Investment and liquidation in renegotiation-proof contracts with moral hazard

Vincenzo Quadrini *
Department of Economics
Stern School of Business
New York University
44 West Fourth Street, 7-85
New York, NY 10012
vquadrin@stern.nyu.edu

July 27, 2003

Abstract

In a long-term contract with moral hazard, the liquidation of the firm can arise as the outcome of the optimal contract. However, if the future production capability or market opportunities remain unchanged, liquidation may not be free from renegotiation. Will the firm ever be liquidated if we allow for renegotiation? This paper shows that the firm can still be liquidated even though liquidation is not free from renegotiation in the long-term contract. In addition to liquidation, the renegotiation-proof contract generates important features of the investment behavior and dynamics of firms observed in the data.

JEL: G32, D82, E22

Keywords: *Financial contract, firm investment, liquidation, renegotiation-proof.*

---

*I would like to thank Rui Albuquerque, Hilary Appel, Gian Luca Clementi and seminar participants at New York University, Stanford University, University of California Los Angeles, University of Pennsylvania, University of Southern California and the 1999 Southern Economic Theory Meeting in Georgetown University. I also thank the National Science Foundation for financial support through Grant SES-0111518. Earlier versions of the paper circulated under the title "Cash-Flow Sensitivity of Investment in Optimal Financial Contracts with Repeated Moral Hazard" and "Investment and default in renegotiation-proof contracts with moral hazard".

1 Introduction

The performance of any economy reflects the level of national investment, a large portion of which is decided at the level of the firm. Hence, understanding the factors that affect the investment decisions of firms and their dissolution has been central to the research agenda in micro and macroeconomics. Particular attention has been devoted to the importance of financial factors in the empirical studies of Fazzari, Hubbard, & Petersen (1988), Gertler & Gilchrist (1994), Gilchrist & Himmelberg (1996, 1998). As summarized by Hubbard (1998), the main findings of these empirical studies are that: (i) all else being equal, the investment of firms is significantly correlated with proxies for changes in net worth or internal funds; and (ii) this correlation is most important for firms likely to face information related, capital-market imperfections. The first finding suggests the existence of frictions in financial markets that affect the investment decision of firms independently of their future production capability or market opportunities. The second finding, instead, underscores the existence of significant heterogeneity in the way in which these frictions affect the investment behavior of firms. In particular, these frictions are more important for certain categories of firms, such as smaller and younger firms, as these studies suggest.

Motivated by the above empirical findings, this paper studies an economy in which investment is financed with optimal contracts between an entrepreneur and an investor in the presence of repeated moral hazard. Moral hazard derives from the existence of a conflict of interest over how resources are allocated in the firm, with the investor unable to fully observe this allocation. To prevent the entrepreneur from implementing a non-cooperative allocation of resources, entrepreneur's payments and investments must be conditional on the realization of revenues or cash flows. More specifically, smaller inputs of capital are financed when current revenues are low, even though these revenues do not affect the future production capability or market opportunities of the firm. However, as the firm grows and the entrepreneur's stake in the firm increases, moral hazard problems become less stringent and investment becomes dependent only on the future productivity of the firm.

An important assumption in the model, in addition to the information asymmetry, is the limited liability of the entrepreneur. Assuming that the parties are able to commit to the future terms of the contract (long-term contract), this assumption makes the liquidation of the firm a possible outcome. Liquidation results in the breakdown of the contract, which is not caused by or linked to the fall in the future productivity or market opportunities of the firm.

Although liquidation can be ex-ante optimal in the sense of improving the expected surplus of the contract, it can be ex-post inefficient for both the investor and the entrepreneur. This is a time-consistency problem: in the event in which the execution of the contract implies liquidation, the parties may find it mutually advantageous to renegotiate ex-post. This observation leads to the question of whether the liquidation of the firm can arise in a renegotiation-proof contract when liquidation is not free from renegotiation in the long-term contract. An important result of this paper is to show that, under certain conditions, the firm can still be liquidated in a renegotiation-proof contract even if liquidation is not free from renegotiation in the long-term contract. Furthermore, there are conditions for which the firm is liquidated in the renegotiation-proof contract even though it is never liquidated in the long-term contract.

While in the strictest sense the liquidation of the firm cannot be interpreted as default, it does share several features of bankruptcy.[1] First, if we think of the value of the contract for the investor as the liability of the firm, the probability of liquidation increases with the firm's liability relative to its total value. Second, in the event of liquidation, all the residual assets of the firm are taken by the investor, which is the usual practice in the bankruptcy procedure. Third, the probability of liquidation decreases with the firm's size. Moreover, since the size of the firm is correlated with its age, liquidation also decreases with age.

The optimal financing of investment with asymmetric information is also studied in Gertler (1992), but in an environment in which the life span of the firm is finite and known (no advance liquidation). There are some similarities with the models developed in Atkeson (1991) and Khan & Revikumar (2001). In these models, however, liquidation cannot be an outcome of the optimal contract. Albuquerque & Hopenhayn (1997) study the optimal contract in an environment

[1] In a strict sense, liquidation cannot be interpreted as "default" or "repudiation of existing liabilities" because the optimal contract is state-contingent and internalizes all possible outcomes. However, in a real contract some of the contingencies—such as those leading to liquidation—may be implicit in the sense that they are not formally stated in the contract. Consequently, when these contingencies arise, the associated outcomes appear to take the form of contract repudiation. But in practise, these events are fully anticipated by the contractual parties. Think for example to a standard debt contract. Although it is not stated explicitly, a debt contract is state-contingent in the sense that the lender internalizes the possibility of receiving lower repayments in case of default.

with limited enforceability. With limited enforceability, however, the long-term contract is always free from renegotiation. Clementi & Hopenhayn (1998) study, independently, the optimal long-term contract in an environment similar to the one studied here but they do not characterize the renegotiation-proof contract. Another related paper is DeMarzo & Fishman (2000). They also consider privately observed revenues but the size of the investment project is fixed.

This study also relates to the literature on optimal insurance contracts with repeated moral hazard. Most of this literature has analyzed the optimal and incentive-compatible allocation of consumption among risk averse agents, when privately observed incomes evolve exogenously or depend on unobservable effort.[2] By contrast, the current paper studies a production economy in which both agents—investors and entrepreneurs—are risk neutral. The problem consists of finding the optimal and incentive-compatible investment schedule that maximizes the net revenues of the firm. In spite of these differences, the two problems have some similarities: the contract solves the trade-off between a higher investment level and a lower investment volatility, that is, investment insurance as opposed to consumption insurance.

The organization of the paper is as follows. Section 2 describes the model and Section 3 provides an informal characterization of some of its properties. The technical derivation of these properties is conducted in Sections 4 and Section 5. After the analysis of the initial stage of contracting (Section 6), Sections 7 and 8 characterizes the dynamic properties of the firm induced by the renegotiation-proof contract. Section 9 discusses the extension to persistent shocks and Section 10 concludes.

2 The model

Consider an entrepreneur with initial wealth a_0 who maximizes the expected utility $E_0 \sum_{t=0}^{\infty} \beta^t c_t$, where c_t is consumption and β is the intertemporal discount factor. The entrepreneur has the managerial skills to run a firm with gross revenue function $F(k, \eta)$, where k is the input of capital decided in the current period and η is a shock realized at the beginning of the next period (and

[2] Examples of these studies include Green (1987), Spear & Srivastava (1987), Thomas & Worrall (1990), Phelan & Townsend (1991), Atkeson & Lucas (1992, 1995), Phelan (1995), Wang (1997), Cole & Kocherlakota (1997).

therefore, after the input of capital has been decided). The shock and the revenue function satisfy the following properties:

Assumption 1 *The shock η is independently and identically distributed in the interval $\mathbf{N} \equiv [\underline{\eta}, \bar{\eta}]$ with density function $g(\eta) > 0$, $\forall\, \eta \in \mathbf{N}$.*

Assumption 2 *The revenue function takes the form $F(k, \eta) = (1-\delta)k + \eta \cdot f(k)$ with $f(k)$ strictly increasing, strictly concave, differentiable and satisfies $f(0) = 0$, $\lim_{k \to 0} f_k(k) = \infty$ and $\lim_{k \to \infty} f_k(k) = 0$.*

The gross revenue results from the sum of two components: the cash-flow $\eta \cdot f(k)$ plus the undepreciated capital $(1-\delta)k$. The concavity of F implies that the revenue function displays decreasing returns to scale and there is an optimal input of capital $\bar{k}$ that maximizes the expected revenue net of the opportunity cost of capital. This is implicitly defined by the first order condition $\beta E F_k(\bar{k}, \eta) = 1$. The assumption that the shock is i.i.d. is made to isolate the impact of financial factors on the investment decision of the firm from the impact of technological or market differences. Section 9, however, will extend the analysis to the case of persistent shocks.

The creation of a new firm requires an initial fixed investment I_0, in addition to the variable investment k. If the firm is liquidated, the residual value of the initial investment is κ. The fixed investment satisfies the following conditions:

Assumption 3 *The fixed investment I_0 and its liquidation value κ satisfy: $\kappa < I_0 < \frac{\beta E F(\bar{k}, \eta) - \bar{k}}{1-\beta}$.*

The first inequality imposes that only part of the initial set-up investment can be recovered in case of liquidation, that is, part of this investment is sunk. As we will see in Section 6, this condition plays an important role in the renegotiation of the contract if the entrepreneur is not excluded from the market. The second inequality insures the existence of a firm. The term $(\beta E F(\bar{k}, \eta) - \bar{k})/(1-\beta)$ is the discounted expected lifetime profits when the firm is always operated at the optimal scale. For the viability of a firm, this value must be greater than the set-up cost I_0.

Once a firm has been created, the entrepreneur has the ability to divert some of the firm's resources (physical and managerial) to generate a private return additive to consumption. Denote by $e \in [0, 1]$ the fraction of resources that the entrepreneur uses to generate revenues and by $1 - e$ the fraction of resources used to generate the private return. Given this allocation of resources, the firm's revenues will be $e \cdot F(k, \eta)$ and the private return $h(1 - e) \cdot F(k, \eta)$. While the revenue is public information, the shock η and the private return are observed only by the entrepreneur. The function h satisfies the following properties:

ASSUMPTION 4 *The diversion function satisfies* $h(0) = 0$, $0 < h'(.) \leq 1$, $h''(.) \leq 0$.

Condition $0 < h'(.) \leq 1$ implies that the private return increases with the amount of resources diverted. Condition $h''(.) \leq 0$ imposes the weak concavity of h. This property is convenient for establishing some basic properties of the contract.

At the moment of starting the project, if the initial entrepreneur's wealth a_0 is not sufficient to self-finance the optimal input of capital $\bar{k}$, the entrepreneur will enter into a contractual relationship with an investor. The discount factor for the investor is also β. For the moment, I assume that the entrepreneur contributes with all personal wealth to the initial financing of the firm. Section 6 will show that this is not simply an assumption but it is optimal for the entrepreneur.

The final assumption defines the reservation values. We have to distinguish the stage before and after signing the contract. For the entrepreneur the initial value of the contract cannot be smaller than his or her wealth a_0. In all subsequent periods it cannot be smaller than a minimum value (limited liability). For simplicity, the lower value is set to zero. For the investor the initial value of the contract cannot be smaller than zero. However, after signing the contract, the investor commits to fulfill any future obligation. The assumption of one-side commitment is not important if we allow for bonding, that is, the ability of the entrepreneur to access a riskless and observable investment at the market interest rate. See Malcomson & Spinnewyn (1988) for details.

The timing of the events is as follows: Capital investment is chosen one period in advance. Therefore, at the beginning of each period the firm starts with k units of capital. At this stage the entrepreneur (but not the investor) observes the shock η and decides how to allocate the firm's resources by choosing e. Given the allocation of resources, the investor observes the revenue. At this point the firm can be liquidated with some probability (randomization). Conditional on liquidation, the investor and the entrepreneur each receive a payment. What is left is used to finance the next period capital (which is zero if the firm is liquidated).[3]

3 The properties of the contract: an informal characterization

Before turning to the technical characterization of the optimal contract, it will be convenient here to describe informally some of its properties.

Let's consider first the optimal long-term contract, that is, the contract that the parties commit not to renegotiate in future dates. The contractual problem can be formulated recursively using the value of the contract for the entrepreneur as a state variable. Let's call this variable q. For any value of q, the contract generates a surplus $S(q)$ which is the sum of the value for the entrepreneur, q, and the investor, $S(q) - q$. The surplus function is plotted in Figure 1. The next section will show that this function is increasing, concave and converges to $\beta\kappa$ as q converges to zero. On the other hand, the surplus function becomes constant for values of q greater than $\bar{q}$, and once the entrepreneur's value reaches $\bar{q}$, it never falls below this value. At this stage agency problems become irrelevant and the firm operates at the optimal scale $\bar{k}$.

These properties imply that for q sufficiently small, $S(q) < \kappa$ and the liquidation of the firm is preferable to its continuation. In fact, if the firm is not liquidated, the investor's value is $S(q) - q$. This is smaller than $\kappa - q$, which is the value the investor would receive if the firm is liquidated (the investor pays q to the entrepreneur and will cash the residual value κ). In Figure 1 the liquidation of the firm is preferable if $q < \hat{q}$.

A strategy that liquidates the firm only if q falls below $\hat{q}$ is preferable to a strategy that never liquidates the firm. However, the adoption of this strategy would make the surplus of the contract convex for low value of q. Therefore, the surplus can be improved by making the liquidation random. More specifically, if q falls below $\underline{q}$, the randomization over the points $q = 0$ (liquidation) and $q = \underline{q}$ (continuation) would be the optimal strategy. With this strategy, the pre-randomization surplus in the interval $[0, \underline{q}]$ becomes the straight line joining κ and $S(\underline{q})$.

[3] The entrepreneur can use the payments either for consumption purposes or for "observable" investments outside the firm with return $1/\beta - 1$. Of course, the entrepreneur can also reinvest these payments in the firm, but this is equivalent to receiving less payments from the contract.

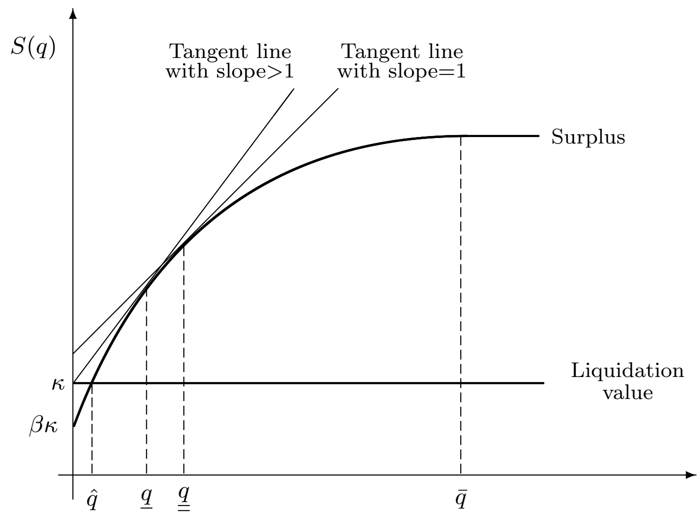

Figure 1: Surplus of a long-term contract and liquidation strategy.

This liquidation strategy is optimal before the realization of the shock. However, after the shock has been realized and the revenue has become public information, this strategy may no longer be optimal and the parties may renegotiate the liquidation of the firm. Whether the liquidation of the firm is free from renegotiation depends on the value of κ. In particular, liquidation is *not* renegotiation-proof if κ is sufficiently small.

