Financial Markets and Firm Dynamics

By Thomas F. Cooley and Vincenzo Quadrini*

Recent studies have shown that the dynamics of firms (growth, job reallocation and exit) are negatively correlated with the initial size of the firm and its age. In this paper we analyze whether financial factors, in addition to technological differences, are important in generating these dynamics. We introduce financial market frictions in a basic model of industry dynamics with persistent shocks to technology and show how the combination of persistent shocks and financial frictions can account for the simultaneous dependence of firm dynamics on size (once we control for age) and on age (once we control for size). In contrast, models with only persistent shocks or financial frictions are unable to account simultaneously for the (conditional) size and age dependence. (*JEL* D21, G3, L2)

Recent studies of the relationship between firm size and growth have overturned the conclusion of Gibrat's Law which holds that firm size and growth are independent. Studies by David S. Evans (1987) and Bronwyn H. Hall (1987) show that the growth rate of manufacturing firms and

---

*Department of Economics, Stern School of Business, New York University, 44 West Fourth Street, New York, NY 10012; Quadrini: also Centre for Economic Policy Research (CEPR). We have received helpful comments from Jeff Campbell, David Chapman, Harold Cole, Tom Cosimano, Joao Gomes, Hugo Hopenhayn, José-Víctor Ríos-Rull, Harald Uhlig and Lu Zhang. We would also like to thank two anonymous referees whom suggestions helped us to improve the paper significantly. This research is supported in part by NSF Grant SBR 9617396.

the volatility of growth is negatively associated with firm size and age. Firm size and age also play an important role in characterizing the dynamics of job reallocation. Steven J. Davis, John C. Haltiwanger, and Scott Schuh (1996) show that the rates of job creation and job destruction in U.S. manufacturing firms are decreasing in firm age and size and that, conditional on the initial size, small firms grow faster than large firms. The empirical regularities of firm dynamics are:[1]

- Conditional on age, the dynamics of firms (growth, volatility of growth, job creation, job destruction and exit) are negatively related to the size of firms.
- Conditional on size, the dynamics of firms (growth, volatility of growth, job creation, job destruction and exit) are negatively related to the age of firms.

We will refer to the first regularity as the "size dependence" and to the second regularity as the "age dependence".

Existing models of the growth of firms explain these features as arising from learning about the technology or from persistent shocks to the technology. Examples are the models studied by Boyan Jovanovic (1982), Hugo Hopenhayn (1992), Hugo Hopenhayn and Richard Rogerson (1993), Jeffrey R. Campbell (1998) and Jeffrey R. Campbell and Jonas D. M. Fisher (2000). These models capture some of the empirical regularities mentioned above but they are unable to simultaneously account for both the size dependence (once we control for the age of the firm) and the age dependence (once we control for the size of the firm).[2]

In this paper we ask whether the introduction of financial frictions in an otherwise standard model of industry dynamics can account for the simultaneous dependence of firm dynamics on both size and age. It seems natural to try to link patterns of firm growth with their financial decisions because there are also important regularities in the financial characteristics of firms that are related to their size. Empirically, the financial behavior of firms is characterized by the following facts: [3]

- Small and younger firms pay fewer dividends, take on more debt and invest more.
- Small firms have higher values of Tobin's q.
- The investment of small firm's is more sensitive to cash flows, even after controlling for their future profitability.

The results of the paper can be summarized as follows. First, we show that a model with financial frictions can capture the features of the financial behavior of firms cited above. We show that smaller and younger firms pay fewer dividends, take on more debt and invest more. Moreover, their investment is more sensitive to cash flows and they have higher values of Tobin's q. Second, we show that the combination of persistent shocks and financial frictions can generate the simultaneous dependence of industry dynamics on size (once we condition on age) and age (once we condition on size). In contrast, a model with only persistent shocks or only financial frictions cannot account for both the size and age dependence.

It is important to emphasize that this paper does not provide a micro foundation for market incompleteness. Rather, we evaluate how market incompleteness affects the dynamics of firm growth. For this reason, we take a simple approach in modeling financial frictions. We assume that firms can finance investment in two ways; with equity—which can be increased by issuing new shares or by reinvesting profits—and with one-period debt. The financial frictions arise because of the following assumptions: (a) there is a cost or premium associated with increasing equity by issuing new shares, compared to reinvesting profits; (b) defaulting on the debt is costly. Although the cost of raising equity is exogenous in the model, it captures the fact that firms prefer to increase equity with internally generated funds and they issue new shares only occasionally (see, for example, Smith (1977) and Ross, Westerfield, & Jordan (1993)). The assumption of a cost to issuing new shares is also made in Joao F. Gomes (1998). The default cost can be justified as a verification cost which is paid in the event of default (costly state verification).

Firms have access to a decreasing return-to-scale technology with inputs of capital and labor. The firm's productivity changes according to a persistent stochastic process. In the absence of financial frictions, the efficiency level of the firm fully determines its size. With financial frictions, however, the size of the firm also depends on its assets (equity). Because there is a cost to issuing new shares and a default cost, equity and debt are not perfect substitutes and the investment choice of the firm depends on the amount of equity it owns. New firms are created with an initial amount of equity which changes endogenously as firms issues new shares and retain earnings. Because the profitability of the firm is stochastic, at each point in time there will be a stationary distribution of heterogeneous firms.

The debt contract is a standard one-period debt contract signed with a financial intermediary. The financial intermediary lends funds at the end of the period and the firm commits to return the borrowed funds plus the interest at the end of the next period. If at that time the firm does not repay the debt, the firm faces a bankruptcy problem. In this case the financial intermediary incurs the cost to verify the financial condition of the firm. The financial intermediary anticipates the possibility that the firm may not repay the debt and the interest rate charged depends on the probability of default.

Firms face a trade-off in deciding the optimal amount of debt: on the one hand, more debt allows them to increase their expected profits by expanding the production scale; on the other, the expansion of the production scale implies a higher volatility of profits. Given that the firm's objective is a concave function of profits, its volatility has a negative impact on the firm's value. This is reinforced by the higher interest rate associated with a higher probability of default (to compensate for the default cost). As the equity of the firm increases, the firm becomes less concerned about the fluctuation of profits (in absolute value) and expands the production scale. But, because of the decreasing returns-to-scale, the increase in production scale is not proportional to the increase in equity and the firm will reduce its leverage as it grows. This financial behavior has important consequences for the dynamics of the firm.

The paper is organized as follows. In section I. we describe the basic model of firm dynamics without financial frictions and analyze its properties. In section II. we extend the model by introducing financial frictions. After describing the firm's problem and deriving some analytical results, we analyze the properties of the model numerically. We look first, in section C., at the case in which the shocks are not persistent. This allows us to illustrate the impact of financial factors on the dynamics of the firm, separately from the impact of persistent shocks. Then in section D. we study the general model with both persistent shocks and financial frictions. As we will see, the model with only persistent shocks and the model with only financial frictions replicate some of the size dependence facts, but are unable to account for the age dependence. In both models, size is the only dimension of heterogeneity. However, the combination of financial frictions and persistent shocks allows the model to account for both the size and age dependence. We then conclude that both financial factors and persistent shocks are important for the properties of the growth of firms.

I. The basic firm dynamics model in the frictionless economy

The basic model of firm dynamics that we start with is a simplified version of the model developed in Hopenhayn (1992). Assume there is a continuum of firms that maximize the expected discounted value of dividends, that is, $E_0\left\{\sum_{t=0}^{\infty}\beta^t d_t\right\}$, where d_t is the dividend distributed at time t, β the discount factor for the firm and E_0 the expectation operator at time zero.

In each period, firms have access to a production technology. We will work directly with the revenue function implied by this technology $y = (z+\varepsilon)G(k,l)$ where y is the revenue, k is the input of capital that depreciates at rate δ, l is the labor input, and the variables z and ε are idiosyncratic shocks that determine the efficiency of the firm. The inputs of capital and labor are decided one period in advance.

Capital and labor are perfect complements, which implies that the capital-labor ratio employed by the firm is always constant.[4] Given this assumption, we write the revenue function as $(z+\varepsilon)F(k)$. Similarly, we will denote by $\phi k = [\delta + w(l/k)]k$, the depreciation of capital plus the cost of labor, where w is the wage rate. Because the wage rate is constant in the model, ϕ will also be constant.

We make the following assumptions about the revenue function and the shocks.

Assumption 1 *The function $F : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ is strictly increasing, strictly concave and continuously differentiable.*

Assumption 2 *The shock z takes values in the finite set $\mathbf{Z} = \{z_0, z_1, .., z_N\}$ and follows a first order Markov process with transition probability $\Gamma(z'/z)$. The shock ε is independently and identically distributed in the set of real numbers $\mathbf{R}$, with zero mean. The density function $f : \mathbf{R} \rightarrow [0,1]$ is continuous, differentiable and $f(\varepsilon) > 0$, $\forall \varepsilon \in \mathbf{R}$.*

The concavity of F implies that the revenue function displays decreasing returns-to-scale. The decreasing returns-to-scale could be rationalized by assuming limited managerial or organizational resources as in Robert E. Lucas (1978). Alternatively, we could assume that these properties derive from the monopolistic nature of the competitive environment where the firm faces a downward sloping demand function.

The structure of the shocks allows us to separate the persistent component, the z, from the non-persistent component, the ε. This in turn allows us to separate the properties of firm growth induced by technological differences (due to persistent shocks) from pure accidental events (due to i.i.d. shocks). The persistent shock, z, is revealed one period in advance, while the i.i.d. shock, ε, is revealed in the current period. Therefore, at the moment of deciding the production plan, the firm knows z but does not know ε.

There are different ways to induce the exit of a firm. As in Jovanovic (1982), for example, we could assume that the firm has some outside opportunity. Alternatively, as in Hopenhayn (1992), we could assume that in addition to the cost for the inputs of capital and labor, there is a fixed cost of production. In both cases, the firm will exit when its value, net of the opportunity or fixed cost, is negative. At this stage, however, it is convenient to assume that the firm's exit is exogenous. Accordingly, we simply assume that with a certain probability the firm becomes unproductive and exits. This is captured in model by assuming that $z_0 = 0$ and $\Gamma(z_0/z_0) = 1$. In section III., we will discuss the properties of the model with endogenous exit when there is a fixed cost of production.

