_{-1})$. The density function for the shock is denoted by $g(z,\eta)$ and it is assumed that $g(z,\eta)$ is stochastically dominated by $g(\hat{z},\eta)$ if $z < \hat{z}$. I refer to z as the "persistent factor". Notice here the notational and timing assumptions. The previous realization of the persistent factor is denoted by z_{-1} while its new realization is denoted by z. The previous realization, z_{-1}, affects the probability distribution for z, and therefore, the density function of the shock $g(z,\eta)$.

I will consider two cases about the observability of z. In the first case the persistent factor z is observed by both parties (public information) while in the second case z is observed only by the entrepreneur (private information).

1 The persistent factor z is public information

When z is public information, the analysis of the previous sections can be easily extended to this case. The contract determines the sequences of investment, liquidation probabilities and payments as functions of the history. The history, conditional on survival, is now defined as the sequence of realizations of z and announcements of η, that is, $\mathbf{h}^t = \{z_0, z_1, \hat{\eta}_1, ..., z_t, \hat{\eta}_t\}$. The contractual problem is formulated recursively by adding a new state, that is, the previous realization of the persistent factor, z_{-1}. After allowing the lower bound q_{min} to depend on z, Proposition 1 becomes:

PROPOSITION 5 *There exists* $\underline{q}(z)$ *and* $\bar{q}(z)$*, with* $q_{min}(z) \leq \underline{q}(z) < \bar{q}(z)$*, such that:*

(a) The schedule $\tilde{q}(z,\eta)$ *is equal to* $\tilde{q}(z,\underline{\eta}) + h'(0)[F(k,\eta) - F(k,\underline{\eta})]$*, with* $\tilde{q}(z,\underline{\eta}) \geq q_{min}(z)$*.*

(b) For each z_{-1}*,* $S(z_{-1},q)$ *is strictly increasing and concave for* $q < \bar{q}(z_{-1})$*, constant for* $q \geq \bar{q}(z_{-1})$*, and differentiable.*

(c) The input of capital is at the optimal level $\bar{k}(z_{-1})$ *if* $q \geq \bar{q}(z_{-1})$*.*

(d) If $\tilde{q}(z,\eta) < \underline{q}(z)$*, then* $p(z,\eta) > 0$*,* $c(z,\eta,\ell) = 0$*,* $q(z,\eta,1) = 0$ *and* $q(z,\eta,0) = \underline{q}(z)$*.*

(e) If $\underline{q}(z) \leq \tilde{q}(z,\eta) < \bar{q}(z)$*, then* $p(z,\eta) = 0$*,* $c(z,\eta,0) = 0$ *and* $q(z,\eta,0) = \tilde{q}(z,\eta)$*.*

(f) If $\tilde{q}(z,\eta) \geq \bar{q}(z)$*,* $p(z,\eta) = 0$ *but there are multiple solutions to* $c(z,\eta,0)$ *and* $q(z,\eta,0) \geq \bar{q}(z)$*.*

PROOF: *Simple extension of the proof of Proposition 1.*

The only difference respect to the case of i.i.d. shocks is that the policy functions are contingent on z in addition to the shock η. Given the properties of the long-term contract, it is also easy to show that under certain conditions the long-term contract is not free from renegotiation and that the renegotiation-proof contract can be defined by imposing lower bounds to the values of $\tilde{q}(z,\eta)$. The equivalent of Propositions 2 and 3 are:

PROPOSITION 6 *Let* $q_{min}(z) = 0$ *for all* z*. If* κ *is sufficiently small, there exists* $\underline{\underline{q}}(z)$ *for some* z*, for which* $\tilde{S}_{\tilde{q}}(z,\underline{\underline{q}}) > 1$*. Moreover, for all* $q \in [\underline{\underline{q}}(z), \bar{q}(z))$*, there is a positive probability that* $\tilde{q}(z,\eta) < \underline{\underline{q}}(z)$ *at some future date.*

PROPOSITION 7 *There exists* $\underline{q_{min}}(z)$ *for which the renegotiation-proof contract is derived by imposing* $q_{min}(z) = \underline{q_{min}}(z)$ *for all* z *in the long-term contract.*

PROOF: *Simple extension of the proofs of Propositions 2 and 3.*

The properties of the optimal and renegotiation-proof contract emphasized in Section 7 also extend to the case of history dependence, albeit with some qualifications.