To see this consider again Figure 1. As will be shown in the next sections, if κ is not too large, the slope of the surplus function is greater than 1 for low values of q. In the figure, this is the case for $q < \underline{\underline{q}}$. However, if the slope of the surplus function is greater than 1, the investor's value, $S(q) - q$, increases with the entrepreneur's value. This implies that the parties would find convenient to renegotiate the contract. The renegotiation of the contract is equivalent to restarting a new long-term contract with initial $q = \underline{\underline{q}}$. Because liquidation takes place only if q falls below $\underline{q}$, which is smaller than $\underline{\underline{q}}$, then the liquidation of the firm is not free from renegotiation.

How can we make the contract renegotiation-proof? This is done by imposing a higher lower bound to the entrepreneur's promised value before randomizing on liquidation. A higher lower bound restricts the set of feasible strategies and, as a result, it reduces the surplus of the optimal contract. As the surplus declines, its maximum slope (before randomizing) also declines. Figure 2 shows that the imposition of this further constraint reduces the surplus for each value of $q < \bar{q}$.

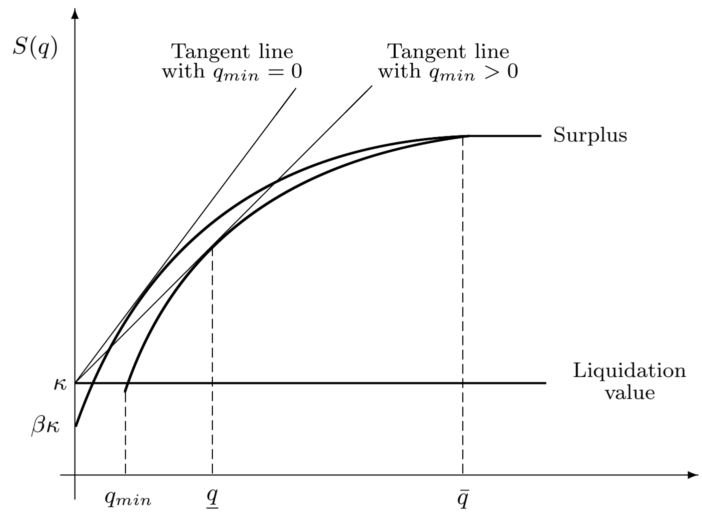

Figure 2: Surplus of a long-term contract after imposing $q_{min} > 0$.

This, in turn, reduces the slope of the tangent line departing from κ. The lower bound q_{min} is increased until the maximum slope equals 1.

Once the surplus function satisfies these properties, it is easy to see why the liquidation of the firm is free from renegotiation. If the investor does not renegotiate the contract, he or she will receive the value κ. If instead the contract is renegotiated, the new contract will restart at $\underline{q}$ and the investor's value is equal to $S(\underline{q}) - \underline{q} = \kappa$ (remember that the slope of $S(q)$ is 1 at $\underline{q}$). Therefore, the investor does not gain from renegotiating the contract, unless he or she receives some transfer from the entrepreneur. However, the entrepreneur is unable to make any transfer because, in case of liquidation, his or her wealth is zero.[4]

Is the firm still liquidated at this point? Proposition 4 establishes that the firm can still be liquidated with some probability. Moreover, if $\kappa = 0$ the firm can also be liquidated in the renegotiation-proof contract even if it is never liquidated in the long-term contract.

In addition to showing that the renegotiation-proof contract can lead to the liquidation of the

[4] At this point the optimal contract is renegotiation-proof even if the entrepreneur is allowed to start a new firm with a new investor (no market exclusion). In fact, the maximum value that the new investor can get from a new contract is κ. But when the entrepreneur has zero wealth, the initial cost for the investor is the set-up cost I_0 which is bigger than κ. As we will see in Section 6, this implies that only entrepreneurs who have a minimum value of wealth are able to start a new firm.

firm, the model also generates important features of the investment behavior and dynamics of firms. These features can be summarized as follows: (i) The investment of smaller and younger firms is sensitive to cash flows, even after controlling for its future production capability or market opportunities. Once the firm reaches the maximum size, however, investment becomes independent of cash-flows. This also implies that smaller (constrained) firms experience higher volatility of investment and growth. (ii) If the employment of the firm is related to investment, then the rate of job reallocation (*i.e.*, job creation and destruction) is also higher for smaller firms. (iii) The probability of liquidation depends negatively on the size of the firm and the share of external financing. Moreover, controlling for the current size, this probability decreases with the performance of the firm. (iv) Constrained entrepreneurs have higher saving rates. This is because the internal return from savings (by investing in the firm) is greater than the market return. These properties will be described in more details in Sections 7 and 8.

4 The long-term contract

Appendices A and B provide a formal definition of a long-term contract and the derivations of some basic properties for the recursive formulation of the optimal contract. Here I start directly with the recursive formulation of the contracting problem. I define first some variables and functions that will be use throughout the paper.

Denote by $p \in [0, 1]$ the probability of liquidation (randomization). Moreover, define $\ell \in \{0, 1\}$ the dummy variable that takes the value of one if the firm is liquidated and zero if the firm is not liquidated. I will refer to this variable as the "liquidation outcome".

Given the true realization of the shock η (unobserved by the investor), the publicly observed revenue is $e \cdot F(k, \eta)$. Denote by $\hat{\eta}$ the shock that would have generated this revenue if the entrepreneur had not diverted resources. This is implicitly defined by the condition $e \cdot F(k, \eta) = F(k, \hat{\eta})$. I will refer to $\hat{\eta}$ as the shock announcement. Because $F(k, \hat{\eta}) = e \cdot F(k, \eta)$, we can define e as a function of k, η and $\hat{\eta}$, that is, $e(k, \eta, \hat{\eta}) = F(k, \hat{\eta})/F(k, \eta)$. The private return from diversion will be denoted by $D(k, \eta, \hat{\eta}) = h(1 - e(k, \eta, \hat{\eta})) \cdot F(k, \eta)$.

As shown in Appendix B, the contractual problem can be formulated recursively by maximizing the surplus of the contract (sum of the values for the investor and the entrepreneur) subject to the entrepreneur's promised value q. Given q, the contract will choose the input of capital, k, the probability of liquidation, $p(\eta)$, and the next period payments and continuation values for the entrepreneur, $c(\eta,\ell)$ and $q(\eta,\ell)$. The liquidation probability is conditional on the announcement of the shock while the entrepreneur's payment and continuation value are also conditional on the liquidation outcome. Notice that the choice variables are functions of the shock announcement $\hat{\eta}$, not the true shock η. However, the imposition of the incentive-compatibility constraints imply that the shock announcement is always revealed and in equilibrium $e=1$. Therefore, I will not distinguish between $\hat{\eta}$ and η, unless explicitly required. The problem can be written as:

$$S(q) \;=\; \max_{\substack{k,\,p(\eta)\\ c(\eta,\ell),\,q(\eta,\ell)}} \int_{\underline{\eta}}^{\bar{\eta}} \Big[-k+\beta F(k,\eta)+\beta\Big(p(\eta)\kappa \;+\; (1-p(\eta))S(q(\eta,0))\Big)\Big]g(\mathrm{d}\eta) \tag{1}$$

subject to

$$p(\eta)\Big[c(\eta,1)+q(\eta,1)\Big]+(1-p(\eta))\Big[c(\eta,0)+q(\eta,0)\Big] \;\geq \tag{2}$$

$$D(k,\eta,\hat{\eta})+p(\hat{\eta})\Big[c(\hat{\eta},1)+q(\hat{\eta},1)\Big]+(1-p(\hat{\eta}))\Big[c(\hat{\eta},0)+q(\hat{\eta},0)\Big] \qquad \forall\,\eta,\hat{\eta}\in\mathbf{N}$$

$$q=\beta\int_{\underline{\eta}}^{\bar{\eta}}\Big(p(\eta)\Big[c(\eta,1)+q(\eta,1)\Big]+(1-p(\eta))\Big[c(\eta,0)+q(\eta,0)\Big]\Big)g(\mathrm{d}\eta) \tag{3}$$

$$p(\eta)\Big[c(\eta,1)+q(\eta,1)\Big]+(1-p(\eta))\Big[c(\eta,0)+q(\eta,0)\Big]\geq q_{min} \tag{4}$$

$$c(\eta,\ell)\geq 0\,,\quad q(\eta,\ell)\geq 0 \tag{5}$$

The function S is the end-of-period surplus of the contract conditional on the survival of the firm. Because the firm is liquidated with probability $p(\eta)$, the expected next period surplus depends on the liquidation outcome. If the firm is liquidated the residual value is κ. If the firm is not liquidated, the next period surplus depends on the entrepreneur's promised value.

Condition (2) is the incentive-compatibility constraint. Given the liquidation probability $p(\eta)$, the value that the entrepreneur receives from reporting the true realization of the shock η cannot be smaller than the value obtained from reporting $\hat{\eta}\neq\eta$. Condition (3) is the promise-keeping constraint and (4) imposes a lower bound q_{min} to the expected value for the entrepreneur before randomizing on liquidation. Although this lower bound was assumed to be zero, in the analysis that follows I will derive all the results for any value of $q_{min} \geq 0$. The formulation of the problem for any value of q_{min} will be convenient for the subsequent analysis of the renegotiation-proof contract.

Before characterizing the main properties of this problem, let's define the function $\tilde{q}(\eta)$ to be the next period value for the entrepreneur (with the exclusion of the private return) before randomizing on liquidation, that is, $\tilde{q}(\eta) = p(\eta)[c(\eta,1) + q(\eta,1)] + (1 - p(\eta))[c(\eta,0) + q(\eta,0)]$. What matters for incentive-compatibility is the function $\tilde{q}(\eta)$, not its components $p(\eta)$, $c(\eta,\ell)$ and $q(\eta,\ell)$. The following lemma establishes a property that $\tilde{q}(\eta)$ has to satisfy for the contract to be incentive compatible.

Lemma 1 *Define* $b(\eta) = \tilde{q}(\underline{\eta}) + h'(0)[F(k,\eta) - F(k,\underline{\eta})]$, *with* $\tilde{q}(\underline{\eta}) \geq q_{min}$. *For the contract to be incentive compatible,* $\tilde{q}(\eta) - b(\eta)$ *cannot be decreasing for all* η.

Proof: *Appendix D.*

The proof of this lemma is based on the fact that, if the entrepreneur has no incentive to falsely report the lowest possible shock $\underline{\eta}$, then he or she will never report any other value different from the true realization. The concavity of the function h is important to get this result. This lemma simplifies the characterization of the optimal long-term contract.

Proposition 1 (Long-term contract) *There exist* $\bar{q}$ *and* $\underline{q}$ *with* $\bar{q} > \underline{q} \geq q_{min}$, *such that:*

(a) The schedule $\tilde{q}(\eta)$ *is equal to* $b(\eta) = \tilde{q}(\underline{\eta}) + h'(0)[F(k,\eta) - F(k,\underline{\eta})]$.

(b) $S(q)$ *is increasing and concave for* $q < \bar{q}$, *constant for* $q \geq \bar{q}$, *and differentiable.*

(c) The input of capital is at the optimal level $\bar{k}$ *if* $q \geq \bar{q}$.

(d) If $\tilde{q}(\eta) < \underline{q}$, *then* $p(\eta) > 0$, $c(\eta,\ell) = 0$, $q(\eta,1) = 0$ *and* $q(\eta,0) = \underline{q}$.

(e) If $\underline{q} \leq \tilde{q}(\eta) < \bar{q}$, *then* $p(\eta) = 0$, $c(\eta,0) = 0$ *and* $q(\eta,0) = \tilde{q}(\eta)$.

(f) If $\tilde{q}(\eta) \geq \bar{q}$, *then* $p(\eta) = 0$ *but there are multiple solutions to* $c(\eta,0)$ *and* $q(\eta,0) \geq \bar{q}$.

PROOF: *Appendix E.*

The properties of the optimal long-term contract summarized in the above proposition are also derived, independently, in Clementi & Hopenhayn (1998). Their analysis, however, do not characterize the renegotiation-proof contract as done in the next section. The proof of Proposition 1 is somewhat involved and thus is relegated to the appendix. Despite the complexity of the proof, however, the properties of the long-term contract have simple intuitions as described below. The reader not interested in these intuitions can skip the remaining part of this section and turn directly to the analysis of renegotiation-proof contract.

Point (a) defines the structure of the schedule $\tilde{q}(\eta)$ which takes a simple form. From this schedule we can see that the value promised to the entrepreneur, q, limits the input of capital. In fact, from the promised-keeping constraint we have $q = \beta E\tilde{q}(\eta) = \beta\tilde{q}(\underline{\eta}) + \beta h'(0)(EF(k,\eta) - F(k,\underline{\eta}))$. Because $\tilde{q}(\underline{\eta})$ cannot be smaller than q_{min}, higher values of q allows for higher inputs of capital. Therefore, higher values of q relaxes the constraints in the problem and allows for higher surpluses. Also notice that, for each q, higher values of q_{min} restrict the inputs of capital. Therefore, we would expect that the surplus function is decreasing in q_{min}. The concavity of the surplus function derives from the concavity of $F(k,\eta)$. When the value of q has reached $\bar{q}$, moral hazard problems disappear and the input of capital is always at the optimal level $\bar{k}$. At this point the entrepreneur's wealth is sufficiently large to self-insure the possible losses generated by the firm when operated at the optimal scale $\bar{k}$. The value promised to the entrepreneur can be larger than $\bar{q}$. However, increasing the entrepreneur's value above $\bar{q}$ implies a simple redistribution of wealth from the investor to the entrepreneur without affecting the surplus.

The same considerations explain the postponement of payments to the entrepreneur before q reaches the upper bound $\bar{q}$ (see points (d) and (e)). Given the constraint on the input of capital imposed by q, it is preferable to increase the entrepreneur's promised value (future consumption), rather than making payments to the entrepreneur (current consumption). This implies that entrepreneurs have high saving rates motivated by the incentive to self-finance their business. Of course, the incentive to increase q vanishes if the firm is liquidated. In this case it is optimal to minimize the entrepreneur's value, that is $q(\eta, 1) = 0$, in order to increase the future values of q

when the firm is not liquidated. This implies that the whole gross revenue $F(k,\eta) = (1-\delta)k + \eta \cdot f(k)$ plus the residual κ is distributed to the investor in case of liquidation.

The result that the probability of liquidation is positive only if $\tilde{q}(\eta) < \underline{q}$ can be explained as follows. Let's observe first that this is possible only if $\underline{q}$ is strictly greater than q_{min}, which in turn is possible if q_{min} is sufficiently small. Consider the extreme case in which $q_{min} = 0$. The surplus function $S(q)$ converges to $\beta\kappa$ as q converges to zero. This implies that, for low values of q, $S(q)$ is smaller than κ. It is then preferable for the investor to pay q to the entrepreneur and liquidate the firm (and claim the residual value κ). Moreover, randomizing over the liquidation choice improves the surplus because liquidation makes the surplus function convex. Given two values of $\tilde{q}(\eta)$ for which it is optimal to set positive probabilities of liquidation, it is optimal in both cases to set the continuation value to $\underline{q}$. This implies that the probability of liquidation is simply equal to $p(\eta) = 1 - \tilde{q}(\eta)/\underline{q}$. Because $\tilde{q}(\eta)$ is increasing in η, the probability of liquidation is higher for smaller realization of η.

5 Renegotiation-proof contract

The optimal contract characterized in the previous section assumes that the parties commit not to renegotiate in future periods, even if renegotiation is ex-post beneficial for both parties. The goal of this section is to show first that the long-term contract is not free from renegotiation (subsection 5.1), and then to characterize the renegotiation-proof contract (subsection 5.2). The formal definition of renegotiation-proof is provided in Appendix C.