For the frictionless economy we assume that $1/\beta - 1 = r$, where r is the market interest rate. This condition can be interpreted as a general equilibrium property of the model. In this economy the Modigliani-Miller theorem applies and whether capital is financed with debt or equity is irrelevant. The problem of the firm is then static and, conditional on surviving, it consists of the maximization of expected profits, that is:

$$\text{(1)} \qquad \max_k \Big\{ \int_\varepsilon (z+\varepsilon)F(k)f(d\varepsilon) - (r+\phi)k \Big\} \;=\; \max_k \Big\{ zF(k) - (r+\phi)k \Big\}$$

The solution is given by the optimal input of capital which depends on the ex-ante productivity of the firm z. We denote it by $k^*(z)$. Given this solution we can then determine the value of an active firm denoted by $V(z)$.

The last feature of the model that needs to be specified is the entrance of new firms. In each period there is a large number of projects, drawn from the invariant distribution of z. The implementation of a project requires a fixed cost κ. In this partial equilibrium analysis, with fixed prices, the mass of new entrant firms is non-degenerate only if the surplus from creating new firms is non-positive, that is, $V(z) - \kappa \leq 0$. Because the surplus from creating new firms with high z is larger than for firms with small z, this condition must be satisfied with equality only for $z = z_N$. In equilibrium it must be the case that $V(z_N) = \kappa$, and all new firms will be of the highest efficiency. Although we do not conduct a general equilibrium analysis, this arbitrage condition will be guaranteed by general equilibrium forces: the entrance of new firms would induce changes in the prices (particularly wages) and in the value of firms until there are no gains from creating new firms. For the frictionless economy, this feature of the model implies that younger firms are larger than older ones. As we will see, however, this is not the case in the model with financial frictions.

A. Industry dynamics properties of the frictionless economy

The fact that z is a sufficient statistic for the size of the firm implies that, once we condition on the firm's size, age becomes irrelevant for its dynamics (growth, volatility of growth and job reallocation). This is formally stated in the following proposition.

Proposition 1 *In the frictionless economy, firms of the same size experience the same dynamics independently of their age.*

This result is intuitive and the formal proof is omitted. What about the dependence of the firm's dynamics on size? To simplify the analysis, let's consider the case in which the variable z takes only three values (z_0, z_1, z_2) where z_0 is the absorbing shock and the transitional probability matrix for $z \in \{z_1, z_2\}$ (conditional on not being z_0) is symmetric. Denote the conditional transitional probability by $\Gamma_c(z'/z)$. We then have the following proposition.

Proposition 2 *Assume that $\Gamma_c(z_1/z_1) = \Gamma_c(z_2/z_2)$. Then small firms grow faster than large firms and the rate of job reallocation is independent of the firm size.*

The proof of the proposition is trivial. In this economy surviving firms are only of two types: small firms with current shock z_1 and large firms with current shock z_2. Neglecting the possibility of exit, small firms will only grow while large firms will only shrink. Moreover, small firms will never destroy jobs (except in the case of exogenous exit) while large firms will never create jobs. Job reallocation is defined as the sum of job creation and job destruction. Because the number of jobs destroyed by a large firm when z switches from z_2 to z_1 is equal to the number of job created by a small firm when z switches from z_1 to z_2, we have that the volume (and rate) of job reallocation is the same for small and large firms. Looking at the individual components of job reallocation, this model is consistent with the observation that job creation is decreasing in the firm's size, but it is inconsistent with the empirical fact that job destruction is also decreasing in the firm's size.

Proposition 2 also holds when the shock z takes more than two values and the transition probability matrix is symmetric with decreasing probability of changing the current z to more distant values. Without some restriction on the transition probability matrix, we cannot derive a general pattern for the dynamics of firms and it is possible to have a non-monotone relation.

To summarize, the basic model of firm dynamics captures some of the size dependence of firm growth but is unable to capture the age dependence once we control for the size of the firm. This result is not affected by the initial size of new entrant firms (determined by the initial value of z). The initial size of new firms affects the size distribution of firms but not their size and age conditional dynamics. [5]

II. The economy with financial frictions

We now extend the basic model of firm dynamics by introducing financial market frictions. At each point in time, firms are heterogeneous in the amount of assets they own as well as their technology level z. Henceforth, the assets of the firms, denoted by e, are referred to as *equity* and it corresponds to the firm's net worth.

The input of capital is financed with equity and by borrowing from a financial intermediary. If we denote the firm's debt by b, the input of capital is $k = e + b$. Financial frictions derive from two assumptions:

(a) There is a cost λ per unit of funds raised by issuing new shares.

(b) The firm can default and the default procedure implies a cost ξ.

The cost of raising funds λ implies that the firm will prefer to increase equity by reinvesting profits, rather than issuing new shares. The default cost increases the cost of borrowing. As we will see, for a given value of equity, the probability of default increases when the firm borrows more because it is more vulnerable to idiosyncratic shocks. This increases the expected default cost and the financial intermediary will demand a higher interest rate.

It is important to note that both assumptions are necessary to have effective financial frictions. If it is costly for the firm to issue new shares, but it can borrow at the market interest rate without limit, then the firm will implement the desired scale of production by borrowing more, whatever the value of its equity. On the other hand, the default cost becomes irrelevant if the firm can increase its equity by issuing new shares without cost.

The particular structure of the debt contract can be justified by asymmetry in information and costly monitoring as in a standard costly state verification model. Assume that at the end of the period, if the firm defaults on the debt, the financial intermediary has the ability to liquidate the firm by paying the cost ξ. Under these conditions, the firm will default only if the end-of-period resources (net worth) are smaller than a certain threshold and the financial intermediary will verify the firm only in the event of default. The threshold point is such that the value of the firm at that level of net worth is exactly zero, so that the firm is indifferent between repaying the debt or defaulting.

In this economy we make a slightly different assumption about the relation between the interest and the discount rates:

Assumption 3 *The risk-free interest rate* r *is such that* $1/\beta - 1 > r > 0$.

This condition can be interpreted as a general equilibrium property of economies with these features. In the frictionless economy the condition $1/\beta - 1 = r$ implies that firms are indifferent between accumulating equity or distributing dividends. In the economy with financial frictions, however, if the interest rate is equal to the discount rate, firms would strictly prefer to reinvest profits, no matter what the value of the equity is. Eventually, after reaching a certain size, part of the equity will be kept in the form of risk-free investments earning the interest rate r. In this case the debt b would be negative meaning that the firm lends money rather than borrowing. This would generate an excessive supply of loans and the subsequent reduction in the lending rate r. With assumption 3, however, there is some upper bound e_{max} which bounds the equity chosen by the firm. At the same time, because the value of the firm is obviously bounded, if the debt of the firm is very large, relative to its equity, the firm will renegotiate the debt. This implies that there is some e_{min} below which the equity of the firm (net worth) will never fall. Therefore, we can restrict the state space for equity to the compact and convex set $[e_{min}, e_{max}]$.

A. The firm's problem

At the end of each period, after the realization of the revenues and the observation of the new z, but before issuing new shares or paying dividends, the firm decides whether to default on its debt. Given the initial equity and the debt, the end-of-period net worth of the firm is:

$$\pi(e, b, z + \varepsilon) = (1 - \phi)(e + b) + (z + \varepsilon)F(e + b) - (1 + \tilde{r})b \tag{2}$$

where $\tilde{r}$ is the interest rate charged by the intermediary. The firm will default if its net worth is such that the value of continuing the firm is less than zero. Denote by $\underline{\underline{e}}(z')$ the value of net worth below which the firm defaults. Figure 1 shows how $\underline{\underline{e}}(z')$ is determined. The figure plots a possible shape for the value of the firm, which is increasing in the value of net worth $\pi(e, b, z + \varepsilon)$. The firm's value is plotted for a particular z'. For very low π, the value of the firm is negative. In this case the firm will default and its liabilities are renegotiated to bring its end-of-period net worth to $\underline{\underline{e}}(z')$.

[Place Figure 1 here]

Associated with the default threshold, there is a value of the shock for which the end-of-period net worth is equal to $\underline{e}(z')$. The threshold shock, denote by $\underline{\varepsilon}(z, e, b, z')$, is defined implicitly by:

$$\pi(e, b, z + \underline{\varepsilon}) \;=\; (1 - \phi)(e + b) + (z + \underline{\varepsilon})F(e + b) - (1 + \tilde{r})b \;=\; \underline{e}(z') \tag{3}$$

The interest rate charged by the intermediary $\tilde{r}$ is implicitly defined by:

$$(1 + r)b \;=\; (1 + \tilde{r})b \int_{\underline{\varepsilon}}^{\infty} f(d\varepsilon) + \int_{-\infty}^{\underline{\varepsilon}} [(1 - \phi)(e + b) + (z + \varepsilon)F(e + b) - \xi] f(d\varepsilon) \tag{4}$$

This simply says that the expected repayment from the loan (expression on the right-hand-side) is equal to the repayment of a riskless loan (expression on the left-hand-side). Therefore, the expected return from the loan is equal to the market interest rate r.

Using (4) to eliminate $\tilde{r}$ in (3), the threshold shock $\underline{\varepsilon}(z, e, b, z')$ is determined by the condition:

$$(1 + r)b + \underline{e}(z') \int_{\underline{\varepsilon}}^{\infty} f(d\varepsilon) + \xi \int_{-\infty}^{\underline{\varepsilon}} f(d\varepsilon) \;=\; (1 - \phi)(e + b) + \zeta(\underline{\varepsilon})F(e + b) \tag{5}$$

where $\zeta(\underline{\varepsilon}) = z + \int_{-\infty}^{\underline{\varepsilon}} \varepsilon f(d\varepsilon) + \underline{\varepsilon} \int_{\underline{\varepsilon}}^{\infty} f(d\varepsilon)$.