Property 1 (Cash-flow sensitivity) *Controlling for z, the investment of constrained firms depends on cash-flows, while the investment of unconstrained firms is independent of cash flows.*

Notice that the investment of unconstrained firms still depends on cash-flows if we do not control for z because the persistent factor affects the next period unconstrained capital, that is, $k' = \bar{k}(z)$.

With history dependence the Tobin's q is no longer a sufficient statistic for the investment of constrained firms. To show this, consider two firms that have the same z_{-1} and q. This implies that these two firms employ the same input of capital. Now suppose that in the next period the first firm gets a high realization of η but a low realization of z. The second firm, instead, gets a low realization of η and a high realization of z. Therefore, the cash flow of the first firm is bigger than for the second firm because $F(k,\eta^1) > F(k,\eta^2)$, Furthermore, let's assume that these realizations of the shock and persistent factor are such that the two firms have the same Tobin's q. Formally, $[F(k,\eta^1) + \tilde{S}(z^1, \tilde{q}(\eta^1))]/(I_0 + k) = [F(k,\eta^2) + \tilde{S}(z^2, \tilde{q}(\eta^2))]/(I_0 + k)$. Even if they have the same Tobin's q, the two firms will choose different investments. Therefore, cash-flows provide further information about the investment of the firm beyond the Tobin's q. For unconstrained firms, instead, the new investment depends only on z and the Tobin's q is a sufficient statistic.

The reason the Tobin's q is no longer a sufficient statistic for constrained firms is because they are now affected by two shocks: z and η. Because these shocks are not perfectly correlated, they have a differential impact on the investment decision of the firm which cannot be summarized by a single variable, that is, the Tobin's q. In this respect the cash-flow sensitivity of investment resembles the results of Abel & Eberly (2002). In that model, even if there are no financial frictions, cash-flows have an additional explanatory power because the firm is affected by multiple shocks.

The result of Abel & Eberly (2002) points out that the cash-flow sensitivity of investment is not necessarily a good proxy for the existence of financial constraints. At the same time, the absence of cash-flow sensitivity (once we control for the Tobin's q) does not imply that the firm is financially unconstrained. In fact, we have seen in Section 7 that the investment of constrained firms is fully explained by the Tobin's q when shocks are i.i.d. These remarks about the inability of the cash-flows sensitivity to signal the presence of financial constraints parallel earlier results by Gomes (2001).

Property 2 (Liquidation pattern) *For any value of z_{-1}, constrained firms face a higher probability of liquidation at some future date than unconstrained firms.*

With i.i.d. shocks, unconstrained firms would never be liquidated. With persistent shocks, however, the expected productivity of the firm may become so low that the expected future profits are smaller than the liquidation value κ. This would depend not only on the current value of the persistent factor z_{-1} but also on its persistence. However, even if the probability of liquidation for unconstrained firms is positive, this probability is smaller than for constrained firms. More importantly, if we control for the future production capability and market opportunities of the firm—that is, the persistent factor z—there are values of z_{-1} for which unconstrained firms are never liquidated while constrained firms will be liquidated with positive probability at some future period. Unconstrained firms will be liquidated only if they experience a large fall in z.

Property 3 (Investor share) *For each z_{-1}, the investor's share of the surplus is strictly decreasing in $q \in (\underline{q}(z_{-1}), \bar{q}(z_{-1}))$.*

This is because in a renegotiation-proof contract the slope of S cannot be greater than 1.

