Financial Globalization, Inequality, and the Raising of Public Debt

Numerical algorithm

To simplify the exposition we describe the numerical procedure when there are only two countries ($N = 2$). We first form a two-dimensional, equally spaced grid over the states B_1 and B_2, which we denote by $\mathbf{S}$. We then solve the model at any grid point for the states $(B_1, B_2) \in \mathbf{S}$ backward, starting from the terminal period $t = T$. Before describing the specific solution at any $t = T, T-1, ..., 1$, we state the following property based on Lemma 2.1, Proposition 2.1, Corollary 2.1 and Proposition 4.1.

Property 1 *Given $B_{j,t}$, $B_{j,t+1}$, $p_{j,t+1}$, $V_{j,t+1}$, $W_{j,t+1}$, $j \in \{1,2\}$, we can solve for all variables at time t using the equations*

$$\eta_t = \frac{\beta}{1 + \beta^{T-t}\left(\sum_{s=1}^{T-t}\beta^{s-1}\right)^{-1}} \tag{1}$$

$$\tilde{b}_{j,t} = \begin{cases} \nu B_{j,t}, & \textit{In autarky} \\ \nu\left(\frac{B_{1,t}+B_{2,t}}{2}\right), & \textit{With mobility} \end{cases} \tag{2}$$

$$\tilde{b}_{j,t+1} = \begin{cases} \nu B_{j,t+1}, & \textit{In autarky} \\ \nu\left(\frac{B_{1,t+1}+B_{2,t+1}}{2}\right), & \textit{With mobility} \end{cases} \tag{3}$$

$$\phi_{j,t} = \mathbb{E}_t\left[\frac{A(z^i_{j,t+1}) + p_{j,t+1}}{A(z^i_{j,t+1}) + p_{j,t+1} + \tilde{b}_{j,t+1}}\right], \tag{4}$$

$$p_{j,t} = \frac{\eta_t\phi_{j,t}(\bar{A} + \tilde{b}_{j,t})}{(1-\eta_t\phi_{j,t})}, \tag{5}$$

$$R_{j,t} = \frac{(1-\eta_t\phi_{j,t})\tilde{b}_{j,t+1}}{\eta_t(1-\phi_{j,t})(\bar{A}+\tilde{b}_{j,t})}, \tag{6}$$

$$\hat{a}^i_{j,t} = A(z^i_{j,t}) + p_{j,t} + \tilde{b}_{j,t}, \tag{7}$$

$$V_{j,t} = \ln(1-\eta_t) + \left(\frac{1}{1-\eta_t}\right)\left[\eta_t \ln\left(\frac{\eta_t\phi_{j,t}}{p_{j,t}}\right) + \mathbb{E}_t \ln \hat{a}^i_{j,t}\right] + \beta V_{j,t+1}, \tag{8}$$

$$W_{j,t} = \ln\left(\bar{w} + \nu\left(\frac{B_{j,t+1}}{R_{j,t}} - B_{j,t}\right)\right) + \beta W_{j,t+1}, \tag{9}$$

where $\nu = \frac{\Phi}{1+\Phi}$, $\bar{A} = \sum_\ell A(z_\ell)\mu_\ell$, $\bar{w} = (1-\theta)\bar{z}^\theta$.

We can solve exactly for the above variables sequentially, once we know $B_{j,t}$, $B_{j,t+1}$, $p_{j,t+1}$, $V_{j,t+1}$ and $W_{j,t+1}$. For the following description of the computational algorithm, it will be convenient to define the vectors $\mathbf{X}_t = \{B_{j,t}, B_{j,t+1}, p_{j,t+1}, V_{j,t+1}, W_{j,t+1}\}_{j=1}^2$ and $\mathbf{Y}_t = \{p_{j,t}, V_{j,t}, W_{j,t}\}_{j=1}^2$. Using Property 1, we can express $\mathbf{Y}_t$ as a function of $\mathbf{X}_t$,

$$\mathbf{Y}_t = \Upsilon(\mathbf{X}_t).$$

We describe next the solution at each time t, starting from the terminal period T.

Solution at $t = T$

In the terminal period governments fully repay their debts. Thus, $B_{j,T+1} = 0$. Furthermore we know that $p_{j,T+1} = V_{1,T+1} = W_{j,T+1} = 0$. Therefore, $\mathbf{X}_T = \{B_{j,T}, 0, 0, 0, 0\}_{j=1}^2$. We can then use the function $\Upsilon(\mathbf{X}_T)$ from property 1 to solve for $\mathbf{Y}_T = \{p_{j,T}, V_{j,T}, W_{j,T}\}_{j=1}^2$ at each grid point $(B_1, B_2) \in \mathbf{S}$.