Note that default does not lead to the exit of the firm. It simply leads to the renegotiation of the debt to the point where the firm would not default. This is because the liquidation of the firm is not in the interest of the financial intermediary. Of course, the firm defaults only if the net worth is negative, $\pi(e, b, z + \varepsilon) < 0$. This implies that, in case of default, the intermediary would not get the full repayment of its debt if the firm is liquidated. On the other hand, by renegotiating the loan and giving the firm the incentive to continue operating, the firm will repay a larger fraction of the debt, either by issuing new shares and/or by contracting a new loan. More specifically, assume that $\pi(e, b, z + \varepsilon) < \underline{e}(z') < 0$. If the firm is liquidated, the intermediary loses $-\pi(e, b, z + \varepsilon)$. If instead the intermediary renegotiates the loan, it loses only $-\pi(e, b, z + \varepsilon) + \underline{e}(z')$.

Using (2) and (3) and taking into account that the debt is renegotiated when $\pi(e, b, z + \varepsilon)$

falls below $\underline{\underline{e}}(z')$, the end-of-period resources of the firm or net worth can be expressed as:

$$(6) \qquad q(e,b,z+\varepsilon,z') = \begin{cases} \underline{\underline{e}}(z') + (\varepsilon - \underline{\underline{\varepsilon}}(z,e,b,z'))F(e+b)\,, & \text{if } \varepsilon \geq \underline{\underline{\varepsilon}}(z,e,b,z') \\ \underline{\underline{e}}(z')\,, & \text{if } \varepsilon < \underline{\underline{\varepsilon}}(z,e,b,z') \end{cases}$$

After the default decision, the firm will decide whether to issue new shares or pay dividends and will choose the next period debt. Although the default choice, the dividend policy, and the choice of the next period debt are all decided at the same time, it is convenient to think of them as decided at different stages. Define $\tilde{\Omega}(z',x)$ to be the end-of-period value of the firm after renegotiating the debt but before issuing new shares or paying dividends. The variable x denotes the corresponding equity (again, after the renegotiation of the debt but before issuing new shares or paying dividends). Also, define $\Omega(z,e)$ to be the value of the firm at the end of the period after issuing new shares or paying dividends, but before choosing the next period debt. The variable e is the end-of-period equity after raising funds with new shares or distributing dividends. The firm's problem can be decomposed as follows:[6]

$$(7) \qquad \Omega(z,e) = \max_b \left\{ \beta \sum_{z'} \int_{\underline{\underline{\varepsilon}}(z,e,b,z')} \tilde{\Omega}(z', q(e,b,z+\varepsilon,z'))\Gamma(z'/z)f(d\varepsilon) \right\}$$

subject to

$$(8) \qquad q(e,b,z+\varepsilon,z') = \begin{cases} \underline{\underline{e}}(z') + (\varepsilon - \underline{\underline{\varepsilon}})F(e+b)\,, & \text{if } \varepsilon \geq \underline{\underline{\varepsilon}} \\ \underline{\underline{e}}(z')\,, & \text{if } \varepsilon < \underline{\underline{\varepsilon}} \end{cases}$$

$$(9) \qquad (1+r)b + \underline{\underline{e}}(z') \int_{\underline{\underline{\varepsilon}}}^{\infty} f(d\varepsilon) + \xi \int_{-\infty}^{\underline{\underline{\varepsilon}}} f(d\varepsilon) = (1-\phi)(e+b) + \zeta(\underline{\underline{\varepsilon}})F(e+b)$$

$$(10) \qquad \tilde{\Omega}(z', \underline{\underline{e}}) = 0$$

$$(11) \qquad \tilde{\Omega}(z',x) = \max_{e'} \left\{ d(x,e') + \Omega(z',e') \right\}$$

subject to

$$d(x, e') = \begin{cases} x - e', & \text{if } x \geq e' \\ (x - e')(1 + \lambda), & \text{if } x < e' \end{cases} \tag{12}$$

Notice that the dynamic program can be solved backward. The second part of the problem defines the new shares or dividend policy of the firm. The function $d(x, z + \varepsilon, e')$ is defined in (12), where x is the end-of-period equity of the firm before the new shares or dividend choices are made. If the firm issues new shares, d is negative. In this case the firm pays the cost λ per unit of equity raised. If the firm pays dividends, d is positive.

Given the function $\tilde{\Omega}(z', x)$, equation (10) defines the value of the net worth below which the firm defaults, that is, the function $\underline{\underline{e}}(z')$. The firm defaults if the value of repaying the debt and continuing the firm is negative. Then, given $\underline{\underline{e}}(z')$, equation (9) determines the threshold shock $\underline{\underline{\varepsilon}}(z, e, b, z')$. Once $\underline{\underline{e}}(z')$ and $\underline{\underline{\varepsilon}}(z, e, b, z')$ are determined, the firm's problem (7) is well defined. The following proposition characterizes some features of the firm's problem.

Proposition 3 *There exists a unique function $\Omega^*(z, e)$ that satisfies the functional equation (7). In addition, if for ϵ_1 and ϵ_2 sufficiently small, $f(\varepsilon) \leq \epsilon_1$ for all $\varepsilon < -\epsilon_2$, then*

(a) the firm's solution is unique, the policy rule $b(z, e)$ is continuous in e;

(b) the input of capital $k = e + b(z, e)$ is increasing in e;

(c) there exist functions $\underline{\underline{e}}(z) < \underline{e}(z) < \bar{e}(z)$, $z \in \mathbf{Z}$, for which the firm renegotiates the loan if the end-of-period resources are smaller than $\underline{\underline{e}}(z)$, will issue new shares if they are smaller than $\underline{e}(z)$, and distribute dividends if they are bigger than $\bar{e}(z)$;

(d) the value function $\Omega^*(z, e)$, is strictly increasing and strictly concave in $[\underline{e}, \bar{e}]$.

Proof: *Appendix.*

The restrictions on the second part of the proposition are motivated by the fact that the end-of-period resource function (net worth), $q(e, b, z + \varepsilon, z')$, is not concave in e and b for all values of ε. In order to assure that the firm's policy is unique, we have to impose some restrictions on the stochastic process for the shock. After imposing these restrictions, the optimal debt policy of the firm is unique as stated in point (a).

Point (c) characterizes the new shares and dividend policies of the firm. The intuition for these policies is provided in Figure 2 which plots the value of a firm as a function of equity, for low and high values of z. If the equity falls below the threshold $\underline{e}(z)$, the firm issues new shares to bring its equity to the level $\underline{e}(z)$, despite the cost of issuing new shares. This is because the concavity of the firm's value (point (d) of the proposition) implies that, when the equity is small, the marginal increase in the firm's value with respect to e is larger than $1 + \lambda$. In the range $(\underline{e}(z), \bar{e}(z))$, instead, the marginal increase in the value of the firm is not sufficient to cover the cost of one unit of new equity. Therefore, no shares are issued. In this range, the marginal increase in the value of the firm is also larger than 1 and the firm prefers to reinvest all the profits. Finally, for values of equity above $\bar{e}(z)$ the marginal increase in the value of the firm is smaller than 1 and the firm distributes dividends.

[Place Figure 2 here]

The figure also shows another interesting feature of the new shares policy. Consider a firm with a low z and suppose that its equity is in the range $(\underline{e}(z_1), \underline{e}(z_2))$. If the productivity switches from z_1 to z_2, the firm will issue new shares. Therefore, in this economy firms issue new shares in two cases: when they are making losses that dissipate their net worth and when their future prospects improve. In both cases, a re-capitalization of the firm is the optimal policy.

The monotonicity of the investment function (point (b) in the above proposition) along with the reinvestment of the firm's profits imply that the investment of firms is sensitive to cash flows even after controlling for the future profitability of the firm (controlling for z') and this sensitivity is greater for smaller firms. In this way the model captures an important empirical regularity of the investment behavior of firms as shown in Fazzari, Hubbard, & Petersen (1988) and Gilchrist & Himmelberg (1995, 1998).

B. Entrance of new firms and invariant distribution of firms

New firms are created with an initial value of equity raised by issuing new shares. The optimal equity of a new firm with initial productivity z is the lower bound $\underline{e}(z)$ as determined in the previous section. Therefore, the cost of creating a new firm with initial productivity z is $\kappa + (1+\lambda)\underline{e}(z)$ and the surplus generated by creating the firm is $\Omega(z, \underline{e}(z)) - \kappa - (1+\lambda)\underline{e}(z)$.

As in the frictionless model, many projects are drawn in each period from the invariant distribution of z, and they will be implemented only if the surplus from creating a new firm is non-negative. In equilibrium, the following arbitrage condition must be satisfied:

$$\Omega(z_N, \underline{e}(z_N)) = \kappa + (1+\lambda)\underline{e}(z_N) \tag{13}$$

As emphasized earlier, this arbitrage condition can be interpreted as a general equilibrium property: the entrance of new firms would induce changes in the prices and in the value of the firm Ω, until there are no gains from creating new firms.[7]

This framework generates complex dynamics and at each point in time the economy is characterized by a certain distribution of firms μ. Technically, μ is a measure of firms over the product set $\prod_{i=1}^{N}[\underline{e}(z_i), \bar{e}(z_i)]$. In the analysis of the next sections, we will concentrate on the invariant distribution of firms denoted by μ^*. The existence of the invariant distribution depends on the properties of the transition function generated by the optimal decision rule $b(z, e)$. The transition function gives rise to a mapping Ψ which returns the next period measure as a function of the current one. The invariant distribution is the fixed point of this mapping, that is, $\mu^* = \Psi(\mu^*)$. In this section we only state the main existence result. The proof requires the introduction of some formal definitions and the derivation of intermediate results which are relatively technical. They are in the appendix.

Proposition 4 *An invariant measure of firms μ^* exists. Moreover, if the probability of default is decreasing in e, then μ^* is unique and the sequence of measures generated by the transition function, $\{\Psi^n(\mu_0)\}_{n=0}^{\infty}$, converges weakly to μ^* from any arbitrary μ_0.*

Proof: *Appendix.*

The convergence result is especially important because it allows us to find this distribution numerically through the repeated application of the mapping Ψ.