Property 4 (Investment volatility) *Controlling for z_{-1}, constrained firms face higher volatility of investment and growth than unconstrained firms.*

This property derives directly from the cash-flows sensitivity (property 1): if we control for z_{-1}, the investment of unconstrained firms remain constant independently of the realization of η. Therefore, they do not experience any volatility of investment and growth. The investment of constrained firms, instead, depends on q' which in turn depends on the realization of the shock.

2 The persistent factor z is private information

The analysis of the previous section can not be easily extended to the case in which the persistent factor z is private information. In this case the contract must be structured such that the entrepreneur reveals the true realizations of both z and η. The source of complication derives from the fact that, if the entrepreneur falsely reports z, the probability distribution of the shock believed by the investor in the subsequent period differs from the true distribution. To make the contract incentive-compatible further restrictions need to be imposed.

The history, conditional on survival, is now defined as the sequence of announcements for z and η, that is, $\mathbf{h}^t = \{z_0, \hat{z}_1, \hat{\eta}_1, ..., \hat{z}_t, \hat{\eta}_t\}$. The initial z_0 is assumed to be public information. The analysis follows Fernandes & Phelan (2000) who show that the contractual problem can be formulated recursively by enlarging the set of state variables. To simplify the analysis I assume that z takes only two values, that is, $z \in \{z_1, z_2\}$. However, it can be easily extended to the case in which z takes more than two values by adding further state variables. (See Fernandes & Phelan (2000) and Doepke & Townsend (2001)).

Define w the value of the contract for the entrepreneur when he or she has reported a value of the persistent factor different from the true realization. Because the shock can take only two values, the misreported value of the persistent factor is the complement of the true z. The complement value will be denoted by z^c. The variable w is conditional on the survival of the firm, after the current payment to the entrepreneur. This is the "threat value". The long-term contractual problem is represented by the following mapping:

$$T(S)(z_{-1}, q, w) = \max_{k, \tilde{q}(z,\eta), \tilde{w}(z,\eta)} \left\{ -k + \beta E_{z_{-1}} F(k, \eta) + \beta E_{z_{-1}} \tilde{S}\Big(z, \tilde{q}(z,\eta), \tilde{w}(z,\eta)\Big) \right\} \tag{1}$$

subject to

$$\tilde{q}(z,\eta) \geq D(k, \eta, \hat{\eta}) + \tilde{q}(z, \hat{\eta}), \qquad \text{for all } \eta, \hat{\eta}, z \tag{2}$$

$$\tilde{q}(z,\eta) \geq \tilde{w}(z^c, \eta), \qquad \text{for all } \eta, z, z^c \tag{3}$$

$$q = \beta E_{z_{-1}} \tilde{q}(z,\eta) \tag{4}$$

$$w = \beta E_{z^c_{-1}} \tilde{q}(z,\eta) \tag{5}$$

$$\tilde{q}(z,\eta) \geq q_{min}(z), \quad \tilde{w}(z,\eta) \geq 0 \tag{6}$$

$$\tilde{S}(z,\tilde{q},\tilde{w}) = \max_{p,c_0,c_1,q',w'} \Big\{ p\cdot\kappa + (1-p)\cdot S\left(z,q',w'\right) \Big\} \tag{7}$$

subject to

$$\tilde{q} = p\cdot c_1 + (1-p)\cdot(c_0+q') \tag{8}$$

$$\tilde{w} = p\cdot c_1 + (1-p)\cdot(c_0+w') \tag{9}$$

The contractual problem is divided in two sub-problems. The first sub-problem is solved before the realization of the shock and before randomizing on liquidation. The decision variables are the input of capital k, the next period promised value $\tilde{q}(z,\eta)$ and the next period threat value $\tilde{w}(z,\eta)$, which are conditional on the announcement of z and η.