5.1 Is the long-term contract free from renegotiation?

To show that the long-term contract is not free from renegotiation, it would be convenient to rewrite the contractual problem (1) by separating the choice of the input of capital and schedule $\tilde{q}(\eta)$, from the choice of the liquidation probability. First notice that, using point (a) of Proposition 1, the promise-keeping constraint can be written as $q = \beta E\tilde{q}(\eta) = \beta\tilde{q}(\underline{\eta}) + \beta h'(0)[EF(k,\eta) - F(k,\underline{\eta})]$. If we use this equation to eliminate $\tilde{q}(\underline{\eta})$ in the schedule $\tilde{q}(\eta) = \tilde{q}(\underline{\eta}) + \beta h'(0)[F(k,\eta) - F(k,\underline{\eta})]$, we obtain $\tilde{q}(\eta) = h'(0)[F(k,\eta) - EF(k,\eta)] + q/\beta$. Therefore, the choice of k fully determines the schedule $\tilde{q}(\eta)$. Because the entrepreneur's payments are zero before reaching $\bar{q}$ and they are not determined for $q > \bar{q}$, the contractual problem can be decomposed as follows:

$$\begin{aligned} S(q) &= \max_k \left\{ -k + \beta E\Big[F(k,\eta) + \tilde{S}(\tilde{q}(\eta))\Big]\right\} \qquad (6) \\ & \quad \text{subject to} \\ \tilde{q}(\eta) &= h'(0)[F(k,\eta) - EF(k,\eta)] + \frac{q}{\beta} \\ q &\geq \beta q_{min} + \beta h'(0)[EF(k,\eta) - F(k,\underline{\eta})] \end{aligned}$$

$$\tilde{S}(\tilde{q}) = \max_p \left\{ p\kappa + (1-p)\, S\left(\frac{\tilde{q}}{1-p}\right)\right\} \qquad (7)$$

The first optimization stage consists of the choice of the input of capital. The first constraint has been derived above while the second is simply the promise-keeping constraint with the term $\tilde{q}(\underline{\eta})$ replaced by q_{min}. Because $\tilde{q}(\underline{\eta}) \geq q_{min}$, the constraint can be satisfied with the inequality sign.

The second optimization stage, represented by (7), consists of the choice of the liquidation probability. The variable $\tilde{q} = \tilde{q}(\eta)$ is the promised value after the announcement of the shock but before randomizing on liquidation. Since there are no payments to the entrepreneur and in case of liquidation $q = 0$, the next period value when the firm is not liquidated must be $q' = \tilde{q}/(1-p)$. The following corollary to Proposition 1 characterizes the function $\tilde{S}(\tilde{q})$.

Corollary 1 *The function $\tilde{S}(\tilde{q})$ is continuous, concave and differentiable for all $\tilde{q} > q_{min}$. Furthermore, it is linearly increasing in $\tilde{q} \in (q_{min}, \underline{q})$ and satisfies $\tilde{S}(\tilde{q}) = S(\tilde{q})$ for $\tilde{q} \geq \underline{q}$.*

Proof: *It follows from the optimal contract characterized in Proposition 1.*

From the above formulation it is easy to see that the problem of renegotiation arises in the second stage, that is, after the observation of the revenue. In the first stage the schedule $\tilde{q}(\eta)$ is chosen optimally to induce the entrepreneur to reveal the true value of the shock (and no diversion). However, once the entrepreneur action has been taken and the shock is revealed, the parties may have an incentive to change, ex-post, the value of $\tilde{q}$ delivered by this schedule. In particular, if the slope of $\tilde{S}(\tilde{q})$ is greater than 1 for some values of $\tilde{q}$ and the delivered value of $\tilde{q}$ is in this range, the parties will have a mutual advantage to change (increase) this value. This is better understood by looking at Figure 3 that plots the function $\tilde{S}(\tilde{q})$. In this figure the contract is renegotiated if $\tilde{q} < \underline{\underline{q}}$.

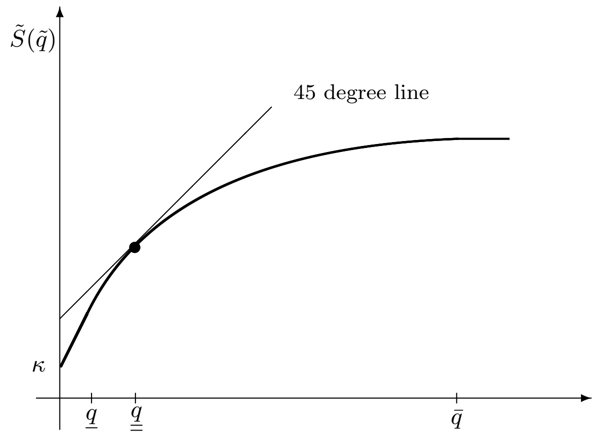

Figure 3: Pre-randomization surplus of the optimal long-term contract.

To prove that the long-term contract is not free from renegotiation we have to show that: (i) there is a region of $\tilde{q}$ for which the slope of $\tilde{S}(\tilde{q})$ is greater than 1, that is, $\underline{\underline{q}} > 0$; and (ii) this region could be reached with positive probability at some future date.[5] This is established in the next proposition.

PROPOSITION 2 *Let $q_{min} = 0$. If κ is sufficiently small, there exists $\underline{\underline{q}} \in [\underline{q}, \bar{q})$ for which $\tilde{S}'(\underline{\underline{q}}) > 1$. Moreover, for all $q \in [\underline{q}, \bar{q})$, there is a positive probability that $\tilde{q} < \underline{\underline{q}}$ at some future date.*

PROOF: *Appendix F.*

To understand the intuition behind this proposition, consider the case in which $\kappa = 0$. In this case the firm will never be liquidated in the long-term contract, that is, $\underline{q} = 0$. This implies that

[5]Fudenberg, Holmstrom, & Milgrom (1990) have shown that a sufficient condition for the renegotiation-proof of the long-term contract is that the utility frontier is downward sloping. In the model this would be the case if the slope of $\tilde{S}(\tilde{q})$ is not greater than 1. The downward property of the utility frontier, however, is only a *sufficient* condition: even if the utility frontier in not downward sloping, the optimal long-term contract may still be free from renegotiation if the upward region is never reached, that is, $\tilde{q}$ never falls below $\underline{\underline{q}}$.

$q' = \tilde{q}$ and $\tilde{S}(q) = S(q)$ for all q. The proof shows that for any $q > 0$, there is some η for which $q' < q$. This implies that after a sequence of bad shocks, q gets arbitrarily close to zero. As q gets close to zero, the promise-keeping constraint implies that k must converge to zero and the marginal revenue to infinity. As a result of this, the marginal increase in the surplus with respect to q must be large. Now consider the case in which $\kappa > 0$, which implies $\underline{q} > 0$. As κ converges to zero, $\underline{q}$ also converges to zero. Therefore, q can take relatively small values after a sequence of bad shocks. High values of F_k then imply that $\tilde{S}'(\tilde{q})$ is large. When κ is large, however, the constraint on the input of capital is not too tight and the slope of the surplus function is always smaller than 1. In this case the long-term contract is free from renegotiation.

5.2 Derivation of the renegotiation-proof contract

Appendix C provides a formal definition of a renegotiation-proofness. Using the recursive formulation of the contracting problem and Proposition 1, this definition can be written as:

DEFINITION 1 *A contract is renegotiation-proof if for all $\tilde{q} \geq q_{min}$ there is no $\hat{q} > \tilde{q}$ such that $\tilde{S}(\hat{q}) - \hat{q} > \tilde{S}(\tilde{q}) - \tilde{q}$.*

This says that, in a renegotiation-proof contract it is not possible to increase the value of the contract for both the entrepreneur and the investor, before randomizing on liquidation. If this condition is satisfied, the contract is also free from renegotiation after randomization. This is obvious given Proposition 1 and Corollary 1.

The renegotiation-proof contract is derived by making endogenous the lower bound q_{min} which, up until this point, has been treated parametrically. The role of q_{min} in characterizing the renegotiation-proof contract is established in the next proposition.

PROPOSITION 3 (RENEGOTIATION-PROOF CONTRACT) *There exists $\underline{q_{min}}$ for which the optimal and renegotiation-proof contract is derived by imposing $q_{min} = \underline{q_{min}}$ in the long-term contract. The value of $\underline{q_{min}}$ is smaller than $\bar{q}$ if $\beta F(\bar{k}, \underline{\eta}) < \bar{k}$.*

PROOF: *Appendix G.*

Therefore, the renegotiation-proof contract can be derived as an optimal long-term contract by imposing the proper lower bound $q_{min} = \underline{q_{min}}$. Thomas & Worrall (1994) and Wang (2000) use a similar procedure to derive the renegotiation-proof contract. The first in a model without information asymmetries while the second in a model with finite horizon. Notice that, if $\underline{q_{min}} < \bar{q}$, renegotiation-proofness does not require that the firm is in the unconstrained status. For this to be the case, it is sufficient that $S(\bar{q}) < \bar{q}$, that is, the total surplus when the firm is unconstrained is smaller than the entrepreneur's value. This will be the case if $\beta F(\bar{k}, \underline{\eta}) < \bar{k}$, that is, if the worst realization of the shock implies negative profits.

The next proposition relates some of the properties of the renegotiation-proof contract to the properties of the long-term contract when $q_{min} = 0$.

Proposition 4 *There exists $\bar{\kappa} > 0$ such that for $q < \bar{q}$:*

(a) If $\kappa < \bar{\kappa}$ the long-term contract is not free from renegotiation.

(b) If $\kappa > 0$ the firm is liquidated with positive probability in both the long-term-contract and the renegotiation-proof contract.

(c) If $\kappa = 0$ the firm is never liquidated in the long-term contract but could be liquidated in the renegotiation-proof contract.

Proof: *Appendix H.*

Therefore, liquidation does not require commitment. In fact, the renegotiation-proof contract always leads to liquidation (with some probability) if the firm is liquidated in the long-term contract. The reverse, however, is not necessarily true. When $\kappa = 0$, the renegotiation-proof contract may lead to liquidation even if the firm is never liquidated in the long-term contract.

It is important to point out that the breakdown of the contract does not derive from the fall in the future production capability or market opportunities of the firm. It is determined by the inability to finance any input of capital for which the investor's value is positive. Using the terminology of Aghion & Bolton (1992) and Hart & Moore (1994), the entrepreneur would not have enough resources to *bribe* the creditor not to liquidate the firm.

6 Initial conditions and existence of a contract

A new contract generates a surplus that will be shared between the two parties according to their bargaining power. Assuming competition in financial markets, the initial value of the contract for the entrepreneur is the maximum value of q that satisfied the participation constraints:

$$S(q) - q \geq I_0 - a_0 \tag{8}$$

$$q \geq a_0 \tag{9}$$

The first constraints imposes that the value of the contract for the investor, $S(q) - q$, cannot be smaller than its costs $I_0 - a_0$ (remember that the initial fixed investment is I_0 and the entrepreneur contributes a_0 to the financing of the contract). The second constraint imposes that the value of the contract for the entrepreneur cannot be smaller than his or her initial wealth.

The initial wealth of the entrepreneur—the variable a_0—plays a crucial role in determining the existence of a renegotiation-proof contract. Because in a renegotiation-proof contract $S(q) - q \leq \kappa$ for all q, condition (8) is not satisfied when $a_0 = 0$ (remember that $I_0 > \kappa$). Therefore, if the entrepreneur has zero wealth, the value of the contract for the investor is negative and the project will not be financed. For the existence of a contract the initial entrepreneur's wealth must be sufficiently large. More specifically, because in a renegotiation-proof contract the maximum value for the investor is $S(\underline{q}) - \underline{q} = \kappa$, the entrepreneur must be able to self-finance the part of the set-up investment that is sunk, that is, $a_0 \geq I_0 - \kappa$. Moreover, even if $a_0 \geq I_0 - \kappa$ and the investor is willing to finance the firm, it may be that the initial q is smaller than a_0, that is, the participation constraint for the entrepreneur (9) is not satisfied. Notice that Assumption 3 guarantees that constraint (9) is satisfied for sufficiently large values of a_0 but not for all $a_0 \geq I_0 - \kappa$. This is another dimension in which the personal wealth of potential entrepreneurs affects the formation of a new businesses. The importance of wealth for entrepreneurial start up is supported by the empirical studies of Evans & Jovanovic (1989), Evans & Leighton (1989), Holtz-Eakin, Joulfaian, & Rosen (1994) and Quadrini (1999).

There is also another feature of the model that should be emphasized. Because $S(q)$ is strictly increasing in $q < \bar{q}$, the marginal increase in the initial value of the contract for the entrepreneur with respect to a_0 is greater than 1. It would be 1 if $S(q)$ is constant. This feature implies that it is optimal for the entrepreneur to contribute with the whole personal wealth to the initial financing of the project.

Finally, it should be noted that in this model renegotiation-proofness also holds if entrepreneurs are not excluded from the market, that is, they are able to start a new firm by signing a new contract with a different investor. To see this let's observe that in case of liquidation the entrepreneur ends up with zero wealth (see Proposition 1). With zero wealth there is no contract that allows the investor to break even. Therefore, the entrepreneur will not be able to start a new firm. Also notice that this conclusion does not hold when $I_0 = \kappa$, that is, when the set up investment is not sunk. Because in this case $S(\underline{q}) - \underline{q} = \kappa$, a new investor will break even by starting the contract at $q = \underline{q}$. However, it is unlikely that the set up investment can be fully recovered when the firm is liquidated.

7 Properties of the optimal and renegotiation-proof contract

This section describes the dynamics properties of the firm induced by a renegotiation-proof contract. Although some of these properties where implicitly derived in the previous sections, it would be useful to restate them here.

Property 1 (Cash-flow sensitivity) *The investment of constrained (small) firms depends on cash-flows, while the investment of unconstrained (large) firms is independent of cash flows.*

This follows directly from Proposition 1. When $q' < \bar{q}$, the next period input of capital is smaller than the optimal level $\bar{k}$ and depends on q'. Because q' depends on $F(k, \eta)$, then k' depends on current cash-flows. Once $q \geq \bar{q}$, the input of capital is always kept at the optimal level $\bar{k}$ independently of $F(k, \eta)$.[6] Therefore, the model seems to generate the heterogeneous cash flows

[6]Although cash flows have on average a positive effect on investment when $q < \underline{q}$, their impact is not necessarily monotone. The cash-flows sensitivity would be monotone if k is strictly increasing in $q \in [\underline{q}, \bar{q})$, but in general this does not have to be the case.

sensitivity of investment emphasized in empirical papers. We should observe, however, that once we control for the Tobin's q—defined as the ratio $[F(k,\eta)+\tilde{S}(\tilde{q}(\eta))]/(I_0+k)$—cash-flows have no explanatory power.

There are two points that need to be clarified regarding this observation. The first point is that the result that the Tobin's q is a sufficient statistic for investment applies only when the shock is i.i.d. As will be argued in Section 9, cash-flows have an explanatory power beyond the Tobin's q when the shock is persistent. In this sense the model seems to be consistent with the empirical studies about the cash flow sensitivity of investment.

The second point is that, with i.i.d. shocks, the Tobin's q is a sufficient statistic for the financial status of the firm. This implies that the use of this variable to isolate the future production capability or market opportunities of the firm may give misleading answers about the importance of financial constraints. More specifically, even though cash-flows are not statistically significant (once we control for the Tobin's q), this does not mean that firms are financially unconstrained or that financial factors are not important for investment. The main problem is that the Tobin's q reflects both the technology or market opportunities of the firm as well as its financial status. In the analysis conducted so far—based on i.i.d. shocks—the Tobin's q reflects only the financial condition of the firm. In the case of persistent shocks it reflects both its financial status as well as its technology or market opportunities.

PROPERTY 2 (LIQUIDATION PATTERN) *Small (constrained) firms face a positive probability of liquidation at some future date, while large (unconstrained) firms are never liquidated.*

This property follows directly from Propositions 1, 2 and 4. As for the previous property, there is no guarantee that the liquidation probability is monotonically decreasing in q. This will be the case if k and the term $h'(0)[F(k,\underline{\eta})-EF(k,\eta)]+q/\beta$ are increasing in q. In the numerical example studied in the next section, this probability decreases monotonically with q.