C. Properties of the economy with financial frictions: the case of i.i.d. shocks

In this section we describe the financial behavior and industry dynamics properties generated by the model with financial market frictions starting with the special case in which z takes only two values: the absorbing shock $z_0 = 0$ and z_1. We will refer to this as the i.i.d. case because, conditional on surviving, the shock is independently and identically distributed. This simple case facilitates an understanding of how the financial mechanism affects the dynamics of firms, as opposed to the dynamics induced by changes in the productivity level. This will also facilitate an understanding of how the interaction between persistent shocks and financial frictions—studied in section D.—affects the dynamics of the firm.

For the frictionless economy the dynamics properties of the firm can be characterized analytically. In the economy with financial frictions we need to use numerical methods. These methods are described in appendix IV..

Parameterization: We parameterize the model assuming that a period is a year and we set the risk-free interest rate r to 4 percent and the depreciation rate to 0.07.

The revenue function is characterized by the function F, the value of z_1 and by the stochastic properties of the shock ε. The function F is specified as $F(k) = k^{\nu}$, and the technology shock ε is assumed to be normally distributed with mean zero and standard deviation σ. The parameter ν determines the degree of returns-to-scale. Studies of the manufacturing sector as in Susantu

Basu and John G. Fernald (1997), find that this parameter is close to 1. We assign a value of 0.975.

In the sample of firms analyzed by Evans (1987), the average probability of exit is about 4.5 percent. Therefore, we assign a value of 0.045 to the probability of the absorbing shock $\Gamma(z_0/z_1)$. The default cost ξ is set to be 1 percent of the value of the equity of the largest firm and the premium for new shares to $\lambda = 0.3$. These two parameters do not affect the qualitative properties of the model.

There are still four parameters to be pinned down. Those are ϕ, β, σ and z_1. The calibration of these parameters is obtained by imposing four conditions: (a) the equity of the largest firm is normalized to 100;[8] (b) The average probability of default is 1 percent; (c) The value of debt of the largest firms is 25 percent of the total value of its assets ($b/k = 0.25$); (d) The capital-output ratio is 2.5. Condition (b) derives from the estimates of Dun and Bradstreet Corporation for the period 1984-92. Condition (c) derives from balance sheet evidence of large firms. For example, in the sample of firms analyzed by Bronwyn H. and Robert E. Hall (1993), the ratio of debt to total assets is about 0.25. Notice that, once we fix ϕ, the wage rate w and the capital-labor ratio can be determined to yield a certain capital income share and a desired range of heterogeneity in employment. For example we can chose a capital income share of 0.36 and have the largest firm employing 2,000 workers. In this way the model generates heterogeneity that is comparable to empirical studies as in Evans (1987). The full set of parameter values is reported in table 1.

[Place Table 1 here]

Financial behavior and invariant distribution: Figure 3 shows the key properties of the financial behavior of firms. These properties can be summarized as follows:

- Small firms take on more debt (higher leverage).
- Small firms face higher probability of default.

- Small firms have higher rates of profits.
- Small firms issue more shares and pay fewer dividends.

[Place Figure 3 here]

The value of debt, plotted in panel (a) of Figure 3, is increasing in the equity of the firm. But, debt as a fraction of equity (leverage) is decreasing in the firm's equity (see panel (b)). To understand why debt is an increasing function of equity, we have to consider the trade-off that firms face in deciding the optimal amount of debt. On the one hand, more debt allows them to expand the production scale and increase their expected profits; on the other, the expansion of the production scale implies a higher volatility of profits and a higher probability of failure. Given that a large fraction of profits is reinvested, and the firm's future value is a concave function of equity (see panel (f)), the firm's objective is a concave function of profits. This implies that the volatility of profits (for a given expected value) has a negative impact on the firm's value. Therefore, in deciding whether to expand the scale of production by borrowing more, the firm compares the marginal increase in the expected profits with the marginal increase in its volatility (and therefore, in the volatility of next period equity). Due to diminishing returns, as the firm increases its equity and implements larger production plans, the marginal expected profits from further increasing the production scale decrease. Consequently, the firm becomes more concerned about the volatility of profits and borrows less in proportion to its equity. As a consequence of higher borrowing, small firms face a higher probability of default as shown in panel (c).

Panels (d) and (e) plot the expected rates of profits, new shares and dividends. The expected profit (as a fraction of equity) is decreasing in the size of the firm. This is a result of the financing policy outlined above and the decreasing returns-to-scale property of the revenue function. The higher profitability of smaller firms implies that they have a greater incentive to reinvest profits and when the equity of the firm falls below a certain threshold, the firm issues new shares even if this requires a premium. In fact, the expected rate of issuance of new shares is decreasing in the equity of the firm while the dividend rate is increasing in the size of the firm. The higher profitability of small firms, associated with their lower dividends, implies that small firms invest more.

Panels (f) and (g) plot the firm's value and the value of Tobin's q. As can be seen from these figures, the firm's value is an increasing and concave function of equity and Tobin's q is decreasing in the firm's size.

Panel (h) plots the invariant distribution of firms. If we exclude the largest size, the shape of this distribution presents a degree of skewness toward small firms which is also an empirical regularity of the data. There is a concentration of firms at the bottom of the distribution because the optimal size of new entrants is small. The concentration of firms in the largest class, instead, follows from the existence, in the model, of an (endogenous) upper bound to the firm's size. In the data we have firms that employ many more workers than the largest firms in the model. Although the number of these firms is relatively small, they account for a large fraction of aggregate production. Accordingly, the largest firms in the model must be interpreted as representing the production of firms employing more than 2,000 workers: the large share in production of these big firms is accounted for in the model by an increase in the number of firms rather than their size.

Finally, Figure 4 plots the joint distribution of firms over size (equity) and age. Because new firms are small, the invariant distribution is characterized by a concentration of firms in small and young classes.

[Place Figure 4 here]

Industry dynamics: Figure 5 shows the key properties of the firms' dynamics. These properties can be summarized as follows:

- Small firms grow faster and experience higher volatility of growth.

- Small firms face higher probability of default.
- Small firms experience higher rates of job reallocation (with some qualification for job destruction).
- Without conditioning on size, young firms experience higher rates of growth, default, and job reallocation (with some qualification for job destruction).

[Place Figure 5 here]

Panel (a) of Figure 5 reports the expected growth rate of equity as a function of the initial size of the firm and panel (b) its standard deviation. The growth rate of the firm is decreasing in size. This derives from the higher rate of profits of small firms, and from their lower dividend payments (see panels (d) and (e) of Figure 3). The standard deviation of growth is also decreasing in the size of the firm except for very small firms. This is because there is a lower bound to the size of the firms. When their equity falls below a certain threshold they issue new shares.

Panel (d) plots the rates of job creation and job destruction. Following Davis et al. (1996), job creation is defined as the sum of employment gains of expanding firms, and job destruction is defined as the sum of employment losses of contracting firms. Both creation and destruction are decreasing in the firm's size. The only exception is the job destruction of very small firms. This is because there is a lower bound $\underline{e}$ to the firm's equity. Therefore, a firm with equity $\underline{e}$ never destroys jobs, with the exception of exit. But, for this model the probability of exit is the same for all firms. As discussed in section III., by making exit endogenous, the probability of exit decreases with the size of the firm. In this way small firms will destroy more jobs due to exit, as opposed to the reduction of the production scale.

The model generates an unconditional age dependence of firm dynamics as shown in panels (e) though (h) of Figure 5. These figures show that growth, variability of growth, failure rates and job reallocation are decreasing in the age of the firm. This dependence, however, derives from the fact that young firms are small, which in turn derives from the small size of new entrants. However, if we control for the size of the firm, these dynamics are independent of age. This is because equity, which determines the size of the firm, is the only dimension of heterogeneity.

To summarize, the model with financial frictions and i.i.d. shocks is able to generate the dependence of the firm dynamics on size but it shares with the basic frictionless model the inability to generate the age dependence, once we control for size. This is because in both models there is only one dimension of heterogeneity (identified by the variable e in the economy with financial friction and i.i.d. shocks, and by the variable z in the basic model with persistent shocks). To account simultaneously for the size and age dependence, another dimension of heterogeneity is needed. As we will see in the next section, this is obtained by combining persistent shocks and financial frictions.

D. Properties of the economy with financial frictions: the case of persistent shocks

Parameterization: Relative to the case of i.i.d. shocks, we only need to parameterize the process for the shock z. We assume that, conditional on surviving, z follows a symmetric two-state Markov process with $\Gamma_c(z_1/z_1) = \Gamma_c(z_2/z_2) = 0.95$. The two values of the shock, are chosen so that in the invariant distribution the employment of the largest firm with current $z = z_1$ is 50 percent the employment of the largest firm with current $z = z_2$. The choice of these parameters and how they affect the dynamic properties of the model are discussed after the presentation of the results.

Financial behavior and invariant distribution: The key properties of the financial behavior of firms are similar to the case of i.i.d. shocks. Now, however, firms differ over two dimensions: equity and productivity z. Panel (a) of Figure 6 plots the value of debt as a function of its equity, for low ($z = z_1$) and high ($z = z_2$) productivity firms. For each equity size, high productivity firms borrow more and implement larger production scales in order to take advantage of their higher productivity. As discussed for the model with i.i.d. shocks, in deciding the scale of production, firms face a trade-off. On the one hand, a larger production scale allows higher expected profits. On the other, a larger production scale implies higher volatility of profits, to which the firm is averse (because the firm's value is a concave function of profits). For a high productivity level z, the marginal expected profit is higher for each production scale. Consequently, the firm is willing to face higher risk by borrowing more and expands the scale of production. As shown in panel (b), firms with a high value of z enjoy higher profits. The higher profits allow these firms to grow faster (see panel (c)). The increase in the volatility of profits induced by higher debt, also implies that their growth rate is more volatile. As a consequence of this, high productivity firms experience higher failure rates (see panel (d)) and higher rates of job reallocation (see panels (e) and (f)).