Constraints (2) and (3) impose incentive-compatibility: the first insures that there is not diversion of resources while the second insures that the true value of z is reported: if the entrepreneur reports the false value $z^c \neq z$, the value of the contract will be $\tilde{w}(z^c,\eta)$ and this must be smaller than the value received by truthfully reporting z, that is, $\tilde{q}(z,\eta)$. Constraints (4) and (5) are the promised-keeping constraints for q and w. Notice that the expectation in equation (5) is conditional on the complement value of the persistent factor z^c_{-1}. This is because w is the value of the contract for the entrepreneur if the reported z_{-1} is different from the true value. Given the schedule $\tilde{q}(z,\eta)$, if the value reported by the entrepreneur z_{-1} is false, then his or her value will be equal to $w = \beta E_{z^c_{-1}}\tilde{q}(z,\eta)$.

The second sub-problem is solved after the announcement of z and η. It chooses the probability of liquidation, p, consumption if the firm is liquidated, c_1, consumption and continuation values if the firm is not liquidated, that is, c_0, q' and w'. I have implicitly assumed that in case of liquidation the promised value to the entrepreneur is zero. Of course, given a certain value promised in case of liquidation, it becomes indifferent whether this value is paid immediately or in future periods. The following proposition establishes some properties of the long-term contract.

Proposition 8 (Long-term contract) *There exist $\underline{q}(z_{-1},w)$ and $\bar{q}(z_{-1},w)$ such that:*

(a) $S(z_{-1},q,w)$ is strictly concave in q,w for $q<\bar{q}(z_{-1},w)$ and constant for $q\geq\bar{q}(z_{-1},w)$.

(b) For $q\geq\bar{q}(z_{-1},w)$, the input of capital is always at the optimal level $\bar{k}(z_{-1})$.

(c) If $\tilde{q}(z,\eta) < \underline{q}(z,\tilde{w})$*, then* $p(z,\eta) > 0$ *and* $q' = \underline{q}(z,\tilde{w})$*.*

(d) If $\underline{q}(z,\tilde{w}) \leq \tilde{q}(z,\eta) < \bar{q}(z,w)$*, then* $p(z,\eta) = 0$*.*

(e) If $\tilde{q}(z,\eta) \geq \bar{q}(z,w)$*, then* $p(z,\eta) = 0$ *and* $q' \geq \bar{q}(z,w)$*. However,* c_0 *and* q' *are not determined.*

PROOF: *See appendix A.*

Several properties of the optimal long-term (commitment) contract are maintained within this framework. In particular, it is still the case that the firm is constrained when the promised value q is small and it becomes unconstrained once q is sufficiently large. Moreover, the firm faces a probability of liquidation if the promised value $\tilde{q}(z,\eta)$ falls below a certain threshold. This threshold, however, depends also on the threat value $\tilde{w}$, in addition to z.

For each value of z_{-1}, the optimal contract is now characterized by two state variables: the promised value q and the threat value w. An "efficient" long-term contract generates an initial surplus given by $S^{Eff}(z_{-1},q) = \max_w S(z_{-1},q,w)$. The properties of the S function guarantee that $S^{Eff}(z_{-1},q)$ is increasing, concave and there is some $\underline{\underline{q}}(z_{-1})$ for which the slope of S^{Eff} is equal to 1 for $q = \underline{\underline{q}}(z_{-1})$ and strictly smaller than 1 for $q > \underline{\underline{q}}(z_{-1})$. Therefore, optimal long-term contracts are initially defined for $q \geq \underline{\underline{q}}(z_{-1})$.