PROPERTY 3 (INVESTOR SHARE) *The investor's share of the surplus is decreasing in $q \in (\underline{q},\bar{q})$.*

Proposition 1 has shown that $S(q)$ is strictly increasing for $q \in (\underline{q}, \bar{q})$. Moreover, for a renegotiation-proof contract the slope of $S(q)$ is not greater than 1, which implies that $S(q) - q$ is not increasing. Therefore, $(S(q) - q)/S(q)$ is strictly decreasing. Notice that this property does not necessarily hold for an optimal long-term contract. This is because the slope of $S(q)$ is not necessarily smaller than 1.

Properties 2 and 3 imply that the probability of liquidation is positive when the entrepreneur's share of the firm is low. Therefore, the liquidation of the firm has several features that resemble a firm's bankruptcy. First, the firm faces a high probability of liquidation when the share value of external investors (relative to the total value of the firm) is high. Second, the external investors have the priority in the assets of the firm in case of liquidation (see Proposition 1). Third, the probability of liquidation is positive for small firms, a feature which is also seen in the data.

PROPERTY 4 (INVESTMENT VOLATILITY) *Small (constrained) firms face higher volatility of investment and growth than large (unconstrained) firms.*

This is obvious given that the investment of small firms depends on cash-flows while it is constant for large firms. If we assume that the firm's employment depends on the input of capital, then small firms also experience higher rates of job turnover (creation and destruction). Because small firms are on average younger (assuming that when they enter they have limited internal funds), it also follows that the job turnover in younger firms is greater than in older firms. This is also another feature of the data as shown in Davis, Haltiwanger, & Schuh (1996).

PROPERTY 5 (SAVINGS) *Entrepreneurs in small (constrained) firms have higher rates of savings.*

We have seen in Proposition 1 that the entrepreneur's consumption is zero before the firm reaches the unconstrained status. This is motivated by the higher internal return before the entrepreneur's wealth reaches $\bar{q}$. The higher incentive to save should also hold if the entrepreneur is risk averse, although in this case consumption would be positive before reaching the unconstrained status. This property is consistent with the evidence of higher entrepreneurial savings as shown in Quadrini (1999, 2000) and Gentry & Hubbard (2000).

8 Other properties of the optimal contract: a numerical analysis

This section characterizes some other properties of the optimal contract, using a parameterized version of the model. The period in the model is one year and the discount factor is 0.95. The gross revenue function takes the form $F(k, \eta) = \eta A k^{\nu} + (1-\delta)k$, with η uniformly distributed in the interval $[0, 2]$, $\nu = 0.85$, $\delta = 0.25$ and $A = 0.356$. The parameter ν determines the rents of the firm generated by decreasing returns to scale and/or monopolistic power. Atkeson, Khan, & Ohanian (1996) provide some arguments suggesting that $\nu = 0.85$ is a reasonable parameterization of this parameter. The term δk is interpreted as the sum of capital depreciation and labor costs. The value of 0.25 is consistent with the standard depreciation rate and the labor income share used in calibrated macro models. The parameter A is such that the optimal input of capital is normalized to $\bar{k} = 1$. With respect to the volatility of η I will conduct a sensitivity analysis.

The diversion function takes the form $h(1-e) = \alpha \cdot (1-e)$. In the baseline model $\alpha = 1$ which implies that the entrepreneur is able to consumes all the hidden cash-flow.

To allow for industry dynamics, that is, entrance and exit, I make an additional assumption. In each period there is the entrance of a fixed mass of new firms with initial $q = \underline{q}$. Therefore, new firms are small initially which is consistent with the data. To generate an invariant distribution of firms I also need to allow for some exogenous exit. Otherwise, for each cohort of new entrants there will be some firms that reach the unconstrained status and never exit. This would imply that the total mass of firms grows over time without bound. Exogenous exit is obtained by assuming that in each period the firm faces a probability ϕ of becoming unproductive and is liquidated. Technically this is obtained by replacing the constraint $p(\eta) \in [0, 1]$ with the constraint $p(\eta) \in [\phi, 1]$. Obviously, this does not change the analytical structure and properties of the model we have studied in the previous sections. The exogenous probability is set to $\phi = 0.04$. Together with the endogenous probability, the average exit rate is about 5 percent. This is consistent with the empirical numbers found in industry dynamics studies such as Evans (1987).

Finally, the liquidation value κ is set so that the initial size of new firms (which is equal to the minimum size in the model) is 25 percent the size of incumbent firms. This is consistent with the numbers reported in OECD (2001). The required value is $\kappa = 0.4$. The initial set-up investment I_0

does not need to be specified. The only requirement is that it cannot be too large (see Assumption 3). Given I_0, the initial assets of the entrepreneur must be $a_0 = I_0 - \kappa$. With respect to the parameters κ and a_0 I will conduct a sensitivity analysis.

Properties of the contract and firm dynamics: Panel a of Figure 4 plots the minimum and maximum values of $\tilde{q}(\eta)$, that is, $\tilde{q}(\underline{\eta})$ and $\tilde{q}(\bar{\eta})$, as functions of the entrepreneur's value q. These values are both increasing in q with the exception of $\tilde{q}(\underline{\eta})$ when q is very small. This is because for small values of q the optimal policy is binding, that is, $\tilde{q}(\underline{\eta}) = \underline{q_{min}}$. The value of q_{min} that makes the optimal contract renegotiation-proof is $\underline{q_{min}} = 0.111$. The other relevant thresholds are $\underline{q} = 0.172$ and $\bar{q} = 3.69$.

The minimum value of $\tilde{q}(\eta)$ is always below the 45 degree line before q reaches $\bar{q}$. Notice that the plots are constructed only for values of $q \leq 1.5$, which is smaller than $\bar{q} = 3.69$. This is because for $q > 1.5$, the investment behavior of the firm is already very close to the investment behavior of unconstrained firms as will be shown in the next panel. The fact that $\tilde{q}(\underline{\eta})$ is below the 45 degree line implies that there is a positive probability that the next period value of q is smaller than the current value. This was formally established in Proposition 2. Because $\underline{q_{min}} < \underline{q}$, this also implies that some firms will be "endogenously" liquidated in the model before they reach $q = \bar{q}$ (remember that the probability of liquidation at some future date is positive if $\tilde{q}(\eta) \in [\underline{q_{min}}, \underline{q})$).

The second panel plots the current input of capital k and the minimum and maximum values for the next period capital, conditional on survival. Because k is monotonically increasing in q, the relation between investment and cash-flows is also monotone, that is, higher cash-flows are associated with higher investments. The monotonicity of $\tilde{q}(\underline{\eta})$ and k implies that the probability that the firm is liquidated at some future date is strictly decreasing in q. Although the monotonicity cannot be generalized for all parameterizations, it holds for a large range of parameters.

Panel c of Figure 4 plots the standard deviation of capital growth as a function of the current size of the firm, k, conditional on survival. As the firm expands and becomes larger, the volatility of investment falls. Finally, panel d plots the invariant distribution given the continuous entrance of new firms with initial $q = \underline{q}$. Eventually, some firms will reach the unconstrained status with $\bar{k} = 1$, which explains the concentration in the largest class.

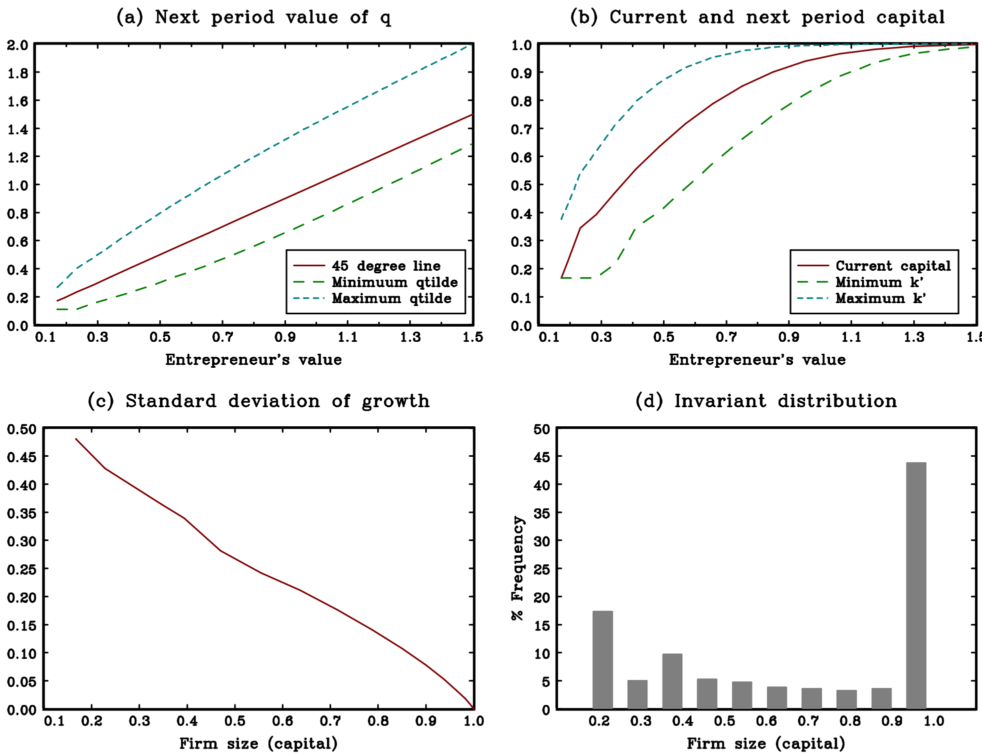

Figure 4: Properties of the contract and invariant distribution

Figure 5 provides information about the age dynamics of firms. After the initial entrance with $q = \underline{q}$, the firm will expand if it realizes a good shock and contracts if it realizes a bad shock. Because new entrants start with $q = \underline{q}$, for new firms is more likely to have $\tilde{q}(\eta) < \underline{q}$. This implies that they face higher probability of exit. This is shown in panel *a*, where the exit rate for newly created firms (firms that are one year old) is more than 10 percent. Because the exogenous probability of exit is 4 percent, for these firms the endogenous exit probability is about 7 percent. After the first period of life, firms that obtain a good shock will have $q > \underline{q}$ and face a lower probability of liquidation. This explains why the average exit probability decreases with age. It is important to point out, however, that age is not an additional explanatory variable for the dynamics of the firms beyond their size. The reason younger firms face lower survival rates is because they are on average smaller. The average size of firms for different age cohort is plotted in panel *b*.

Panel *c* plots the fraction of surviving firms that are "unconstrained". In this graph unconstrained firms are defined as those for which the input of capital is at least 99 percent the optimal

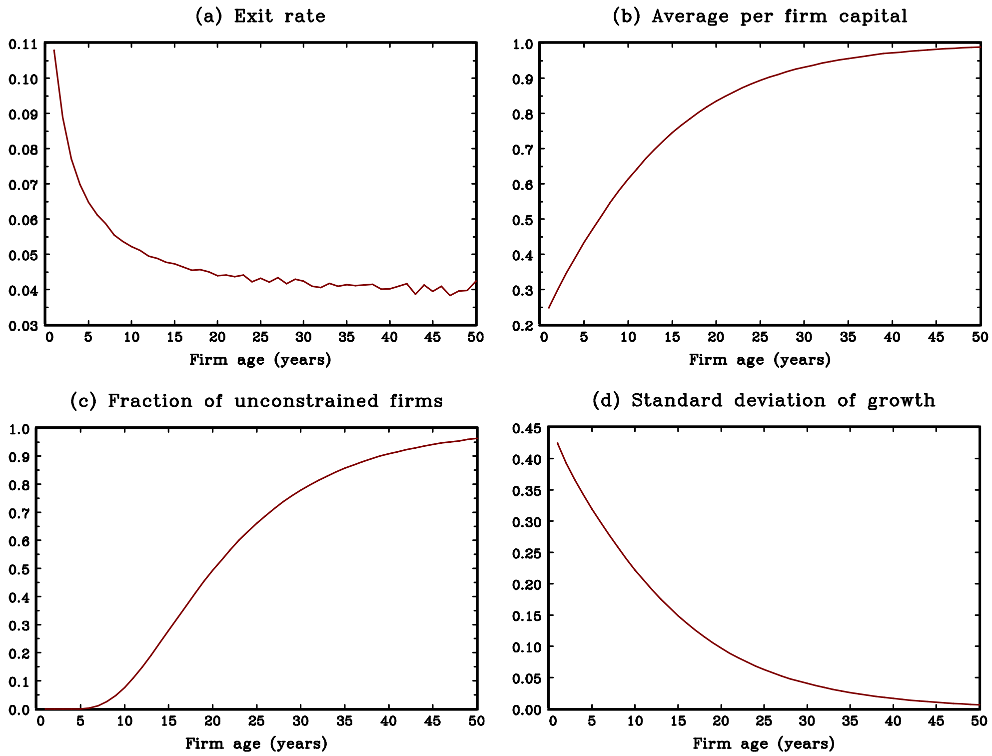

Figure 5: Age dependence.

level $\bar{k} = 1$. The fraction of these firms grows over time and converges to 1. After 20 years, about half of the surviving firms operate at or are close to the optimal scale. This figure provides information about the average time necessary to reach or being close to the optimal scale.

The last panel plots the standard deviation of capital growth. As more and more firms approach the unconstrained status, the volatility of investment declines. This pattern also shows that the sensitivity of investment to cash flows tends to decline for older firms.

Commitment versus renegotiation-proof contracts: Although long-term contracts that are not free from renegotiation cannot be used as a positive theory of actual behavior, it would be interesting to know how the equilibrium would change if commitment were possible.

Figure 7 plots several variables capturing the dynamics of firms for different age cohorts. Two cases are considered: when agents are able to commit to long-term contracts and when contracts are renegotiation-proofs. In the case of renegotiation-proof contracts new firms enter with initial

$q = \underline{q}$. This implies that, at the moment of signing a new contract, the personal wealth of the entrepreneur is $a_0 = I_0 - \kappa$. In the case of commitment it is assumed that new entrepreneurs have the same initial wealth $a_0 = I_0 - \kappa$. The initial entrepreneur's value is then determined as described in Section 6, that is, as the maximum value of q that does not violate the participation constraint for the investor. Because with commitment the surplus of the contract is higher for any $q \in [\underline{q}, \bar{q})$, the initial q is larger than in the case of renegotiation-proof contracts. The initial determination of q is shown in Figure 6. The figure plots the value of the contract for the investor, $S(q) - q$, with and without commitment. In the figure, q^{RP} is the initial q in the environment with renegotiation-proof contracts and q^{CO} is the initial q in the environment with commitment.

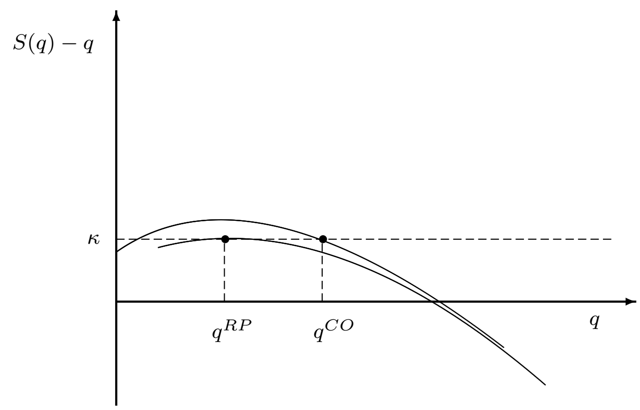

Figure 6: Initial conditions of the optimal contract.