[Place Figure 6 here]

The last two panels of Figure 6 plot the fraction of low and high productivity firms in the invariant distribution as a function of equity (panel (g)) and as a function of age (panel (h)). Of course, above a certain equity size, all firms are of the high productivity type. This is because the maximum size of the firm increases with z. Below this size, the fraction of low productivity firms tends to increase because new firms are small and have high productivity. After entering, some of them switch to low productivity but at a slow rate. In the interim they growth so that, when they switch, they have more equity. This process also implies that the fraction of low productivity firms increases with age as shown in panel (h).

Industry dynamics: This heterogeneous behavior among firms of different productivity, introduces an age dependence in the dynamics of firms. Figure 7 plots the average growth rate, the default rate and the rates of job reallocation (creation and destruction), as a function of the firm's size and age. In order to separate the size effect from the age effect, these variables are plotted for different age classes of firms (left panels) and for different size classes (right panels). These variables are all decreasing in the size of the firm and in its age, even after controlling, respectively, for age and size. The only exception is for the job destruction of very small firms. As observed previously, there is a lower bound to the size of firms. Consequently, firms that are initially close to this bound destroy very few jobs. We have also observed previously that this feature could be corrected if we introduce endogenous exit. In that case small firms will destroy more jobs because they face higher rates of exit.

[**Place Figure 7 here**]

The size dependence in this analysis, is driven mainly by the same factors that generated this dependence in the model with independent shocks. In contrast, the age dependence derives from the technological composition of firms in each age class. As shown in panel (h) of Figure 6, the fraction of young firms with low productivity is smaller than old firms. This is because new entrants have high productivity. Now because firms with $z = z_2$ experience higher rates of growth, failure and job reallocation, we also have that for each size class, younger firms grow faster and face higher rates of failure and job reallocation. Thus, in an economy with persistent shocks to technology, in order to have a significant dependence of firm dynamics on age, there must be a heterogeneous composition of firm types in each age class of firms. In this respect the degree of persistence plays an important role. If the shock is not highly persistent, the heterogeneous composition of firms becomes insignificant after a few periods. With a very persistent shock, instead, the heterogeneity vanishes slowly and the age dependence is maintained for a large range of ages.

Quantitatively, the age dependence is small, but this may depend on the simple structure chosen for the stochastic process of z. Also notice that the age effect is more important among the class of small firms and it almost disappears for very large firms.

The initial productivity of new firms plays an important role. In the model new entrant firms are of the high productivity type. This is consistent with the view that new entrants possess better technologies and perform better than incumbent firms as in Jeremy Greenwood and Boyan Jovanovic (1999). Alternatively we could assume that new firms initially enter with low productivity, that is, $z = z_1$. This assumption would be consistent with a learning-by-doing interpretation of the technology shock z. However, if new entrant firms are of the low productivity type, then the age dependence would be of the wrong sign, with old firms experiencing higher rates of growth, job reallocation and failure.

To summarize, the integration of a basic model of firms dynamics (frictionless economy) with a model of financial frictions (economy with frictions and i.i.d. shocks) is able to account for most of the stylized facts about the financial behavior and the growth of firms. In particular, this more general model is able to generate the simultaneous dependence of industry dynamics on size and age while the other two models—the frictionless model and the model with financial frictions and i.i.d. shocks—can only account for the size dependence.

III. Endogenous exit

In this section we briefly discuss the extension of the model that allows for endogenous exit. Endogenous exit can be introduced by assuming that, in each period, the firm faces a fixed cost of production ζ.

Consider first the frictionless economy. In this economy the value of a firm is increasing in the value of z. If z is highly persistent, for small values of z' the value of the firm will be negative. In this case the firm will exit. Suppose that the shock z can take N values. Then we can identify $\underline{z}$ for which the firm will exit if $z' \leq \underline{z}$ and will continue operating if $z' > \underline{z}$. Now, the probability that $z' \leq \underline{z}$ decreases with the current value of z. Because firms with a higher value of z are larger firms, we have that the exit rate decreases with the size of the firm. This is basically the result of Hopenhayn (1992). Although the model is able to capture the size dependence, if we control for the current size of the firm (the current value of z), the survival probability of firms is independent of their age. Therefore, the model is not able to generate the (conditional) dependence of exit on age.

Now consider the model with financial frictions. We have seen that the value of the firm is increasing in e and z. This is also the case when there is a fixed cost. Consequently, in this model, exit decreases with the size of the firm. The age dependence, however, cannot be established in general but will depend on the parameterization of the model. From the analysis of the previous section we have seen that a larger proportion of young firms are of the high productivity type. On the one hand, this implies that younger firms borrow more and face higher probabilities of falling to lower values of equity for which the value of the firm is negative. This may imply that younger firms face a higher probability of exit. This mechanism also explains why younger firms face higher probability of default, as we have seen in the previous section. On the other hand, because of their high productivity, younger firms face a lower probability of falling to lower values of z for which the value of the firm becomes negative. This decreases the exit probability of young firms. A priori, we cannot say which effect dominates. But in principle, for certain parameter values, the exit probability might be decreasing in the age of the firms, even after controlling for their size.

IV. Conclusion

Existing models of industry dynamics that abstract from financial market frictions are unable to account simultaneously for the dependence of the firm dynamics on size and age. A model with only persistent shocks to technology, as in Hopenhayn (1992), is able to account for the size dependence but does not capture the (conditional) age dependence. The learning model of Jovanovic (1982) is able to generate the age dependence but does not capture the (conditional) size dependence. In this paper we introduce financial market frictions in an otherwise standard industry dynamics model and we show that the integration of persistent shocks and financial market frictions allows the model to account for the simultaneous dependence of the firm dynamics from size (once we control for age) and age (once we control for size). More importantly, the integration of these two features helps to reconcile the characteristics of firms growth with many of the financial features related to size that are observed in the data.

This paper can be viewed as a first step toward the study of the importance of financial market frictions for the dynamics of the firm. It is a first step because we consider only one period debt contracts. We leave for future research the goal of studying the dynamics of the firm when financial contracts are not limited to one-period debt. Examples of this approach are Rui Albuquerque and Hugo Hopenhayn (1997) and Vincenzo Quadrini (1999). The first studies optimal dynamic contracts in an environment in which information is symmetric and financial frictions derive from the limited enforceability of these contracts; the second studies an environment in which financial frictions derive from information asymmetries which generate moral hazard problems.

Appendix A: Analytical proofs

LEMMA 1 *Let (e_1, b_1) and (e_2, b_2) be two arbitrary points in the feasible space, let (e_α, b_α) be a convex combination of these two points, and define the function $\psi(\varphi)$ as:*

$$\text{(A1)} \qquad \psi(z+\varepsilon, z') = q(e_\alpha, b_\alpha, z+\varepsilon, z') - \alpha \cdot q(e_1, b_1, z+\varepsilon, z') - (1-\alpha) \cdot q(e_2, b_2, z+\varepsilon, z')$$

where $q(e_1, b_1, z+\varepsilon, z')$ is the end-of-period resource function defined in (6). Then, under the conditions of proposition 3, there exists $\hat{\varepsilon}(z, z') < \infty$ such that, for each $z, z' \in \mathbf{Z}$, $\psi(z+\varepsilon, z') \le 0$ if $\varepsilon \le \hat{\varepsilon}(z, z')$, and $\psi(z+\varepsilon, z') > 0$ if $\varepsilon > \hat{\varepsilon}(z, z')$. Moreover, $\int \psi(z, \varphi, z') f(d\varepsilon) > 0$.

PROOF: The end-of-period resource function is equal to:

$$\text{(A2)} \qquad q(e, b, z+\varepsilon, z') = \begin{cases} \underline{\underline{e}}(z') + (\varepsilon - \underline{\underline{\varepsilon}}) F(e+b)\,, & \text{if } \varepsilon \ge \underline{\underline{\varepsilon}} \\ \underline{\underline{e}}(z')\,, & \text{if } \varepsilon < \underline{\underline{\varepsilon}} \end{cases}$$

Therefore, q is a linear function of ε with slope $F(e+b)$. Because F is strictly concave, we have that $F(e_\alpha + b_\alpha) > \alpha F(e_1 + b_1) + (1-\alpha) F(e_2 + b_2)$. This implies that the slope of $q(e_\alpha, b_\alpha, z+\varepsilon, z')$ is greater than the slope of $\alpha q(e_1, b_1, z+\varepsilon, z') + (1-\alpha) q(e_2, b_2, z+\varepsilon, z')$. Therefore, if for small value of ε the function $q(e_\alpha, b_\alpha, z+\varepsilon, z')$ is smaller than $\alpha q(e_1, b_1, z+\varepsilon, z') + (1-\alpha) q(e_2, b_2, z+\varepsilon, z')$, as we increase ε the value of the first function gets closer to the second function until they cross. In the case in which the function $q(e_\alpha, b_\alpha, z+\varepsilon, z')$ is greater than $\alpha q(e_1, b_1, z+\varepsilon, z') + (1-\alpha) q(e_2, b_2, z+\varepsilon, z')$, even for small values of ε, then the first function will be greater than the second for all values of ε.

Let's now prove the second part of the lemma. Simple algebra gives us:

$$\text{(A3)} \qquad \int q(e, b, z+\varepsilon, z') f(d\varepsilon) \;=\; (e+b)(1-\phi) - (1+r)b + zF(e+b) - \xi \int_{-\infty}^{\underline{\underline{\varepsilon}}(z,e,b,z')} f(d\varepsilon)$$

Because F is strictly concave, the first part of the above expression is strictly concave. The term $\xi \int_{-\infty}^{\underline{\underline{\varepsilon}}(z,e,b,z')} f(d\varepsilon)$, however, is not necessarily concave. But, under the conditions of proposition 3, the sensitivity of this term to changes in e and b are very small. For sufficiently small ϵ_1, these changes are negligible and $\int \psi(z+\varepsilon, z') f(d\varepsilon) > 0$.

Proof of proposition 3: Let's start the proof by assuming that z is a constant. The extension to the case of persistent shocks is trivial. First we can restrict the feasible value of e to the set $[e_{min}, e_{max}]$, where e_{min} is sufficiently small and e_{max} sufficiently large so that the equity of the firm will never be outside this interval. We also observe that the choice of b is bounded for each value of e. Denote by k^* the optimal input of capital in the absence of financial frictions. Of course, $b \leq k^* - e$ (it is never optimal to expand the production scale beyond the optimal scale). At the same time $e + b \geq 0$ (capital cannot be negative). Therefore, the correspondence that defines the feasible set for b is continuous, compact and convex valued. We will denote this correspondence by $\mathbf{B}(e) \equiv \{\, b \,|\, -e \leq b \leq k^* - e\}$.