Even if we start with $q \geq \underline{\underline{q}}(z_{-1})$, there is no guarantee that $\tilde{q}(z,\eta)$ will be above $\underline{\underline{q}}(z)$, which may lead to renegotiation. This possibility of renegotiation is similar to the case of i.i.d. shocks. With history dependence, however, there is also another source of renegotiation. Even if $\tilde{q} = \tilde{q}(z,\eta)$ will be above $\underline{\underline{q}}(z)$, $\tilde{w}$ may be different from $\arg\max_{\tilde{w}} \tilde{S}(z,\tilde{q},\tilde{w})$. To make the contract renegotiation-proof, it is not sufficient to impose lower bounds to $\tilde{q}(z,\eta)$ as done in the case in which z is public information. The values of $\tilde{w}(z,\eta)$ also need to be constrained for each value of z and $\tilde{q}$. More specifically, the constraint $\tilde{w}(z,\eta) \geq 0$ must be replaced by

$$\tilde{w}(z,\eta) \;\geq\; w_{min}(z,\tilde{q}) \tag{10}$$

If a renegotiation-proof contract exists, this contract will be characterized by lower bound functions $\underline{q_{min}}(z)$ and $\underline{w_{min}}(z,\tilde{q})$. After imposing these constraints, there will be a correspondence that relates values of w to values of q at any history contingency. This correspondence is given by $\psi(z_{-1}, q) = \arg\max_w S(z_{-1}, q, w)$. By the definition of a renegotiation-proof contract, future values of w are always in the set $\psi(z_{-1}, q)$.

In this paper I do not establish the existence of the renegotiation-proof, that is, I do not establish the existence of the lower bound functions $\underline{q_{min}}(z)$ and $\underline{w_{min}}(z, \tilde{q})$. However, if this contract exists, it is easy to show that the properties of such contract are similar to the properties established in the previous subsection for the case in which the persistent factor is public information.

Appendix

A Proof of Proposition 8

First notice that in sub-problem (7) there is an indeterminacy in the choice of consumption conditional on liquidation. To show this, suppose that the solution is p, c_1 and c_0. Now consider $\hat{c}_1 \neq c_1$ and $\hat{c}_0 \neq c_0$ that satisfy $pc_1 + (1-p)c_0 = p\hat{c}_1 + (1-p)\hat{c}_0$. Obviously this is also a solution. What matters is the expected consumption. Therefore, instead of maximizing over c_1 and c_0, we maximize over the expected value of consumption $c = pc_1 + (1-p)c_0$, and without loss of generality the last two constraints of sub-problem (7) can be replaced by:

$$\tilde{q} = c + (1-p)q' \tag{11}$$

$$\tilde{w} = c + (1-p)w' \tag{12}$$

It can be verified that the mapping T satisfies the Blackwell conditions for a contraction. Therefore, there is a unique fixed point S. Now let's show that there is $\bar{q}(z_{-1}, w)$ for which, once the firm reaches this value, capital is always at the optimal level $\bar{k}(z_{-1})$. Consider the following contract. In every period the investor pays to the entrepreneur the transfer $\max_{z_{-1}}\{\bar{k}(z_{-1})\} - \min_{z_{-1}}\{F(\bar{k}(z_{-1}), \underline{\eta})\}$ and the entrepreneur retains all the revenues. Basically, the entrepreneur gets paid the maximum possible losses that the firm can realize independently of whether these losses are effectively realized. Because the transfers are independent of the shock and, once added to the revenues, they are sufficient to finance the optimal input of capital, the contract that invests $\bar{k}(z_{-1})$ in all periods is incentive compatible. The value of this contract for the entrepreneur depends on z_{-1} and w and it is denoted by $\bar{q}(z_{-1}, w)$. If q is bigger than this upper bound, the surplus does not change. It simply entitles the entrepreneur to higher transfers from the investor. Therefore, for $q > \underline{q}(z_{-1}, w)$, the surplus only depends on z_{-1} and w.

I show now that T maps concave functions of q, w into concave functions and the fixed point S is concave. To see this, consider first sub-problem (7). If S is concave in q', w' for each value of z, then $\tilde{S}$ is also concave in $\tilde{q}, \tilde{w}$.