As shown in Figure 7, in the environment with long-term contracts firms experience lower exit rates initially (panel a). This is because the initial q is larger than $\underline{q}$. Of the initial cohort, some firms experience low realizations of η and they get close to $\underline{q}$. This will increase the average probability of exit. Over time, however, fewer firms will have values of q that are close to $\underline{q}$ and the average exit rate is approximately equal to the exogenous rate ϕ.

As a result of the higher initial q, firms reach the optimal scale faster and for each age cohort there is a larger fraction of firms operating at the optimal scale (see panel b). Panel c shows that the average firm size is larger for each age cohort and panel d shows that the volatility of growth is smaller. Therefore, the ability to commit reduces the tightness of financial constraints for the whole economy and reduces the firm level volatility.

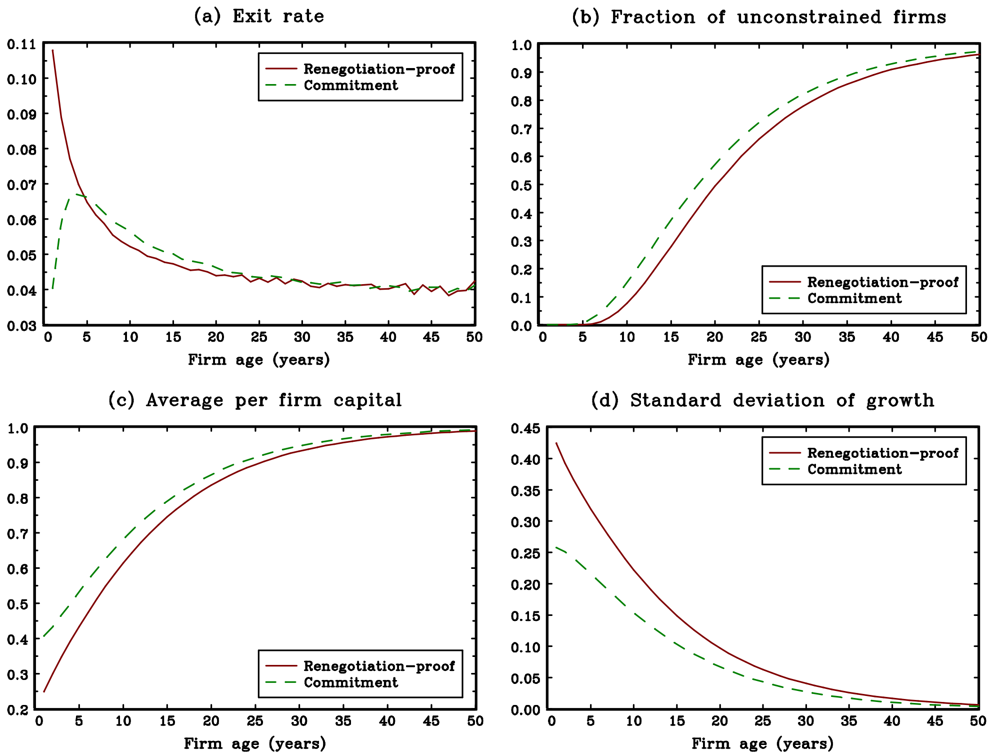

Figure 7: Equilibrium with and without commitment.

Sensitivity analysis: Figure 7 conducts a sensitivity analysis with respect to several parameters. The first two panels plot the average capital and the fraction of unconstrained firms for different values of the repudiation parameter α. Lower values of α means that the entrepreneur gets lower utility from diversion. This relaxes the tightness of the agency problems and, as a result, the average size of firms and the fraction of unconstrained firms increase. This parameter can be interpreted as an index of investor protection. The model would then predicts that higher investor protection increases capital investment. In this sense the paper is related to the literature in law and finance. See Demirguc-Kunt & Levine (2001) for a comprehensive review of the main empirical findings.

Panels *b.1* and *b.2* shows the sensitivity with respect to the variability of the shock. Lower variability reduces the severity of agency problems and allows for higher investment. This is because it is more difficult to divert a large fraction of resources when there is lower uncertainty at the firm level. In the extreme case in which the range of the shock is zero and there is not uncertainty, the firm will be unconstrained from the beginning.

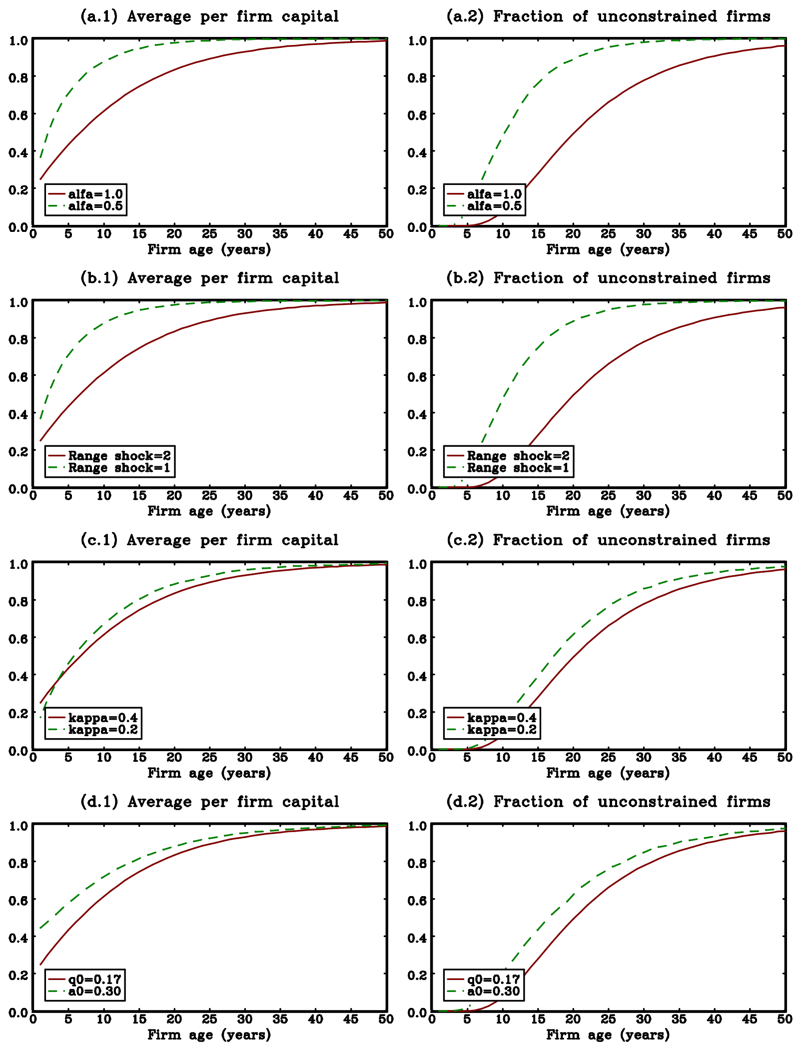

Figure 8: Sensitivity analysis.

Panels *c.1* and *c.2* show the sensitivity with respect to the liquidation value κ. Lower values of κ are associated with higher values of $\underline{q_{min}}$, that is, the lower bond that makes the contract free from renegotiation. This will also increase the value of $\underline{q}$. Because the initial size of new firms is assumed to be $\underline{q}$, this implies that new entrants are initially larger. The larger size of new entrants associated with smaller values of κ tends to persist for several age cohorts of firms.

The last two panels show what happens if we increase the initial wealth of the entrepreneur a_0. In the baseline model we assumed that the initial wealth is just enough to pay for the sunk part of the set-up investment $I_0 - \kappa$. This implies that the initial state of the firm is $q = \underline{q}$. If we increase a_0, the initial q will also increase. Therefore, if potential entrepreneurs are endowed with higher values of wealth, the initial size of firms is larger. The larger size of firms tend to persist for several age cohorts as shown in panel *d.1*. This also explains why the fraction of unconstrained firms increases (see panel *d.2*).

9 History dependence

In this section I briefly describe how the analysis of the previous sections can be extended to the case in which there is history dependence. For economy of space, the technical analysis is contained in a supplemental appendix available upon request from the author. Here I simply describe the details of the model and summarize the main properties.

Assume that the shock still takes values in the set $\eta \in [\underline{\eta}, \bar{\eta}]$ and is observed only by the entrepreneur. However, the probability density depends on a variable z that follows a finite state Markov chain with transition probabilities $\Gamma(z/z_{-1})$. The density function for the shock is denoted by $g(z, \eta)$ and it is assumed that $g(z, \eta)$ is stochastically dominated by $g(\hat{z}, \eta)$ if $z < \hat{z}$. I refer to z as the "persistent factor".

In the supplemental appendix I study two cases. In the first case z is observed by both parties (public information) while in the second case z is observed only by the entrepreneur (private information). Although the analysis of these two cases require different tools, the qualitative properties of the optimal and renegotiation-proof are similar. These properties are also similar to the properties emphasized in Section 7 for the case of i.i.d. shocks, albeit with some qualifications.

Property 1 (Cash-flow sensitivity) *Controlling for z, the investment of constrained firms depends on cash-flows, while the investment of unconstrained firms is independent of cash flows.*

Notice that the investment of unconstrained firms still depends on cash-flows if we do not control for z because the persistent factor affects the next period unconstrained capital, that is, $k' = \bar{k}(z)$.

With history dependence the Tobin's q is no longer a sufficient statistic for the investment of constrained firms. The reason is because firms are now affected by two shocks: z and η. Because these shocks are not perfectly correlated, they have a differential impact on the investment decision of the firm which cannot be summarized by a single variable, that is, the Tobin's q. In this respect the cash-flow sensitivity of investment resembles the results of Abel & Eberly (2002). In their model cash-flows have an additional explanatory power even if there are no financial frictions because the firm is affected by multiple shocks.

The result of Abel & Eberly (2002) points out that the cash-flow sensitivity of investment may not be a good proxy for the existence of financial constraints. At the same time, the absence of cash-flow sensitivity (once we control for the Tobin's q) does not imply that the firm is unconstrained. In fact, we have seen in Section 7 that with i.i.d. shocks the investment of constrained firms is fully explained by the Tobin's q. These remarks parallel earlier results by Gomes (2001).

PROPERTY 2 (LIQUIDATION PATTERN) *For any value of z_{-1}, constrained firms face a higher probability of liquidation at some future date than unconstrained firms.*

With i.i.d. shocks, unconstrained firms would never be liquidated. With persistent shocks, however, the expected productivity of the firm may become so low that the expected future profits are smaller than the liquidation value κ. This would depend not only on the current value of the persistent factor z_{-1} but also on its persistence. However, even if the probability of liquidation for unconstrained firms is positive, this probability is smaller than for constrained firms. More importantly, if we control for the future production capability and market opportunities of the firm—that is, the persistent factor z—there are values of z_{-1} for which unconstrained firms are never liquidated while constrained firms will be liquidated with positive probability at some future date. Unconstrained firms will be liquidated only if they experience a large fall in z.

PROPERTY 3 (INVESTMENT VOLATILITY) *Controlling for z_{-1}, constrained firms face higher volatility of investment and growth than unconstrained firms.*

This property derives directly from the cash-flows sensitivity (property 1): if we control for z_{-1}, the investment of unconstrained firms remain constant independently of the realization of η. Therefore, they do not experience any volatility of investment and growth. The investment of constrained firms, instead, depends on q' which in turn depends on the realization of the shock.

10 Conclusion

The study of financially distressed firms has always received special attention due to the concern that, in case of liquidation, viable productive units would be inefficiently destroyed. However, if the liquidation of the firm is decided within optimal arrangements, it seems counterintuitive that these arrangements would allow for the liquidation of viable projects. Is it possible that in optimal contracts firms with good technologies and market opportunities are liquidated? The answer is yes if there are agency problems induced by information asymmetry and if the contractual parties are able to commit to the initial terms of the contract (long-term contract). In order to induce the desired action from the entrepreneur, it is optimal to make the liquidation of the firm conditional on its performance. However, once the entrepreneur has taken his or her action and the investor has observed the firm's performance, liquidation may be inefficient and the parties have an incentive to renegotiate. This conclusion may lead to the conjecture that liquidation is never observed in a renegotiation-proof contract, unless driven by a deterioration of the firm technology or market opportunities. One of the results of this paper is to invalidate this conjecture. More specifically, this paper shows that a firm can be liquidated in the renegotiation-proof contract even if liquidation is not free from renegotiation in the long-term contract.

In addition to providing a positive theory of the firm's liquidation based on financial factors, this paper also shows that the renegotiation-proof contract can generate several features of the firms' investment behavior and dynamics observed in the data. Thus, financial market frictions, in addition to technological factors, are important for understanding the life-cycle dynamics of a firm as well as its investment behavior.

Appendix

A Definition of a Long-term contract and recursive formulation

Denote by $\mathbf{h}^t$ the history of shock announcements conditional on the survival of the firm, that is, $\mathbf{h}^t \equiv \{\hat{\eta}_1, ..., \hat{\eta}_t\}$. Denote by $\mathbf{H}^t$ the collection of all possible histories up to time t. Conditional on survival, a contract specifies an input of capital k used in period t as a function of $\mathbf{h}^{t-1}$, a liquidation probability p as a function of $(\mathbf{h}^{t-1}, \hat{\eta}_t)$ and a payment to the entrepreneur c as a function of $(\mathbf{h}^{t-1}, \hat{\eta}_t, \ell_t)$. The payments to the investor are determined residually from the budget constraint. Henceforth, I will use the following notational convention: $(\mathbf{h}^{t-1}, \hat{\eta}_t)$ refers to the history up to time t, conditional on the survival of the firm to t, but before randomizing on liquidation. Instead, $\mathbf{h}^t$ refers to the history up to time t after randomizing, conditional on the survival of the firm to $t+1$. Therefore, the difference between $(\mathbf{h}^{t-1}, \hat{\eta}_t)$ and $\mathbf{h}^t$ is that in the former the uncertainty about the survival of the firm to the next period has not been resolved yet, while in the latter it has been resolved in favor of continuation. A formal definition of the contract follows:

Definition 2 (Contract) *A contract* $\sigma = (\sigma^k, \sigma^p, \sigma^c)$ *consists of a sequence of functions* $\{\sigma_t\}_{t=0}^{\infty} = \{\sigma_t^k, \sigma_t^p, \sigma_t^c\}_{t=0}^{\infty}$, *with* $\sigma_t^k : \mathbf{H}^t \rightarrow \mathbf{R}_+$, $\sigma_t^p : \mathbf{H}^t \times \mathbf{N} \rightarrow [0,1]$, $\sigma_t^c : \mathbf{H}^t \times \mathbf{N} \times \{0,1\} \rightarrow [0, \bar{c}]$.

Implicit in the definition of the contract is its feasibility which requires that the inputs of capital and the payments to the entrepreneur are non-negative. In addition, I have imposed that the payments to the entrepreneur take values in the compact set $\mathbf{C} = [0, \bar{c}]$. The upper bound $\bar{c}$ is sufficiently large that it is never binding in the model.