Consider the firm's problem as defined in (7):

$$\text{(A4)} \qquad T(\Omega)(e) = \max_{b \in \mathbf{B}(e)} \left\{ \beta \int_{\underline{\underline{\varepsilon}}(e,b)} \tilde{\Omega}(q(e, b, z+\varepsilon)) f(d\varepsilon) \right\}$$

subject to

$$\text{(A5)} \qquad q(e, b, z+\varepsilon) = \begin{cases} \underline{\underline{e}} + (\varepsilon - \underline{\underline{\varepsilon}}) F(e+b)\,, & \text{if } \varepsilon \geq \underline{\underline{\varepsilon}} \\ \underline{\underline{e}}\,, & \text{if } \varepsilon < \underline{\underline{\varepsilon}} \end{cases}$$

$$\text{(A6)} \qquad (1+r)b = (1-\phi)(e+b) + (z+\varepsilon)F(e+b) - \xi \int_{-\infty}^{\underline{\underline{\varepsilon}}} f(d\varepsilon) - \underline{\underline{e}} \int_{\underline{\underline{\varepsilon}}}^{\infty} f(d\varepsilon)$$

$$\text{(A7)} \qquad \tilde{\Omega}(\underline{\underline{e}}) = 0$$

$$\text{(A8)} \qquad \tilde{\Omega}(x) = \max_{e'} \left\{ d(x, e') + \Omega(e') \right\}$$

subject to

$$\text{(A9)} \qquad d(x, e') = \begin{cases} x - e', & \text{if } x \geq e' \\ (x - e')(1+\lambda), & \text{if } x < e' \end{cases}$$

Notice that the mapping is solved backward. Problem (A8) defines the function $\tilde{\Omega}$. Even if the function Ω is decreasing for some values of e', the function $\tilde{\Omega}$ is always strictly increasing. Then equation (A7) determines the default value of equity $\underline{\underline{e}}$. Because $\tilde{\Omega}$ is strictly increasing, $\underline{\underline{e}}$ is unique. Given $\underline{\underline{e}}$ equation (A6) uniquely defines the default threshold for the shock $\underline{\underline{\varepsilon}}$ and problem (A4) is well defined.

We prove first that T maps bounded and continuous functions into itself and there is a unique continuous and bounded function Ω^* that satisfies the functional equation $\Omega^* = T(\Omega^*)$. The fact that T

maps continuous and bounded functions into itself is proved by verifying the conditions for the theorem of the maximum. If Ω is a continuous and bounded function, then the boundedness and continuity of $\int q(e, b, z + \varepsilon) f(d\varepsilon)$ and $\Omega(e)$ implies that the objective function is continuous and bounded. Because the correspondence $B(e)$ is continuous, compact and convex valued, and the set $[e_{min}, e_{max}]$ is compact and convex, the maximum exists and the function resulting from the mapping $T(\Omega)(e)$ is continuous and bounded. The fact that there is a unique fixed point of the mapping T is proved by showing that T is a contraction. This, in turn, is shown by verifying that T satisfies the Blackwell conditions of monotonicity and discounting.

We want to show now that, under certain conditions, Ω^* is strictly increasing and concave in the interval $[\underline{\underline{e}}, \bar{e}]$, where $\underline{\underline{e}}$ is the value of equity below which the firm defaults, and $\bar{e}$ is the value of equity above which the firm distributes dividends. To show this, we study first a slightly different mapping. Then by showing that for a certain range of equity the two mappings have the same solution (fixed point), we can characterize the properties of the first mapping by studying the second. The modified mapping is obtained by replacing (A4) with the following:

$$T_2(\Omega)(e) \;=\; \begin{cases} \max_{b \in \mathbf{B}(e)} \left\{ \beta \displaystyle\int_{\underline{\underline{\varepsilon}}(e,b)} \tilde{\Omega}(q(e, b, z + \varepsilon)) f(d\varepsilon) \right\}, & \text{if } e \geq \underline{\underline{e}} \\ \\ \max_{b \in \mathbf{B}(\underline{\underline{e}})} \left\{ \beta \displaystyle\int_{\underline{\underline{\varepsilon}}(\underline{\underline{e}},b)} \tilde{\Omega}(q(\underline{\underline{e}}, b, z + \varepsilon)) f(d\varepsilon) \right\} - (\underline{\underline{e}} - e)\,, & \text{if } e < \underline{\underline{e}} \end{cases}$$

In (A10) we have added an extra term as if the firm, before borrowing, issues new shares anytime its initial equity are below $\underline{\underline{e}}$. The new mapping (A10) is also a contraction and has a unique fixed point, denoted by Ω^{**}. We show now that $\Omega^*(e) = \Omega^{**}(e)$ for $e \geq \underline{\underline{e}}$.

Consider problem (A10) where the function $\tilde{\Omega}$ is derived from (A8) after substituting Ω^*. Of course, for $e \geq \underline{\underline{e}}$, $T(\Omega^*)(e) = \Omega^*(e)$. This implies that $\Omega^{**}(e) = \Omega^*(e)$ for $e \geq \underline{\underline{e}}$. For $e < \underline{\underline{e}}$, the two fixed points are different. But we are interested only in values of $e \geq \underline{\underline{e}}$.

After establishing the equivalence between the two fixed points for the relevant range of e, we now characterize the properties of Ω^{**}. By doing so we also characterize the properties of Ω^* for $e \geq \underline{\underline{e}}$. If Ω is concave and satisfies $\Omega(0) \geq 0$, then the function $\tilde{\Omega}$ defined in (A8) is strictly increasing and concave. In addition, if $q(e, b, z + \varepsilon)$ was strictly concave for each ε, then $\int \tilde{\Omega}(q(e, b, z + \varepsilon)) f(d\varepsilon)$ would be strictly concave in e and b and the maximizing value of b would be unique. This would also imply that $T_2(\Omega)$ is strictly concave and satisfies $T_2(\Omega)(0) \geq 0$ in $[e_{min}, e_{max}]$. Lemma 1, however, shows that $q(e, b, z+\varepsilon)$ is strictly concave only for ε above a certain threshold. The problem is that, in the neighbor of the failing shock, the end-of-period resource function is not concave. However, if the density probability satisfies the restrictions of proposition 3, then T_2 will map increasing and concave functions in strictly increasing and concave functions.

This point can be shown as follows: Take two points for equity e_1, e_2 and two points for debt b_1, b_2 and define e_α, b_α to be a convex combination of these two points with $0 < \alpha < 1$. Then, by lemma 1, if the density function of the shock satisfies the conditions of proposition 3, there exists $\hat{\varepsilon}$ such that the term $\psi_\alpha(z+\varepsilon) = q(e_\alpha, b_\alpha, z+\varepsilon) - \alpha q(e_1, b_1, z+\varepsilon) - (1-\alpha)q(e_2, b_2, z+\varepsilon)$ is greater than zero if $\varepsilon > \hat{\varepsilon}$, and non-positive if $\varepsilon \leq \hat{\varepsilon}$. The concavity of $\int q(e, b, z+\varepsilon)f(d\varepsilon)$, however, implies that the concave points dominates the non-concave ones and $\int \psi_\alpha(z+\varepsilon)f(d\varepsilon) > 0$

Now consider the term:

$$\tilde{\Omega}(q(e_\alpha, b_\alpha, z+\varepsilon)) - \alpha\tilde{\Omega}(q(e_1, b_1, z+\varepsilon)) - (1-\alpha)\tilde{\Omega}(q(e_2, b_2, z+\varepsilon)) \tag{A10}$$

Because $\tilde{\Omega}$ is increasing and concave, this term is greater than zero if $\psi_\alpha(z+\varepsilon) > 0$, but may be smaller than zero if $\psi_\alpha(z+\varepsilon) < 0$. Although the points for which $\psi_\alpha(z+\varepsilon) > 0$ dominates the points for which $\psi_\alpha(z+\varepsilon) < 0$ and $\int \psi_\alpha(z+\varepsilon)f(d\varepsilon) > 0$, this is not necessarily the case for the function defined in (A10). For this to be true, we need to restrict the quantitative importance of ε for those values for which the firm defaults. The firm will never default if $\varepsilon > 0$. Therefore, we have to impose that $f(\varepsilon)$ is relatively small for $\varepsilon < 0$ as assumed in proposition 3. We then have:

$$\begin{aligned} \int [\tilde{\Omega}(q(e_\alpha + b_\alpha, z+\varepsilon)) - \alpha\tilde{\Omega}(q(e_1 + b_1, z+\varepsilon)) - (1-\alpha)\tilde{\Omega}(q(e_2 + b_2, z+\varepsilon))]f(d\varepsilon) \quad &\geq \\ \int [q(e_\alpha + b_\alpha, z+\varepsilon) - \alpha q(e_1 + b_1, z+\varepsilon) - (1-\alpha)q(e_2 + b_2, z+\varepsilon)]f(d\varepsilon) \quad &> \quad 0 \end{aligned}$$

The first inequality comes from the imposed restriction on f, while the second inequality comes from the strict concavity of the expected value of q. Also notice that the slope of $\tilde{\Omega}$ is always between 1 and $1-\lambda$. Therefore, there is a limit to the possible amplification of non concave points. Given this result, it is easy to prove that the mapping T_2 maps concave functions into strictly concave functions and the fixed point $\Omega^{**}(e)$ is strictly concave in e. Moreover, the optimal solution for b is obviously unique (this is simply a problem of maximizing a continuous and concave function over a compact and convex set). The theorem of the Maximum will then guarantee that the solution is continuous in e. We can also show that the optimal value of b is such that $k = e + b$ is non-decreasing in e. Simply observe that, due to the concavity of the expected value of $\tilde{\Omega}$, the marginal return from k is decreasing in k. Because a larger e relaxes the constraint on the feasible k, the reduction in k is not optimal.