Take $(\tilde{q}_1, \tilde{w}_1)$ and $(\tilde{q}_2, \tilde{w}_2)$ and assume that the respective solutions are p_1, c_1, and p_2, c_2. Denote by $\tilde{q}_\theta$, $\tilde{w}_\theta$, p_θ and c_θ the linear combinations of these points. Because S is concave, there exists a linear function $H(z, \tilde{q}, \tilde{w})$ such that $S(z, \tilde{q}, \tilde{w}) = H(z, \tilde{q}, \tilde{w})$ for $\tilde{q} = (\tilde{q}_\theta - c_\theta)/(1-p_\theta)$ and $\tilde{w} = (\tilde{w}_\theta - c_\theta)/(1-p_\theta)$ but $S(z, \tilde{q}, \tilde{w}) \leq H(z, \tilde{q}, \tilde{w})$ for $\tilde{q} \neq (\tilde{q}_\theta - c_\theta)/(1-p_\theta)$ and/or $\tilde{w} \neq (\tilde{w}_\theta - c_\theta)/(1-p_\theta)$. In other words, the function

H is the hyperplane above S. Then,

$$\begin{aligned}
\tilde{S}(z, \tilde{q}_\theta, \tilde{w}_\theta) &\geq p_\theta \kappa + (1-p_\theta) S(z, (\tilde{q}_\theta - c_\theta)/(1-p_\theta), (\tilde{w}_\theta - c_\theta)/(1-p_\theta)) \\
&= p_\theta \kappa + \theta(1-p_1) H(z, (\tilde{q}_\theta - c_\theta)/(1-p_\theta), (\tilde{w}_\theta - c_\theta)/(1-p_\theta)) + \\
&\quad (1-\theta)(1-p_2) H(z, (\tilde{q}_\theta - c_\theta)/(1-p_\theta), (\tilde{w}_\theta - c_\theta)/(1-p_\theta)) \\
&= p_\theta \kappa + \theta(1-p_1) H(z, (\tilde{q}_1 - c_1)/(1-p_1), (\tilde{w}_1 - c_1)/(1-p_1)) + \\
&\quad (1-\theta)(1-p_2) H(z, (\tilde{q}_2 - c_2)/(1-p_2), (\tilde{w}_2 - c_2)/(1-p_2)) \\
&\geq p_\theta \kappa + \theta(1-p_1) S(z, (\tilde{q}_1 - c_1)/(1-p_1), (\tilde{w}_1 - c_1)/(1-p_1)) + \\
&\quad (1-\theta)(1-p_2) S(z, (\tilde{q}_2 - c_2)/(1-p_2), (\tilde{w}_2 - c_2)/(1-p_2)) \\
&= \theta \tilde{S}(z, \tilde{q}_1 - c_1, \tilde{w}_1 - c_1) + (1-\theta)\tilde{S}(z, \tilde{q}_2, \tilde{w}_2)
\end{aligned}$$

To show the third step, notice that the function H is linear. Therefore, we can always write

$$H\left(z, \frac{\tilde{q}_\theta - c_\theta}{1-p_\theta}, \frac{\tilde{w}_\theta - c_\theta}{1-p_\theta}\right) = H\left(z, \frac{\tilde{q} - c}{1-p}, \frac{\tilde{w} - c}{1-p}\right) + \alpha_q \left[\frac{\tilde{q}_\theta - c_\theta}{1-p_\theta} - \frac{\tilde{q} - c}{1-p}\right] + \alpha_w \left[\frac{\tilde{w}_\theta - c_\theta}{1-p_\theta} - \frac{\tilde{w} - c}{1-p}\right]$$

where α_q and α_w are the slopes of the hyperplane H in the direction of q and w. The equivalence, then, follows trivially after some rearrangement. Therefore, $\tilde{S}$ is concave if S is concave.