To simplify the notation, I first define the following functions:

$$\bar{\sigma}^c(\mathbf{h}^{t-1}, \hat{\eta}_t) = \sigma^p(\mathbf{h}^{t-1}, \hat{\eta}_t) \cdot \sigma^c(\mathbf{h}^{t-1}, \hat{\eta}_t, 1) + (1 - \sigma^p(\mathbf{h}^{t-1}, \hat{\eta}_t)) \cdot \sigma^c(\mathbf{h}^{t-1}, \hat{\eta}_t, 0) \tag{10}$$

$$\pi(\mathbf{h}^{t-1}, \hat{\eta}_t) = -\sigma^k(\mathbf{h}^{t-1}) + \beta\Big[F(\sigma^k(\mathbf{h}^{t-1}), \hat{\eta}_t) + \sigma^p(\mathbf{h}^{t-1}, \hat{\eta}_t) \cdot \kappa - \bar{\sigma}^c(\mathbf{h}^{t-1}, \hat{\eta}_t)\Big] \tag{11}$$

The first function is the expected payment to the entrepreneur, given the strategies $\sigma^p(\mathbf{h}^{t-1}, \hat{\eta}_t)$ and $\sigma^c(\mathbf{h}^{t-1}, \hat{\eta}_t, \ell_t)$. The second can be interpreted as the flow return for the investor given the strategies $\sigma^k(\mathbf{h}^{t-1})$, $\sigma^p(\mathbf{h}^{t-1}, \hat{\eta}_t)$ and $\sigma^c(\mathbf{h}^{t-1}, \hat{\eta}_t, \ell_t)$: the investor anticipates $\sigma^k(\mathbf{h}^{t-1})$ at $t-1$ and for each $\hat{\eta}_t$ announced in the next period, he or she expects to receive $F(\sigma^k(\mathbf{h}^{t-1}), \hat{\eta}_t) + \sigma^p(\mathbf{h}^{t-1}, \hat{\eta}_t) \cdot \kappa - \bar{\sigma}^c(\mathbf{h}^{t-1}, \hat{\eta}_t)$.

A contract must be *individually rational* and *incentive compatible*. Acceptability or individual rationality requires that the expected benefits for both parties are non smaller than their reservation values. At the moment of signing the contract, the reservation value for the investor is zero while for the entrepreneur is the value of his or her wealth a_0. After signing the contract, the investor commits to fulfill any future obligation while the reservation value for the entrepreneur, conditional on continuation, cannot be smaller than $q_{min} \geq 0$. The definition of individual rationality follows.

Definition 3 (Individual rationality) *A contract σ is individually rational if:*

$$E_0 \sum_{t=1}^{\infty} \beta^{t-1} \pi(\mathbf{h}^{t-1}, \hat{\eta}_t) \geq I_0 - a_0 \tag{12}$$

$$E_0 \sum_{t=1}^{\infty} \beta^{t} \bar{\sigma}^c(\mathbf{h}^{t-1}, \hat{\eta}_t) \geq a_0 \tag{13}$$

$$E_t \sum_{s=t}^{\infty} \beta^{s-t} \bar{\sigma}^c(\mathbf{h}^{s-1}, \hat{\eta}_s) \geq q_{min} \qquad \forall \ \mathbf{h}^{t-1} \in \mathbf{H}^{t-1}, \, t \geq 1 \tag{14}$$

Conditions (12) and (13) are the participation constraints for the investor and the entrepreneur at the moment of signing the contract, while (14) is the acceptability condition for the entrepreneur at each subsequent history in which the firm is not liquidated. Given the assumption of commitment, for the investor there is no other acceptability constraint beyond the first period. Notice that the expectation is conditional on $\mathbf{h}^{t-1}$ (which is conditional on survival) and $\hat{\eta}_t$.

Definition 4 (Incentive-compatibility) *A contract σ is incentive compatible if for all $\mathbf{h}^{t-1} \in \mathbf{H}^{t-1}$, $\eta_t, \hat{\eta}_t \in \mathbf{N}$, $t \geq 1$,*

$$E_t \sum_{s=t}^{\infty} \beta^{s-t} \bar{\sigma}^c(\mathbf{h}^{s-1}, \eta_s) \geq D(\sigma^k(\mathbf{h}^{t-1}), \eta_t, \hat{\eta}_t) + \bar{\sigma}^c(\mathbf{h}^{t-1}, \hat{\eta}_t) + E_t \sum_{s=t+1}^{\infty} \beta^{s-t} \bar{\sigma}^c(\hat{\mathbf{h}}^{s-1}, \eta_s)$$

where $\hat{\mathbf{h}}^t = \{\mathbf{h}^{t-1}, \hat{\eta}_t\}$ and $\hat{\mathbf{h}}^s = \{\hat{\mathbf{h}}^{s-1}, \eta_s\}$ for $s > t$.

Incentive-compatibility requires that, for each history of shocks and conditional on the survival of the firm, the entrepreneur has no incentive to misreport the shock and divert resources, assuming that the entrepreneur follows the truth-telling strategy in all future periods. The return from announcing a smaller shock (and diverting resources) is given by the unobservable private return $D(k, \eta, \hat{\eta}) = h(1 - e(k, \eta, \hat{\eta}))F(k, \eta)$, plus the expected payment received with the announcement of $\hat{\eta}_t$ and the continuation value following the history $\hat{\mathbf{h}}^t$. If incentive-compatibility for one-shot deviation is satisfied, it will be satisfied for any finite number of deviations. (See theorem 4.2 in Fudenberg & Tirole (1993).)

To use a more compact notation, denote by $U^I(\sigma)$ the expected value of the contract σ for the investor and $U^E(\sigma)$ the expected value for the entrepreneur. They are defined as:

$$U^I(\sigma) = E_0 \sum_{t=1}^{\infty} \beta^{t-1} \pi(\mathbf{h}^{t-1}, \hat{\eta}_t) \tag{15}$$

$$U^E(\sigma) = E_0 \sum_{t=1}^{\infty} \beta^{t} \bar{\sigma}^c(\mathbf{h}^{t-1}, \hat{\eta}_t) \tag{16}$$

The properties of individual rationality and incentive-compatibility are:

$$U^I(\sigma) \geq I_0 - a_0 \tag{17}$$

$$U^E(\sigma) \geq a_0 \tag{18}$$

$$U^E(\sigma|\mathbf{h}^{t-1}, \hat{\eta}_t) \geq q_{min}, \qquad \forall\ \mathbf{h}^{t-1} \in \mathbf{H}^{t-1},\ \hat{\eta}_t \in \mathbf{N},\ t \geq 1 \tag{19}$$

$$U^E(\sigma|\mathbf{h}^{t-1}, \eta_t) \geq U^E(\sigma|\mathbf{h}^{t-1}, \hat{\eta}_t), \qquad \forall\ \mathbf{h}^{t-1} \in \mathbf{H}^{t-1},\ \eta_t, \hat{\eta}_t \in \mathbf{N},\ t \geq 1 \tag{20}$$

While conditions (19) and (20) have to be satisfied at each point in time and for each history, conditions (17) and (18) have to be satisfied only at the moment of signing the contract. It would be convenient then to characterize implementable contracts disregarding, for the moment, conditions (17) and (18). Henceforth, I will refer to a contract that satisfies conditions (19) and (20) simply as "long-term contract".

B Set of implementable contracts and recursive formulation

Define $\mathbf{V}$ the set of values achievable by the investor and the entrepreneur with long-term contracts.

$$\mathbf{V} = \left\{ (U^I(\sigma), U^E(\sigma)) \in \mathbf{R}^2 \,\middle|\, \sigma \text{ satisfies (19)-(20)} \right\}$$

The next proposition establishes an important property of this set.

Proposition 5 *The set* $\mathbf{V}$ *is not empty and compact.*

Proof: *See Abreu, Pearce, & Stacchetti (1990).*

The fact that the set $\mathbf{V}$ is not empty is easy to prove. Consider the contract in which the investor makes a fixed payment to the entrepreneur in every period and the entrepreneur retains and reinvests optimally all the profits. Of course, this contract is incentive compatible. It may violate the initial acceptability condition but, for the moment, the definition of a long-term contract does not take into account this constraint. The compactness proof is more complex and can be obtained by following the steps used by Abreu et al. (1990).

Define $\mathbf{S}^E$ the set of values achievable by the entrepreneur with long-term contracts, that is,

$$\mathbf{S}^E = \left\{ q \in \mathbf{R} \,\middle|\, \exists\, u \in \mathbf{R} \;\; s.t. \;\; (u, q) \in \mathbf{V} \right\} \tag{21}$$

Moreover, define $\mathbf{S}^I(q)$ the set of values achievable by the investor with long-term contracts, conditional on the entrepreneur receiving the value $q \in \mathbf{S}^E$. Formally, for all $q \in \mathbf{S}^E$,

$$\mathbf{S}^I(q) = \left\{ u \in \mathbf{R} \,\middle|\, (u, q) \in \mathbf{V} \right\} \tag{22}$$

The utility frontier, defined over $q \in \mathbf{S}^E$, is given by the function:

$$UF(q) = \max u \in \mathbf{S}^I(q) \tag{23}$$

Even if a contract delivers *initially* values that are located on the utility frontier, the continuation values at some future history may be interior to the set $\mathbf{V}$. In the remaining part of this section, I will show that future values delivered by an optimal long-term contract are always located on the frontier. This result allows me to characterize the optimal contract recursively using a Bellman's equation.

In defining the recursive problem, it is convenient to work with the total surplus generated by a long-term contract, that is, the sum of values for the investor and the entrepreneur. Therefore, instead of working with the function $UF(q)$, I will use the function $S^*(q) = UF(q) + q$. Of course, if the surplus of a long-term contract satisfies this function, then the contract values are located on the utility frontier.

Define $C(q) = \{k(q), p(q)(\hat{\eta}), c(q)(\hat{\eta}, \ell), v(q)(\hat{\eta}, \ell)\}$ the functions that, for each state $q \in \mathbf{S}^E$, give the invested capital $k \in \mathbf{K}$, the liquidation probability $p \in [0, 1]$, the payment to the entrepreneur $c \in \mathbf{C}$ and the next period value for the entrepreneur $q' \in \mathbf{S}^E$. Moreover, for any bounded function $S : \mathbf{R}_+ \to \mathbf{R}$, define the mapping T as:

$$T(S)(q) = \max_{C(q)} \int_{\underline{\eta}}^{\bar{\eta}} \Big[-k + \beta[F(k,\eta) + p(\eta)\kappa + (1-p(\eta))S(v(\eta,0))]\Big] g(\mathrm{d}\eta) \tag{24}$$

subject to

$$p(\eta)[c(\eta,1) + v(\eta,1)] + (1-p(\eta))[c(\eta,0) + v(\eta,0)] \geq \tag{25}$$

$$D(k,\eta,\hat{\eta}) + p(\hat{\eta})[c(\hat{\eta},1) + v(\hat{\eta},1)] + (1-p(\hat{\eta}))[c(\hat{\eta},0) + v(\hat{\eta},0)] \qquad \forall\, \eta, \hat{\eta} \in \mathbf{N}$$

$$q = \beta \int_{\underline{\eta}}^{\bar{\eta}} \Big(p(\eta)[c(\eta,1) + v(\eta,1)] + (1-p(\eta))[c(\eta,0) + v(\eta,0)]\Big) g(\mathrm{d}\eta) \tag{26}$$

$$p(\eta)[c(\eta,1) + v(\eta,1)] + (1-p(\eta))[c(\eta,0) + v(\eta,0)] \geq q_{min} \tag{27}$$

The function S is the surplus generated by the contract if the firm is not liquidated. In defining the above mapping, I have dropped the dependence of the policy functions from q to simplify the notation. Observe that, by imposing the incentive-compatibility constraints (25), the entrepreneur will never divert resources. Consequently, the contractual problem is specified assuming $\hat{\eta} = \eta$. Also observe that k denotes the capital invested today which will give the revenue $F(k,\eta)$ in the next period, after the realization of the next period shock η. Therefore, the current net value of revenues is $-k + \beta E F(k,\eta)$.

PROPOSITION 6 *The surplus function $S^*(q)$ is the fixed point of the mapping T, that is, $S^*(q) = T(S^*)(q)$, for all $q \in \mathbf{S}^E$.*

PROOF: *To prove that $S^* = T(S^*)$, I show that both $S^* \geq T(S^*)$ and $S^* \leq T(S^*)$ hold. To show that $S^* \geq T(S^*)$, it is enough to prove that there exists a long-term contract σ such that $U^E(\sigma) = q$ and $U^I(\sigma) + U^E(\sigma) = T(S^*)$. Consider the contract $\sigma = \{\sigma^k(\mathbf{h}^0), \sigma^p(\mathbf{h}^0,\eta), \sigma^c(\mathbf{h}^0,\eta,\ell), \sigma|\mathbf{h}^1\}$ where the first components are defined as:*

$$\begin{aligned} \sigma^k(\mathbf{h}^0) &= k(q_0) \\ \sigma^p(\mathbf{h}^0,\eta) &= p(q_0)(\eta) \\ \sigma^c(\mathbf{h}^0,\eta,\ell) &= c(q_0)(\eta,\ell) \end{aligned}$$

and the continuation satisfies:

$$\begin{aligned} U^I(\sigma|\mathbf{h}^1) &= S^*(v(q_0)(\eta,0)) - v(q_0)(\eta,0) \\ U^E(\sigma|\mathbf{h}^1) &= v(q_0)(\eta,0) \end{aligned}$$

Notice that this is possible because $S^*\Big(v(q_0)(\eta,0)\Big) - v(q_0)(\eta,0) \in \mathbf{V}^I$ *and* $v(q_0)(\eta,0) \in \mathbf{V}^E$*. By construction the above contract satisfies* $U^I(\sigma) + U^E(\sigma) = T(S^*)$*. Because* $U^I(\sigma) + U^E(\sigma)$ *cannot be bigger than* S^**, I have proved that* $S^* \geq T(S^*)$*.*

I show now that $S^* \leq T(S^*)$*. For any contract* σ *for which* $(U^I(\sigma|q), U^E(\sigma|q)) \in \mathbf{V}(q)$*, we have:*

$$U^I(\sigma|q) + U^E(\sigma|q) \leq T(S^*)(q) \tag{28}$$

This must also hold for the contract that maximizes $U^I(\sigma|q) + U^E(\sigma|q)$*, which is* $S^*(q)$*. Because this holds for all* q*, then* $S^* \leq T(S^*)$*.* *Q.E.D.*

C Definition of renegotiation-proof contract

DEFINITION 5 (RENEGOTIATION-PROOF CONTRACT) *A contract* σ *is renegotiation-proof if there is no other implementable (feasible, individually rational and incentive-compatible) contract* $\hat{\sigma}$ *for which*

$$U^E(\hat{\sigma}|\mathbf{h}^{t-1}, \eta_t) > U^E(\sigma|\mathbf{h}^{t-1}, \eta_t) \qquad \text{and} \qquad U^I(\hat{\sigma}|\mathbf{h}^{t-1}, \eta_t) > U^I(\sigma|\mathbf{h}^{t-1}, \eta_t),$$
$$U^E(\hat{\sigma}|\mathbf{h}^{t}) > U^E(\sigma|\mathbf{h}^{t}) \qquad \text{and} \qquad U^I(\hat{\sigma}|\mathbf{h}^{t}) > U^I(\sigma|\mathbf{h}^{t})$$

for any $\mathbf{h}^t \in \mathbf{H}^t$*,* $\eta_t \in \mathbf{N}$*, and* $t \geq 1$*.*

The two conditions impose renegotiation-proofness in the two stages of the contract: before and after randomization. This definition is consistent with several definitions adopted in the literature. See Bergin & MacLeod (1993) for a review.

D Proof of Lemma D

Let $D(k,\eta,\hat{\eta}) = h(1 - e(k,\eta,\hat{\eta}))F(k,\eta)$ the private return from diverting resources when the true shock is η and the announcement is $\hat{\eta}$. The marginal gain in private return from decreasing the shock announcement from $\hat{\eta}$ is equal to the negative of the derivative of $D(k,\eta,\hat{\eta})$ with respect to $\hat{\eta}$. Because $e(k,\eta,\hat{\eta}) = F(k,\hat{\eta})/F(k,\eta)$, the marginal gain is $-R_{\hat{\eta}}(k,\eta,\hat{\eta}) = h'(1 - e(k,\eta,\hat{\eta}))F_\eta(k,\hat{\eta})$.