The solution obtained for $\mathbf{Y}_T$ at each grid point $(B_1, B_2) \in \mathbf{S}$ is used to form the approximate function

$$\mathbf{Y}_T = \Gamma_T(B_{1,T}, B_{2,T})$$

The reason we need to create this approximate function is because, when we move to the next step $t = T - 1$, we need to determine $\mathbf{Y}_T$ also for values of $B_{1,T}$ and $B_{2,T}$ that are not on the grid $\mathbf{S}$. The approximate function is created with bilinear interpolation of the solutions $\mathbf{Y}_T$ obtained at the grid points $(B_1, B_2) \in \mathbf{S}$. Armed with the approximate function $\Gamma_T(B_{1,T}, B_{2,T})$, we can move to period $t = T - 1$.

Solution at $t < T$

The main difference from the terminal period T is that now we need to solve for the optimal debts $B_{1,t+1}$ and $B_{2,t+1}$ chosen by governments. To find the optimal debt chosen by each government at each grid point $(B_1, B_2) \in \mathbf{S}$, we implement the following steps.

1. We first solve for the optimal response functions to the debt chosen by the other country. To find the optimal response function we need to find the government objective $O_{j,t}(B_{1,t+1}, B_{2,t+1}) = \Phi W_{j,t} + V_{j,t}$. The response function of country 1 and country 2 are defined, respectively, as

$$\begin{aligned} \varphi_{1,t}(B_{2,t+1}) &= \max_{B_{1,t+1}} O_{1,t}(B_{1,t+1}, B_{2,t+1}), \\ \varphi_{2,t}(B_{1,t+1}) &= \max_{B_{2,t+1}} O_{2,t}(B_{1,t+1}, B_{2,t+1}). \end{aligned}$$

   The optimization is performed by searching over an equally spaced grid $\mathbf{B}$. This grid is finer than the two dimensional grid for the states $\mathbf{S}$ so that we obtain a more accurate approximation to the maximization problem. The detailed steps are as follows:

   (a) Given $(B_{1,t+1}, B_{2,t+1}) \in \mathbf{B} \times \mathbf{B}$, we find $\mathbf{Y}_{t+1}$ using the approximate function $\Gamma_{t+1}(B_{1,t+1}, B_{2,t+1})$ found in the previous step $t + 1$.

   (b) Given $\mathbf{Y}_{t+1}$ we have all the terms we need to construct the vector $\mathbf{X}_t$. Thus we can find $\mathbf{Y}_t$ using the function $\Upsilon_t(\mathbf{X}_t)$ from Property 1.

   (c) The vector $\mathbf{Y}_t$ contains the necessary elements to compute the government objectives $O_{j,t}(B_{1,t+1}, B_{2,t+1})$ for $(B_{1,t+1}, B_{2,t+1}) \in \mathbf{B} \times \mathbf{B}$.

(d) Now that we know the government objectives at the grid points, we compute the optimal response functions as

$$\begin{aligned} \varphi_{1,t}(B_{2,t+1}) &= \max_{B_{1,t+1} \in \mathbf{B}} O_{1,t}(B_{1,t+1}, B_{2,t+1}), \\ \varphi_{2,t}(B_{1,t+1}) &= \max_{B_{2,t+1} \in \mathbf{B}} O_{2,t}(B_{1,t+1}, B_{2,t+1}), \end{aligned}$$

which are defined only over the grid $\mathbf{B}$. To make the response functions continuous, we join the grid values with piece-wide linear segments.

2. The optimal response functions allows us to compute the equilibrium policies chosen by the two governments. They are the fix point $(B^*_{1,t+1}, B^*_{2,t+1})$ to

$$\begin{aligned} B^*_{1,t+1} &= \varphi_{1,t}(B^*_{2,t+1}) \\ B^*_{2,t+1} &= \varphi_{2,t}(B^*_{1,t+1}) \end{aligned}$$

After making the response function continuous, we find the solution $(B^*_{1,t+1}, B^*_{2,t+1})$ using a nonlinear solver. Of course, the solution is not necessarily on the grid $\mathbf{B} \times \mathbf{B}$.

3. Given the solution for $(B^*_{1,t+1}, B^*_{2,t+1})$, we can compute $\mathbf{Y}_{t+1} = \Gamma_{t+1}(B^*_{1,t+1}, B^*_{2,t+1})$ and construct the vector $\mathbf{X}_t$ associated with the equilibrium policies. This allows us to compute $\mathbf{Y}_t = \Upsilon_t(\mathbf{X}_t)$.

Once we have completed the above steps and found the vector $\mathbf{Y}_t$ for each grid point $(B_1, B_2) \in \mathbf{S}$, we can then construct the approximate (bi-linearly interpolated) function

$$\mathbf{Y}_t = \Gamma_t(B_{1,t}, B_{2,t}).$$

We can then move to the earlier step until we reach the initial period $t = 1$.