Given the concavity of $\tilde{S}$, from the first part of the mapping—sub-problem (1)—we can also verify that $T(S)$ is strictly concave in $q < \bar{q}(z_{-1}, w)$. In fact, take two points (q_1, w_1) and (q_2, w_2). Associated with (q_1, w_1) there are solutions x_1, $\tilde{q}_1(\eta, z)$ and $\tilde{w}_1(\eta, z)$ and with (q_2, w_2) there are solutions x_2, $\tilde{q}_2(\eta, z)$ and $\tilde{w}_2(\eta, z)$. Notice that as in the proof of Proposition 1, I used the change of variable for the optimal investment: Instead of maximizing with respect to k, the maximization is over the variable $x = f(k)$. With this change of variable, the function $F(f^{-1}(x), \eta) - F(f^{-1}(x), \underline{\eta}) = (\eta - \underline{\eta})x$ is linear in x.

I want to show now that if we take a convex combination of (q_1, w_1) and (q_2, w_2), that is, $q_\theta = \theta q_1 + (1-\theta)q_2$ and $w_\theta = \theta w_1 + (1-\theta)w_2$, the convex combination of the two solutions is feasible when $q = q_\theta$ and $w = w_\theta$. Let's prove first that constraint (2) is satisfied by the convex combination. Define $b(z, \eta) = \tilde{q}(z, \underline{\eta}) + h'(0)[F(f^{-1}(x), \eta) - F(f^{-1}(x), \underline{\eta})] = \tilde{q}(z, \underline{\eta}) + h'(0)(\eta - \underline{\eta})x$, with $\tilde{q}(\underline{\eta}) \geq q_{min}(z)$. Following the same steps of Lemma 1 in the proof of Proposition 1, we can show that constraint (2) is satisfied if $\tilde{q}(\eta, z) - b(\eta, z)$ is not decreasing for all η and for each value of z. Therefore, what I need to show is that $\tilde{q}_\theta(z, \eta) - b_\theta(z, \eta)$

is not decreasing in η. Take two values of the shock, $\eta_a < \eta_b$. We know that:

$$\tilde{q}_1(z, \eta_a) - b_1(z, \eta_a) \leq \tilde{q}_1(z, \eta_b) - b_1(z, \eta_b) \tag{13}$$

$$\tilde{q}_2(z, \eta_a) - b_2(z, \eta_a) \leq \tilde{q}_2(z, \eta_b) - b_2(z, \eta_b) \tag{14}$$

From this it must also be the case that:

$$\tilde{q}_\theta(z, \eta_a) - \theta b_1(z, \eta_a) - (1-\theta) b_2(z, \eta_a) \leq \tilde{q}_\theta(z, \eta_b) - \theta b_1(z, \eta_b) - (1-\theta) b_2(z, \eta_b) \tag{15}$$

Because $b(z, \eta)$ is linear in x, then $\theta b_1(z, \eta_a) + (1-\theta) b_2(z, \eta_a) = b_\theta(z, \eta_a)$ and $\theta b_1(z, \eta_b) + (1-\theta) b_2(z, \eta_b) = b_\theta(z, \eta_b)$. Therefore,

$$\tilde{q}_\theta(z, \eta_a) - b_\theta(z, \eta_a) \leq \tilde{q}_\theta(z, \eta_b) - b_\theta(z, \eta_b) \tag{16}$$

Because this holds for any values of η_a and η_b, the incentive-compatibility constraint (2) is also satisfied for the convex combination of the solutions.

The other constraints (3)-(6) are obviously satisfied at $q = q_\theta$ and $w = w_\theta$ by the convex combinations x_θ, $\tilde{q}_\theta(z, \eta)$ and $\tilde{w}_\theta(z, \eta)$. Therefore, the convex combination of the solution is feasible. We can then show that $T(S)(z_{-1}, q_\theta, w_\theta) > \theta T(S)(z_{-1}, q_1, w_1) + (1-\theta) T(S)(z_{-1}, q_2, w_2)$. Therefore, the fix point of T is concave in (q, w). Given the properties of the surplus function, it is then easy to prove the other points of the proposition. *Q.E.D.*

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