Incentive-compatibility requires that the marginal gain from diverting is not larger than the marginal loss in promised value from the schedule $\tilde{q}(\hat{\eta})$. In particular, this must be satisfied at $\hat{\eta} = \eta$. When $\hat{\eta} = \eta$, $e(k,\eta,\eta) = 1$ and the marginal gain from diversion is $h'(0)F_\eta(k,\eta)$. Now consider the function

$b(\hat{\eta}) = \tilde{q}(\underline{\eta}) + \int_{\underline{\eta}}^{\hat{\eta}} h'(0)F_\eta(k,\eta) = \tilde{q}(\underline{\eta}) + h'(0)[F(k,\hat{\eta}) - F(k,\underline{\eta})]$. This is the integral from $\underline{\eta}$ to $\hat{\eta}$ of the marginal gain from diverting at the true shock, added to the term $\tilde{q}(\underline{\eta})$. If the schedule $\tilde{q}(\hat{\eta})$ is equal to $b(\hat{\eta})$, then the entrepreneur does not have an incentive to marginally divert resources from the true realization of the shock η. However, this does not exclude that the entrepreneur has an incentive to announce a value of $\hat{\eta}$ smaller than η. The next step is to show that under the schedule $\tilde{q}(\hat{\eta}) = b(\hat{\eta})$, the entrepreneur has no incentive to announce any $\hat{\eta} < \eta$. It is enough to show that the marginal gain evaluated at $\hat{\eta}$ is (weakly) decreasing in the true realization of the shock η. This simply means that the marginal gain from further decreasing $\hat{\eta}$ is smaller the farther is $\hat{\eta}$ from the true shock η. Let η_1 and η_2 be two different true realizations of the shock and $\hat{\eta}$ a possible shock announcement. They satisfy $\hat{\eta} \leq \eta_1 < \eta_2$. The marginal gains evaluated at η when the true shock is respectively η_1 and η_2 are:

$$h'(1 - e(k,\eta_1,\hat{\eta}))F_\eta(k,\hat{\eta}) \qquad \text{if } \eta = \eta_1 \tag{29}$$

$$h'(1 - e(k,\eta_2,\hat{\eta}))F_\eta(k,\hat{\eta}) \qquad \text{if } \eta = \eta_2 \tag{30}$$

Because $e(k,\eta_1,\hat{\eta}) > e(k,\eta_2,\hat{\eta})$ and $h(.)$ is concave, we have that $h'(1 - e(k,\eta_1,\hat{\eta}))F_\eta(k,\hat{\eta}) \geq h'(1 - e(k,\eta_2,\hat{\eta}))F_\eta(k,\hat{\eta})$. This means that the entrepreneur gains less in private return from marginally decreasing $\hat{\eta}$, when the true shock η is higher. Because the schedule $b(\hat{\eta})$ guarantees that the gain in promised value from not diverting at $\hat{\eta} = \eta$ is equal to the gain from marginally diverting, it must still be true that the marginal gain in private return is not larger than the marginal loss in promised value from marginally decreasing $\hat{\eta}$ when $\hat{\eta} < \eta$.

The function $b(\hat{\eta})$ is not the only incentive-compatible schedule. It only sets lower bounds to its slope. Therefore, any schedule $\tilde{q}(\hat{\eta})$ such that $\tilde{q}(\hat{\eta}) - b(\hat{\eta})$ is not decreasing in $\hat{\eta}$, is also incentive compatible. *Q.E.D.*.

E Proof of Proposition 1

Let's assume that $b(\eta)$ is the optimal schedule, that is, $\tilde{q}(\eta) = \tilde{q}(\eta) = \tilde{q}(\underline{\eta}) + h'(0)[F(k,\eta) - F(k,\underline{\eta})]$. I will show later that this is not only an assumption but it is optimal.

In addition to incentive-compatibility, the entrepreneur must also get the promised value, that is, $q = \beta E\tilde{q}(\eta)$. Substituting the optimal schedule in the promised keeping constraint, we get $q = \beta\tilde{q}(\underline{\eta}) + \beta h'(0)[EF(k,\eta) - F(k,\underline{\eta})]$. Because $\tilde{q}(\underline{\eta})$ cannot be smaller than q_{min}, this constraint imposes an upper bound to k. This constraint also makes clear that smaller is q, and smaller is the upper bound on k. Using this equation to eliminate $\tilde{q}(\underline{\eta})$ in $\tilde{q}(\eta) = \tilde{q}(\underline{\eta}) + h'(0)[F(k,\eta) - F(k,\underline{\eta})]$ we get:

$$\tilde{q}(\eta) = h'(0)[F(k,\eta) - EF(k,\eta)] + \frac{q}{\beta} \tag{31}$$

Equation (31) defines the schedule for the next period promised value $\tilde{q}(\eta)$ (before randomization), given the input of capital. Notice that $\tilde{q}(\eta)$ is fully determined by the choice of k.

Consider now the following mapping:

$$\begin{aligned} T(S)(q) &= \max_{k\geq 0, p(\eta), c(\eta,\ell), q(\eta,\ell)} \Big\{ -k + \beta E\left[F(k,\eta) + p(\eta)\kappa + (1-p(\eta))S\left(q(\eta,0)\right)\right] \Big\} \\ & \quad \text{subject to} \\ \tilde{q}(\eta) &= h'(0)[F(k,\eta) - EF(k,\eta)] + \frac{q}{\beta} \\ \tilde{q}(\eta) &= p(\eta)[c(\eta,1) + q(\eta,1)] + (1-p(\eta))[c(\eta,0) + q(\eta,0)] \\ q &\geq \beta q_{min} + \beta h'(0)[EF(k,\eta) - F(k,\underline{\eta})] \end{aligned} \tag{32}$$

The last constraint is the promised-keeping constraint after replacing $\tilde{q}(\underline{\eta})$ with q_{min}. Because $\tilde{q}(\underline{\eta}) \geq q_{min}$, the equality sign has been replaced with the inequality sign. The above mapping satisfies the Blackwell conditions for a contraction. Therefore, there is a unique fixed point S^*. In addition, this mapping preserves monotonicity (increasing) and concavity. To show this, separate the choice of k from the choice of $p(\eta)$, $c(\eta,\ell)$ and $q(\eta,\ell)$ and rewrite the mapping as follows:

$$\begin{aligned} T(S)(q) &= \max_{x} \Big\{ -f^{-1}(x)[1-\beta(1-\delta)] + \beta E\eta x + \beta E\tilde{S}(\tilde{q}(\eta)) \Big\} \\ & \quad \text{subject to} \\ \tilde{q}(\eta) &= h'(0)(\eta - E\eta)x + \frac{q}{\beta} \\ q &\geq \beta q_{min} + \beta h'(0)(E\eta - \underline{\eta})x \end{aligned} \tag{33}$$

$$\begin{aligned} \tilde{S}(\tilde{q}) &= \max_{p, c(\ell), q(\ell)} \Big\{ p\cdot\kappa + (1-p)\cdot S\left(q(0)\right) \Big\} \\ & \quad \text{subject to} \\ \tilde{q} &= p(\eta)[c(1) + q(1)] + (1-p)[c(0) + q(0)] \end{aligned} \tag{34}$$

Notice that I have made a change of variables in the first part of the problem. Instead of maximizing k, I maximize $x = f(k)$. Because the function $f(k)$ is monotonically increasing, there is a unique correspondence between x and k. The advantage of this change of variable is that the schedule $\tilde{q}(\eta)$ is linear in x.

Consider first problem (34). This is the problem after the announcement of the shock but before randomizing on liquidation. I will show now that if S is increasing and concave, $\tilde{S}$ is also increasing and concave in $\tilde{q}$. To see this, observe first that if S is increasing and concave, then for any liquidation probability p, an optimal solution is $c(0) = c(1) = q(1) = 0$ and $q(0) = \tilde{q}/(1-p)$ (although not necessarily unique). Therefore, I can write the objective function as:

$$\tilde{S}(\tilde{q}) = \max_{p} \left\{ p \cdot \kappa + (1-p) \cdot S\left(\frac{\tilde{q}}{1-p}\right)\right\} \tag{35}$$

From this formulation is easy to see that $\tilde{S}$ is concave. Take $\tilde{q}_1$ and $\tilde{q}_2$ and assume that the optimal solutions are respectively p_1 and p_2. Denote by $\tilde{q}_\theta$ and p_θ the linear combinations of these points. Because S is concave, there exists a linear function $H(\tilde{q})$ such that $S(\tilde{q}) \leq H(\tilde{q})$ for $\tilde{q} \neq \tilde{q}_\theta/(1-p_\theta)$ and $S(\tilde{q}) = H(\tilde{q})$ for $\tilde{q} = \tilde{q}_\theta/(1-p_\theta)$. Then,

$$\begin{aligned}
\tilde{S}(\tilde{q}_\theta) &\geq p_\theta\kappa + (1-p_\theta)S(\tilde{q}_\theta/(1-p_\theta)) \\
&= p_\theta\kappa + \theta(1-p_1)H(\tilde{q}_\theta/(1-p_\theta)) + (1-\theta)(1-p_2)H(\tilde{q}_\theta/(1-p_\theta)) \\
&= p_\theta\kappa + \theta(1-p_1)H(\tilde{q}_1/(1-p_1)) + (1-\theta)(1-p_2)H(\tilde{q}_2/(1-p_2)) \\
&\geq p_\theta\kappa + \theta(1-p_1)S(\tilde{q}_1/(1-p_1)) + (1-\theta)(1-p_2)S(\tilde{q}_2/(1-p_2)) \\
&= \theta\tilde{S}(\tilde{q}_1) + (1-\theta)\tilde{S}(\tilde{q}_2)
\end{aligned}$$

To show the third step, notice that the function H is linear. Therefore, we can always write $H(\tilde{q}_\theta/(1-p_\theta)) = H(\tilde{q}/(1-p)) + \alpha[\tilde{q}_\theta/(1-p_\theta) - \tilde{q}/(1-p)]$ where α is the slope of H. The equivalence, then, trivially follows.

I can show now that if $\tilde{S}$ is concave, then S is strictly concave. This follows from the fact that in problem (33) the law of motion $\tilde{q}(\eta)$ is linear in x and q, for each η, and the return function $-f^{-1}(x)[1-\beta(1-\delta)]+\beta E\eta x$ is strictly concave in x. Therefore, the surplus function is strictly concave. We can also show that $S(q)$ converges to $\beta\kappa$ as q converges to zero. In fact, in the limit case in which $q = 0$, the incentive-compatibility constraint is satisfied only if $x = k = 0$. Moreover, because $q' = q$, this must be the case also in future periods. If there is no possibility of production, the best strategy is to liquidate the firm. Because this will be done in the next period, the liquidation value is discounted by β. The differentiability of S can be proved by verifying the conditions of Theorem 9.10 in Stokey, Lucas, & Prescott (1989).

The concavity of F implies that there exists a maximum amount of capital $\bar{k}$ which is optimal. Define $\bar{q}$ as the value of q that satisfies $\bar{q} = h'(0)[F(\bar{k}, \underline{\eta}) - EF(\bar{k}, \eta)] + \bar{q}/\beta$. This is the incentive-compatibility constraint after imposing $k = \bar{k}$ and $\eta = \underline{\eta}$. If $q = \bar{q}$, then the next period value of q cannot be smaller than $\bar{q}$ even if $k = \bar{k}$. Consequently, once q has reached $\bar{q}$, the optimal input of capital will always be $\bar{k}$. This is the unconstrained status for the firm. Further increases in q cannot change the surplus because k is already at the optimal level. On the other hand, if $q < \bar{q}$ and $k = \bar{k}$, q' can be smaller than $\bar{q}$. After a sequence of small realizations of η, the value of q will be sufficiently small that the firm can be either liquidated and/or is unable to use $k = \bar{k}$. Therefore, if q is smaller than $\bar{q}$, the firm is dynamically constrained.

To show that there is some $\underline{q}$ for which $q(0) = \underline{q}$ when $\tilde{q} < \underline{q}$ and $q(0) = \tilde{q}$ when $\tilde{q} \geq \underline{q}$, consider again problem (34). We have already seen that this problem can be simplified as in (35). Given the strict concavity of S, it is easy to verify that the solution is unique. Moreover, if the solution is interior ($0 < p < 1$), this solution is characterized by the first order condition:

$$\kappa + S'\left(\frac{\tilde{q}}{1-p}\right) \cdot \left(\frac{\tilde{q}}{1-p}\right) - S\left(\frac{\tilde{q}}{1-p}\right) = 0 \tag{36}$$

Consider the points $\tilde{q}_1$ and $\tilde{q}_2$ for which the solutions are interior. Let p_1 and p_2 be the solutions. Because there is a unique value of $\tilde{q}/(1-p)$ that satisfies the first order condition, we must have that $\tilde{q}_2/(1-p_2) = \tilde{q}_1/(1-p_1)$. Because $\tilde{q}/(1-p) = q(0)$, this implies that $q(0)$ is the same for all $\tilde{q}$ having interior solutions. This point is denoted by $\underline{q}$. This also shows that, for $\tilde{q} \geq \underline{q}$, the solution cannot be interior and $q(0) = \tilde{q}$. Given this result, it can be verified that the maximum slope of $\tilde{S}$ is $(\tilde{S}(\underline{q}) - \kappa)/\underline{q}$ and $(\tilde{S}(q) - \kappa)/q \leq (\tilde{S}(\underline{q}) - \kappa)/\underline{q}$ for all q. This property will be useful in the proofs of later propositions.

The previous proofs assumed that $\tilde{q}(\eta) = b(\eta)$. In general, however, the incentive-compatible schedule can be written as $\tilde{q}(\eta) = b(\eta) + m(\eta)$ where $m(\eta)$ is not decreasing and satisfies $Em(\underline{\eta}) = 0$. Now consider a possible deviation from $m(\eta) = 0$. Keeping constant the input of capital, the volatility of next period q increases. Given the concavity of S, this cannot be optimal. *Q.E.D.*

F Proof of Proposition 2

Consider problem (6) and define λ the Lagrange multiplier associated with the promised keeping constraint (second constraint). The envelope condition is $S_q = \lambda + E\tilde{S}_{\tilde{q}}$. If $\lambda > 0$, then $\tilde{q}(\underline{\eta}) = q_{min} = 0$ and clearly $\tilde{q}(\eta) < q$ for some η, unless $q = q_{min} = 0$. If $\lambda = 0$, then $S_q = E\tilde{S}_{\tilde{q}}$. Remember that $q \geq \underline{q}$ and for $q \geq \underline{q}$, $\tilde{S}(q) = S(q)$. Because S is strictly concave in $q \in (\underline{q}, \bar{q})$, $S_q = E\tilde{S}_{\tilde{q}}$ cannot be satisfied if $\tilde{q}(\eta) \geq q$ for all η. Now consider the case in which $\kappa = 0$. In this case the firm is never liquidated. Consequently $\underline{q} = q_{min} = 0$ and $\tilde{q}(\eta) = q'$. Given the previous result, if $q < \bar{q}$, there is always the probability that future values of q get arbitrarily close to zero. From the promised-keeping constraint, this also implies that the input of capital gets arbitrarily close to zero and F_k converges to infinity. From the first order conditions we can derive $[1 - h'(0)S_q]\beta EF_k + \beta h'(0)ES_{\tilde{q}}F_k = 1$. Because $h'(0) \leq 1$, if $S_q \leq 1$, this cannot be satisfied as k converges to zero. By continuity this result applies to sufficiently small values of κ. *Q.E.D.*

G Proof of Proposition 3

It is enough to show that: i) the maximum value of $\tilde{S}_q$ (the derivative of the function $\tilde{S}$) is not increasing in q_{min}; and ii) there exists some $q_{min} < \bar{q}$ for which the maximum value of $\tilde{S}_q < 1$. I prove first that the maximum value of $\tilde{S}_q$ is decreasing in q_{min}.