Given the strict concavity of Ω^*, the dividend policy assumes a simple form. More specifically, there exists a lower and upper bound $\underline{e}$ and $\bar{e}$, with $\underline{e} < \bar{e}$, for which dividends are negative (the firm issues new shares) when the end-of-period resources are smaller than $\underline{e}$, and positive when the end-of-period resources are larger than $\bar{e}$.

With persistent shocks, the proof follows exactly the same steps. The only difference is that the lower and upper bounds for the next period equity and the failure value of equity depend on the next period z, that is, $\underline{\underline{e}}(z') \leq \underline{e}(z') \leq \bar{e}(z')$.

Proof of proposition 4: Let's start the proof by assuming that, conditional on surviving, z is constant. The extension of the proof to the case of persistent shocks is trivial. Let $Q(e, \mathbf{A}) : [\underline{e}, \bar{e}] \times \mathcal{A} \to [0, 1]$ be the transition function, where $\mathcal{A}$ is the collection of all Borel sets that are subsets of $[\underline{e}, \bar{e}]$, and $\mathbf{A}$ is one of its elements. Note that we can define the transition function over the measurable space $([\underline{e}, \bar{e}], \mathcal{A})$ given the optimal policy of the firm characterized in proposition 3. The function Q delivers the following distribution function for the next period equity e', given the current value of e:

$$\text{(A11)} \qquad \int_{\underline{e}}^{x} Q(e, de') = \begin{cases} (1-\eta) \displaystyle\int_{-\infty}^{\underline{\varepsilon}} f(d\varepsilon) + \eta & \text{if} \quad x = \underline{e} \\ (1-\eta) \displaystyle\int_{-\infty}^{\underline{\underline{\varepsilon}} + \frac{(x - \underline{\underline{e}})}{F(e+b)}} f(d\varepsilon) + \eta & \text{if} \quad \underline{e} \leq x < \bar{e} \\ 1 & \text{if} \quad x > \bar{e} \end{cases}$$

where η is the mass of new entrant firms which is equal to the mass of exiting firms. In this way, the total mass of firms is constant. By normalizing the total mass of firms to 1, the distribution of firms is represented by a probability measure. Given the current probability measure μ_t, the function Q delivers a new probability measure μ_{t+1} through the mapping $\Psi : M([\underline{e}, \bar{e}], \mathcal{A}) \to M([\underline{e}, \bar{e}], \mathcal{A})$, where $M([\underline{e}, \bar{e}], \mathcal{A})$

is the space of probability measures on $([\underline{e}, \bar{e}], \mathcal{A})$. The mapping Ψ is defined as:

$$\mu_{t+1}(\mathbf{A}) = \Psi(\mu_t)(\mathbf{A}) = \int_e Q(e, \mathbf{A}) d\mu \tag{A12}$$

An invariant probability measure μ^* is the fixed point of Ψ, *i.e.*, $\mu^* = \Psi(\mu^*)$. In the following lemma we prove that Q has the Feller property. This property turns out to be useful in the subsequent proof of the existence of an invariant probability measure.

Lemma 2 *The transition function Q has the Feller property.*

Proof 1 *The transition function Q has the Feller property if the function $T(Q)(e)$ defined as:*

$$T(Q)(e) = \int_{\underline{e}}^{\bar{e}} v(e') Q(e, de') \tag{A13}$$

is continuous for any continuous and bounded function v. Conditional on being productive (which happens with probability $1-\eta$), the next period equity is given by:

$$e' = \begin{cases} \underline{e} & \text{if } \varepsilon \leq \underline{\varepsilon} \\ \underline{e} + (\varepsilon - \underline{\underline{\varepsilon}}) F(e+b) & \text{if } \underline{\underline{\varepsilon}} < \varepsilon < \bar{\varepsilon} \\ \bar{e} & \text{if } \varepsilon \geq \bar{\varepsilon} \end{cases} \tag{A14}$$

Therefore, the function $T(Q)(e)$ can be written as:

$$T(Q)(e) = \eta v(\underline{e}) + (1-\eta) \Bigg[v(\underline{e}) \int_{-\infty}^{\underline{e}} f\left(\underline{\underline{\varepsilon}}(e,b) + \frac{(e' - \underline{\underline{e}})}{F(e+b)} \right) F(e+b) de' + \int_{\underline{e}}^{\bar{e}} v(e') f\left(\underline{\underline{\varepsilon}}(e,b) + \frac{(e' - \underline{\underline{e}})}{F(e+b)} \right) F(e+b) de' + v(\bar{e}) \int_{\bar{e}}^{\infty} f\left(\underline{\underline{\varepsilon}}(e,b) + \frac{(e' - \underline{\underline{e}})}{F(e+b)} \right) F(e+b) de' \Bigg] \tag{A15}$$

Because $b(e)$ is a continuous function, then $F(e + b(e))$ is also continuous. If in addition $\underline{\underline{\varepsilon}}(e, b(e))$ is continuous in e, then the continuity of f implies that $T(Q)$ is continuous. So we only need to prove that $\underline{\underline{\varepsilon}}$ is continuous.

The function $\underline{\underline{\varepsilon}}(e,b)$ is implicitly defined by:

$$(1+r)b + \underline{e}(z') \int_{\underline{\varepsilon}}^{\infty} f(d\varepsilon) + \xi \int_{-\infty}^{\underline{\underline{\varepsilon}}} f(d\varepsilon) \; = \; (1-\phi)(e+b) + \zeta(\underline{\underline{\varepsilon}}) F(e+b) \tag{A16}$$

where $\zeta(\underline{\underline{\varepsilon}}) = z + \int_{-\infty}^{\underline{\underline{\varepsilon}}} \varepsilon f(d\varepsilon) + \underline{\underline{\varepsilon}} \int_{\underline{\underline{\varepsilon}}}^{\infty} f(d\varepsilon)$, is strictly increasing and continuous in $\underline{\underline{\varepsilon}}$ under assumption 2. This implies that ζ is invertible and the inverse function is continuous. Given this, it can be easily verified that for the relevant range of e and b, $\underline{\underline{\varepsilon}}(e, b)$ is continuous (singleton) function of e and b.

Because the probability measure of firms has support in the compact set $[\underline{e}, \bar{e}]$ and, as proved in lemma 2 the transition function Q has the Feller property, then Theorem 12.10 in Nancy L. Stokey, Robert E. Lucas with Edward C. Prescott (1989) guarantees that there exists an invariant distribution μ^*. In order to prove that μ^* is unique we need extra conditions. Theorem 12.12 in Stokey, Lucas, & Prescott (1989) establishes that, if Q is monotone and satisfies a mixing condition, then the invariant probability measure μ^* is unique. We want to show then that under the assumption made in the second part of the proposition, Q is monotone and it satisfies the mixing condition.

- (**Monotonicity**) To prove that Q is monotone, we have to show that $Q(e_1, .)$ is dominated by $Q(e_2, .)$ $\forall\, e_1, e_2 \in [0, \bar{e}]$, with $e_1 \leq e_2$. The dominance means that for all bounded increasing functions v, $\int v(e')Q(e_2, de') \geq \int v(e')Q(e_1, de')$. When the state space is defined in $\mathbf{R}^1$, then $Q(e_2, .)$ dominates $Q(e_1, .)$ if and only if $\int_{\underline{e}}^{x} Q(e_2, de') \leq \int_{\underline{e}}^{x} Q(e_1, de')$, $\forall x \in [\underline{e}, \bar{e}]$. Given the monotonicity of $k(e) = e + b(e)$ stated in proposition 3, if the default probability is decreasing in e ($\underline{\underline{\varepsilon}}(e, b(e))$ is decreasing in e), it can be verified that Q, defined in $(A11)$, is monotone.

- (**Mixing condition**) We have to prove that there exists $\hat{e} \in [\underline{e}, \bar{e}]$, $\epsilon > 0$, and $N \geq 1$ such that $\Psi^N(\underline{e}, [\hat{e}, \bar{e}]) \geq \epsilon$ and $\Psi^N(\bar{e}, [\underline{e}, \hat{e}]) \geq \epsilon$. Because we are assuming that $f(\varepsilon) > 0\ \forall \varepsilon \in \mathbf{R}$, then the mixing condition is obviously satisfied.

Appendix B: Computational procedure

The computational procedure is based on value function iteration, after the discretization of the state space e. Following is the description of the individual steps.

1. Guess default thresholds $\underline{\underline{e}}(z)$ and equity bounds $\underline{e}(z)$ and $\bar{e}(z)$. Then for each $z \in \mathbf{Z}$, choose a grid in the space of firms' equity, that is, $e \in \mathbf{E}(z) \equiv \{e_1(z), ..., e_N(z)\}$. In this grid, $e_1(z) = \underline{e}$ and $e_N(z) = \bar{e}(z)$.

2. Guess initial steady state values of debt $b_i^*(z)$, for $i = 1, .., N$ and $z \in \mathbf{Z}$.

3. Guess initial steady state values of firm's value $\Omega_i(z)$, for $i = 1, .., N$ and $z \in \mathbf{Z}$.

4. Approximate with a second order Taylor expansion the function $\tilde{\Omega}_i(z, b)$, for $i = 1, .., N$, around the guessed points for the steady state values $b_i^*(z)$. The value function $\Omega(z)$ is approximated with peace-wise linear functions joining the grid points in which the value function is evaluated. The definition of $\tilde{\Omega}_i(z, b)$ takes as given the dividend policy of the firm consisting in retaining issuing new shares in the interval $[\underline{\underline{e}}(z), \underline{e}(z)]$ and retaining all profits until the firm reaches the size $e_N(z)$.

5. Solve for the firm's policy b_i by differentiating the function $\tilde{\Omega}_i(z, b)$ with respect to b and update the default threshold using the condition $(\underline{e}(z) - \underline{\underline{e}}(z))(1 + \lambda) = \Omega(z, \underline{e})$.

6. Eliminate b from $\tilde{\Omega}_i(z, b)$ using the policy rules found in the previous step. The found values are the new guesses for $\Omega_i(z, b)$. The procedure is then restarted from step 4 until converged.