Consider a value of q_{min} for which the contract is not renegotiation-proof. Denote this value by q^1_{min} and let's use the superscript 1 to denote all functions and values associated with q^1_{min}. The maximum value of $\tilde{S}^1_q$ is equal to $(S^1(\underline{q}^1) - \kappa)/\underline{q}^1 > 1$. Now consider $q^2_{min} > q^1_{min}$. The maximum value of $\tilde{S}^2_q$ associated with q^2_{min} will be $(S^2(\underline{q}^2) - \kappa)/\underline{q}^2$. We have the following inequality:

$$\frac{S^1(\underline{q}^1) - \kappa}{\underline{q}^1} \geq \frac{S^1(\underline{q}^2) - \kappa}{\underline{q}^2} \geq \frac{S^2(\underline{q}^2) - \kappa}{\underline{q}^2} \tag{37}$$

The first inequality derives from the fact that, for any value of q_{min}, $(S(q) - \kappa)/q < (S(\underline{q}) - \kappa)/\underline{q}$ for all $q \neq \underline{q}$. This in turn derives from the definition of $\underline{q}$ (the randomizing point) and the strict concavity of $S(q)$. The second inequality derives from the fact that $S^2(q)$ cannot be bigger than $S^1(q)$ for any value of q (remember that by increasing q_{min} we restrict the feasible values of k). Therefore, the maximum slope of $\tilde{S}$ cannot increase in q_{min}.

I now show that there is some $q_{min} < \bar{q}$ for which the maximum value of $\tilde{S}_q$ is smaller than 1. Consider a point $\hat{q} \in (\underline{q}, \bar{q})$ for which $S(\hat{q}) < \kappa + \hat{q}$. It can be verified that $S(q) < \kappa + q$ for all $q \geq \hat{q}$. I show now that if we impose $q_{min} = \hat{q}$, $\tilde{S}_q < 1$. The function $\tilde{S}(\tilde{q})$ is defined as:

$$\tilde{S}(\tilde{q}) = \max_p \left\{ p \cdot \kappa + (1 - p) \cdot S\left(\frac{\tilde{q}}{1 - p}\right) \right\} \tag{38}$$

and the first order conditions are:

$$\kappa + \left(\frac{\tilde{q}}{1 - p}\right) \cdot S_q\left(\frac{\tilde{q}}{1 - p}\right) - S\left(\frac{\tilde{q}}{1 - p}\right) \leq 0, \qquad (< \text{if } p = 0) \tag{39}$$

Notice that $\tilde{q}/(1 - p) = q'$ and the first order conditions can be written as $q' \cdot S_q(q') \leq S(q') - \kappa$. After imposing $q_{min} = \hat{q}$, we know that $S(q - \kappa) < q$ for all q. This is because, with the initial q_{min}, $S(q) - \kappa < q$ for all $q \geq \hat{q}$ and $S(q)$ cannot increase with a higher value of q_{min}. Together with the first order conditions this implies that $q' \cdot S_q(q') < q'$, which is satisfied only if $S_q(q') < 1$. Because $q' \geq \underline{q}$ and for $q' \geq \underline{q}$, we have that $\tilde{S}(q') = S(q')$ (see Corollary 1). Therefore, we also have that $\tilde{S}_q < 1$.

Because the maximum value of S_q is not increasing in q_{min} and there is a value of q_{min} for which it becomes smaller than 1, there must be some $\underline{q_{min}} \in (0, \bar{q})$ for which the maximum value is exactly 1. This is the value of q_{min} that defines the renegotiation-proof contract. *Q.E.D.*

H Proof of Proposition 4

I prove first that the maximum slope of $\tilde{S}$—which is equal to $(S(\underline{q}) - \kappa)/\underline{q}$—is decreasing in κ. Consider κ^1 and κ^2, with $\kappa^1 < \kappa^2$. The superscript will also be used to differentiate the variables and functions associated with the two values of κ. We have the following inequalities:

$$\frac{\tilde{S}^1(\underline{q}^1) - \kappa^1}{\underline{q}^1} \geq \frac{\tilde{S}^1(\underline{q}^2) - \kappa^1}{\underline{q}^2} > \frac{\tilde{S}^2(\underline{q}^2) - \kappa^2}{\underline{q}^2} \tag{40}$$

The first inequality was explained in the proof of Proposition 3. The second derives from the fact that, for each $q < \bar{q}$, $\kappa^2 - \kappa^1 > \tilde{S}^2(q) - \tilde{S}^1(q)$. This means that an increase in κ increases the pre-randomization surplus less than the increase in κ. Therefore, $\tilde{S}_q$ is decreasing in κ.

A contract is renegotiable if the slope of $\tilde{S}$ is greater than 1. As shown above, the maximum slope of $\tilde{S}$ is decreasing in κ and obviously converges to zero as κ converges to $\bar{\kappa} = S(\bar{q})$. Therefore, there is some κ for which $\tilde{S}_q \leq 1$, which is the condition for renegotiation-proof. This value of κ is denoted by $\bar{\kappa}$.

To show that there is a positive probability of liquidation when $\kappa > 0$, it is enough to show that the contract can be improved by allowing for liquidation. Assume that the firm is never liquidated, that is, $\underline{q} = q_{min} = 0$. Under this assumption $\lim_{q \to 0} S(q) = \beta\kappa$. Therefore, there is some $\hat{q}$ for which $S(q) < \kappa$ and the contract can be improved by liquidating the firm when $q \leq \hat{q}$.

To show that the contract can also be liquidated in the renegotiation-proof contract when $\kappa \in (0, \bar{\kappa})$ it is enough to show that $\underline{q}$ cannot decrease in q_{min} and it is strictly greater if we set q_{min} equal to the previous $\underline{q}$ (the one associated with the previous smaller value of q_{min}). A higher value of q_{min} makes the constraints on k tighter for each value of q. In fact, combining the incentive-compatibility constraints with the promised-keeping constraint, the input of capital is constrained by $q \geq \beta q_{min} + \beta h'(0)[EF(k, \eta) - F(k, \underline{\eta})]$ (see problem (6)). Therefore, the marginal impact of q on $S(q)$ cannot be smaller for higher values of q_{min}, that is, $S_q^1(q) \leq S_q^2(q)$ for all q. Because condition (37) implies that $S_q^1(\underline{q}^1) \geq S_q^2(\underline{q}^2)$ (remember that $S_q(\underline{q})$ is equal to the maximum slope of $\tilde{S}$), then we must have that $\underline{q}^2 \geq \underline{q}^1$. What is left to prove is that if we set $q_{min}^2 = \underline{q}^1$, then $\underline{q}^2 > \underline{q}^1$. If we set $q_{min} = \underline{q}^1$, the input of capital has to decrease (strictly) at

$q = q_{min}$. Therefore, $S^2(\underline{q}^1) < S^1(\underline{q}^1)$. This implies that $(S^2(\underline{q}^1) - \kappa)/\underline{q}^1 < (S^1(\underline{q}^1) - \kappa)/\underline{q}^1$. Because $S_q(\underline{q}) = (S(\underline{q}) - \kappa)/\underline{q}$, if $\underline{q}^2 = \underline{q}^1$ we would have:

$$S_q^2(\underline{q}^1) \;=\; \frac{S^2(\underline{q}^1) - \kappa}{\underline{q}^1} \;<\; \frac{S^1(\underline{q}^1) - \kappa}{\underline{q}^1} \;=\; S_q^1(\underline{q}^1) \tag{41}$$

which violates the condition $S_q^2(\underline{q}^1) \geq S_q^1(\underline{q}^1)$. Therefore, $\underline{q}^2$ must be strictly greater than $\underline{q}^1$.

What is left to prove is that for $\kappa = 0$ the firm can be liquidated in the renegotiation-proof contract but not in the long-term contract. Consider the long-term contract. Obviously, for $q > 0$ the surplus function is strictly greater than zero. Therefore, the firm is never liquidated. The only case in which the firm is liquidated is when $q = 0$. In this case, in fact, the surplus is zero because k must be zero. Because $q' = \tilde{q}(\eta) = \tilde{q}(\underline{\eta}) + h'(0)[F(k, \eta) - F(k, \underline{\eta})]$, the probability that $q' = 0$ is zero almost surely for any $q > 0$.

In the renegotiation-proof contract randomization is still possible. This is because the restriction $\tilde{q}(\eta) \geq q_{min}$ reduces the surplus $S(q)$ and increases its slope. Therefore, by increasing q_{min} we may reach a point in which the randomization over the liquidation of the firm my become optimal. However, this cannot be generalized and depends on the parametrization of the model. *Q.E.D.*

REFERENCES

Abel, A. B. & Eberly, J. C. (2002). Q Theory Without Adjustment Costs & Cash Flow Effects Without Financing Constraints. Unpublished manuscript, Wharton School and Kellogg School of Management.

Abreu, D., Pearce, D., & Stacchetti, E. (1990). Toward a Theory of Discounted Repeated Games With Imperfect Monitoring. *Econometrica*, *58*(5), 1041–63.

Aghion, P. & Bolton, P. (1992). An Incomplete Contracts Approach to Financial Contracting. *Review of Economic Studies*, *59*(3), 473–94.

Albuquerque, R. & Hopenhayn, H. (1997). Optimal Dynamic Lending Contracts With Imperfect Enforceability. Forthcoming *Review of Economic Studies.*

Atkeson, A. (1991). International Lending With Moral Hazard and Risk of Reputation. *Econometrica*, *59*(4), 1069–89.

Atkeson, A., Khan, A., & Ohanian, L. (1996). Are Data on Industry Evolution and Gross Job Turnover Relevant for Macroeconomics?. *Carnegie-Rochester Conference Series on Public Policy*, *44*(1), 216–50.

Atkeson, A. & Lucas, R. E. (1992). On Efficient Distribution With Private Information. *Review of Economic Studies*, *59*(4), 427–53.

Atkeson, A. & Lucas, R. E. (1995). Efficiency and Equality in a Simple Model of Efficient Unemployment Insurance. *Journal of Economic Theory*, *66*(1), 64–88.

Bergin, J. & MacLeod, W. B. (1993). Efficiency and Renegotiation in Repeated Games. *Journal of Economic Theory*, *61*(1), 42–73.

Clementi, G. & Hopenhayn, H. A. (1998). Optimal Lending Contracts and Firms' Survival with Moral Hazard. Unpublished manuscript, Carnegie Mellon University and University of Rochester.

Cole, H. L. & Kocherlakota, N. (1997). Efficient Allocations With Hidden Income and Hidden Storage. Research Department Staff Report 238, Federal Reserve Bank of Minneapolis.

Davis, S., Haltiwanger, J., & Schuh, S. (1996). *Job Creation and Distruction*. MIT Press, Cambridge, Massachusetts.

DeMarzo, P. M. & Fishman, M. J. (2000). Optimal Long-Term Financial Contracting with Privately Observed Cash Flows. Unpublished manuscript, Stanford University and Northewestern University.

Demirguc-Kunt, A. & Levine, R. (2001). *Financial Structure and Economic Growth*. MIT Press, Cambridge, Massachusetts.

Evans, D. S. & Jovanovic, B. (1989). An Estimated Model of Entrepreneurial Choice under Liquidity Constraints. *Journal of Political Economy*, *97*(4), 808–27.

Evans, D. S. & Leighton, S. L. (1989). Some Empirical Aspects of Entrepreneurship. *American Economic Review*, *79*(3), 519–35.

Evans, D. S. (1987). Tests of Alternative Theories of Firm Growth. *Journal of Political Economy*, *95*(4), 657–74.

Fazzari, S. M., Hubbard, G. R., & Petersen, B. C. (1988). Financing Constraints and Corporate Investment. *Brookings Papers on Economic Activity*, *88*(1), 141–95.

Fudenberg, D., Holmstrom, B., & Milgrom, P. (1990). Short-Term Contracts and Long-Term Agency Relationships. *Journal of Economic Theory*, *51*(1), 1–31.

Fudenberg, D. & Tirole, J. (1993). *Game Theory*. MIT Press, Cambridge, Massachusetts.

Gentry, W. M. & Hubbard, R. G. (2000). Entrepreneurship and Household Saving. Unpublished manuscript, Columbia University.

Gertler, M. (1992). Financial Capacity and Output Fluctuations in an Economy With Multiperiod Financial Relationships. *Review of Economic Studies*, *59*(2), 455–72.

Gertler, M. & Gilchrist, S. (1994). Monetary Policy, Business Cycles, and the Behavior of Small Manufacturing Firms. *Quarterly Journal of Economics*, *CIX*(2), 309–40.

Gilchrist, S. & Himmelberg, C. P. (1996). Evidence on the Role of Cash Flow for Investment. *Journal of Monetary Economics*, *36*(3), 541–72.

Gilchrist, S. & Himmelberg, C. P. (1998). Investment: Fundamentals and Finance. In Bernanke, B. S. & Rotemberg, J. (Eds.), *NBER Macroeconomics Annual*, pp. 223–62. MIT Press, Cambridge, Massachusetts.

Gomes, J. F. (2001). Financing Investment. *American Economic Review*, *91*(5), 1263–85.

Green, E. J. (1987). Lending and the Smoothing of Uninsurable Income. In Prescott, E. C. & Wallace, N. (Eds.), *Contractual Arrangements for Intertemporal Trade*, pp. 3–25. University of Minneapolis Press, Minneapolis.

Hart, O. & Moore, J. (1994). A Theory of Debt Based on the Inalienability of Human Capital. *Quarterly Journal of Economics*, *109*, 841–79.

Holtz-Eakin, D., Joulfaian, D., & Rosen, H. S. (1994). Sticking it Out: Entrepreneurial Survival and Liquidity Constraints. *Journal of Political Economy*, *102*(1), 53–75.

Hubbard, R. G. (1998). Capital-Market Imperfections and Investment. *Journal of Economic Literature*, *36*(1), 193–225.

Khan, A. & Revikumar, B. (2001). Growth and Risk-Sharing with Private Information. *Journal of Monetary Economics*, *47*, 499–521.

Malcomson, J. M. & Spinnewyn, F. (1988). The Multiperiod Principal-Agent Problem. *Review of Economic Studies*, *55*, 391–407.

OECD (2001). Productivity and Firm Dynamics: Evidence from Microdata. *Economic Outlook*, *69*(1), 209–223.

Phelan, C. (1995). Repeated Moral Hazard and One-Sided Commitment. *Journal of Economic Theory*, *66*(2), 488–506.

Phelan, C. & Townsend, R. (1991). Computing Multiperiod Information-Constrained Optima. *Review of Economic Studies*, *58*(4), 853–81.

Quadrini, V. (1999). The Importance of Entrepreneurship for Wealth Concentration and Mobility. *Review of Income and Wealth*, *45*(1), 1–19.

Quadrini, V. (2000). Entrepreneurship, Saving and Social Mobility. *Review of Economic Dynamics*, *3*(1), 1–40.

Spear, S. & Srivastava, S. (1987). On Repeated Moral Hazard With Discounting. *Review of Economic Studies*, *54*(4), 599–818.

Stokey, N. L., Lucas, R. E., & Prescott, E. C. (1989). *Recursive Methods in Economic Dynamics*. Harvard University Press, Cambridge, Massachusetts.

Thomas, J. & Worrall, T. (1990). Income Fluctuations and Asymmetric Information: An Example of the Repeated Pricipal-Agent Problem. *Journal of Economic Theory*, *51*, 367–390.

Thomas, J. & Worrall, T. (1994). Foreign Direct Investment and The Risk of Expropriation. *Review of Economic Studies*, *61*, 81–108.

Wang, C. (1997). CEO Compensation and Shareholder Wealth in a Dynamic Agency Model. *Journal of Economic Theory*, *76*(1), 72–105.

Wang, C. (2000). Renegotiation-Proof Dynamic Contracts with Private Information. *Review of Economic Dynamics*, *3*(3), 396–422.