7. After value function convergence, check whether the firm policies found in step 5 reproduce the guesses for the steady state values of debt $b_i^*(z)$. If not, update this guesses and restart the procedure from step 3 until convergence.

8. Check the optimality of the lower and upper bounds $\underline{e}(z)$ and $\bar{e}(z)$ by verifying $\beta \frac{\partial \Omega(z,e)}{\partial e}\Big|_{e=\underline{e}(z)} = 1 + \lambda$ and $\beta \frac{\partial \Omega(z,e)}{\partial e}\Big|_{e=\bar{e}(z)} = 1$. To check for these conditions, compute the numerical derivative of $\Omega(z, e)$ at $e_1(z)$ and $e_N(z)$, taking as given the value of $b_1^*(z)$ and $b_N^*(z)$ found previously. If the condition is not satisfied, update the initial guesses for $\underline{e}(z)$ and $\bar{e}(z)$ and restart the procedure from step 1 until convergence.

9. Given the decision rules, the transition function for the distribution of firms is well defined and the invariant distribution of firms is determined by iterating on the law of motion for this distribution.

REFERENCES

Albuquerque, Rui and Hopenhayn, Hugo (1997). Optimal Dynamic Lending Contracts With Imperfect Enforceability. Unpublished manuscript, University of Rochester.

Basu, Susantu and Fernald, John G. (1997). Returns to Scale in U.S. Production: Estimates and Implications. Journal of Political Economy, 105(2), 249–83.

Campbell, Jeffrey R. (1998). Entry, Exit, Embodied Technology, and Business Cycle. Review of Economic Dynamics, 1(2), 371–408.

Campbell, Jeffrey R. and Fisher, Jonas D. M. (2000). Aggregate Employment Fluctuations With Microeconomic Asymmetries. American Economic Review, 90(5), 1323–45.

Davis, Steven; Haltiwanger, John; and Schuh, Scott (1996). Job Creation and Distruction. MIT Press, Cambridge, Massachusetts.

Evans, David S. (1987). Tests of Alternative Theories of Firm Growth. Journal of Political Economy, 95(4), 657–74.

Fazzari, Stephen M.; Hubbard, Glenn R.; and Petersen, Bruce C. (1988). Financing Constraints and Corporate Investment. Brookings Papers on Economic Activity, 88(1), 141–95.

Gilchrist, Simon and Himmelberg, Charles P. (1995). Evidence on the Role of Cash Flow for Investment. Journal of Monetary Economics, 36(3), 541–72.

Gilchrist, Simon and Himmelberg, Charles P. (1998). Investment: Fundamentals and Finance. In Bernanke, Ben S. and Rotemberg, Julio (Eds.), NBER Macroeconomics Annual. MIT Press, Cambridge, Massachusetts.

Gomes, Joao F. (1998). Financing Investment. Forthcoming in *American Economic Review.*

Greenwood, Jeremy and Jovanovic, Boyan (1999). The Information-Technology Revolution and the Stock Market. American Economic Review, 89(2), 116–22.

Hall, Bronwyn H. (1987). The Relationship Between Firm Size and Firm Growth in The U.S. Manufacturing Sector. Journal of Industrial Economics, 35, 583–606.

Hall, Bronwyn H. and Hall, Robert E. (1993). The Value and Performance of U.S. Corporations. Brookings Papers on Economic Activity, 93(93), 1–49.

Hopenhayn, Hugo (1992). Entry, Exit, and Firm Dynamics in Long Run Equilibrium. Econometrica, 60(5), 1127–50.

Hopenhayn, Hugo and Rogerson, Richard (1993). Job Turnover and Policy Evaluation: A General Equilibrium Analysis. Journal of Political Economy, 101(5), 915–38.

Jovanovic, Boyan (1982). The Selection and Evolution of Industry. Econometrica, 50(3), 649–70.

Lucas, Robert E. (1978). On The Size Distribution of Business Firms. Bell Journal of Economics, 9(4), 508–23.

Quadrini, Vincenzo (1999). Investment and Default in Optimal Financial Contracts With Repeated Moral Hazard. Unpublished manuscript, Duke University.

Ross, Sthephen A.; Westerfield, Randolph W.; and Jordan, Bradford D. (1993). Fundamentals of Corporate Finance. Irwin Press. 2nd Edition.

Smith, Clifford W. (1977). Alternative Methods for Raising Capital: Rights Versus Underwritten Offerings. Journal of Financial Economics, 5, 273–307.

Stokey, Nancy L.; Lucas, Robert E.; and Prescott, E. C. (1989). Recursive Methods in Economic Dynamics. Harvard University Press, Cambridge, Massachusetts.

Notes

[1]Some of these empirical facts are shown using establishment data while others are shown using firm level data. However, many of the empirical facts based on establishment data, also hold for single-unit establishments (i.e., establishments that are firms).

[2]These models can generate an *unconditional* dependence of the firm dynamics on size and age. In other words, without conditioning on age, the firm dynamics is negatively related to its size, and without conditioning on size, the firm dynamics is negatively related to the firms age. But, they cannot account simultaneously for the *conditional* dependence on both size and age. Consider, for example, the models developed in Jovanovic (1982) and Hopenhayn (1992). They are the primary models of industry dynamics that emphasize learning and persistent productivity shocks. In the learning model of Jovanovic, age is the main dimension of heterogeneity: For a certain distribution of the shock (log-normal) firms of the same age experience the same growth rate and variability of growth, independently of their size. For a more general distribution of the shock it is not clear whether size has a positive, negative or non-monotone impact on the firm dynamics. (See Jovanovic (1982), page 656). In Hopenhayn's model size is the only dimension of heterogeneity: firms of the same size experience the same dynamics independently of their age. (See Hopenhayn (1992), page 1141). The models developed in Hopenhayn & Rogerson (1993), Campbell (1998) and Campbell & Fisher (2000) are similar to Hopenhayn's model.

[3]See Clifford W. Smith (1977), Stephen A. Ross, Randolph W. Westerfield, and Bradford D. Jordan (1993), Stephen Fazzari, Glenn R. Hubbard, and Bruce Petersen (1988) and Simon Gilchrist and Charles Himmelberg (1995,1998).

[4]This is without loss of generality. Given that the wage rate is constant, the capital-labor ratio would be constant even if the two inputs were substitutable.

[5]It will affect the unconditional dependence on age. When new entrants are small, the growth

rate of firms is negatively related to their age (if we do not control for size). If they are large, the growth rate is positively related to the age of the firm (again, if we do not control for its size).

[6]Notice that the exogenous probability of exit is implicitly accounted in the formulation of the problem by assuming that z_0 is an absorbing shock, that is, $z_0 = 0$ and $\Gamma(z_0/z_0) = 1$.

[7]Condition (13) implies that all new firms are of the high productivity type. As we will see, this feature of the model has important consequences for the dynamics of firms. Although we consider this to be the relevant case, we will also discuss the alternative case in which new entrants are of the low productivity type.

[8]Alternatively, we could fix z_1 and the equity of the largest firm would be determined endogenously.

Table 1. Calibration values for the model parameters.

Lending rate	r	0.040
Intertemporal discount rate	β	0.956
Return to scale parameter	ν	0.975
Depreciation rate	δ	0.070
Standard deviation of the shock ε	σ	0.280
Productivity parameter	z_1	0.428
Probability of exogenous exit	$\Gamma(z_0/z_1)$	0.045
Default cost	ξ	1.000
New shares premium	λ	0.300

Figure 1. Value of the firm as a function of end-of-period net worth (equity).

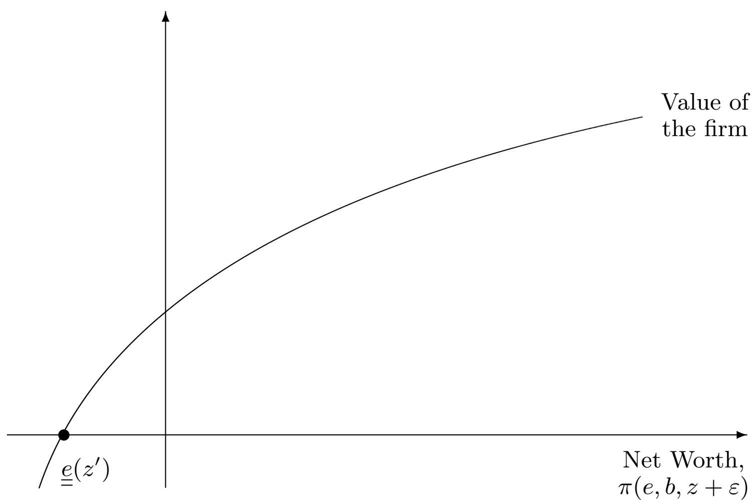

Figure 2. Firm's value as a function of equity for low and high values of z.

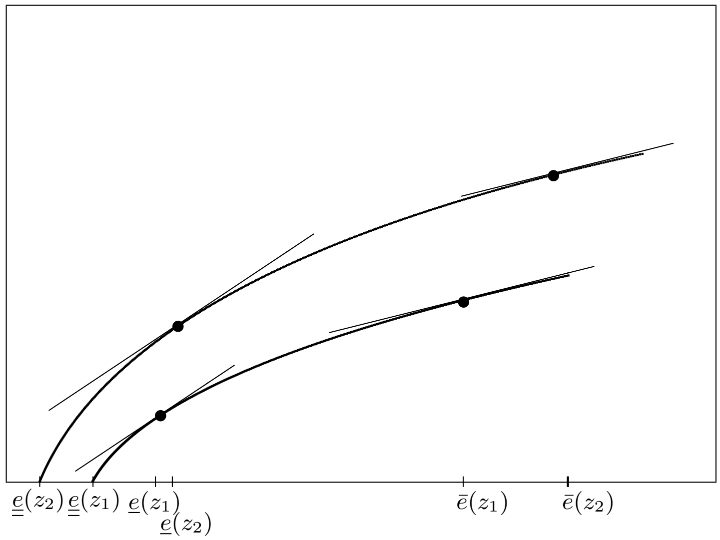

Figure 3. Financial behavior with i.i.d. shocks.

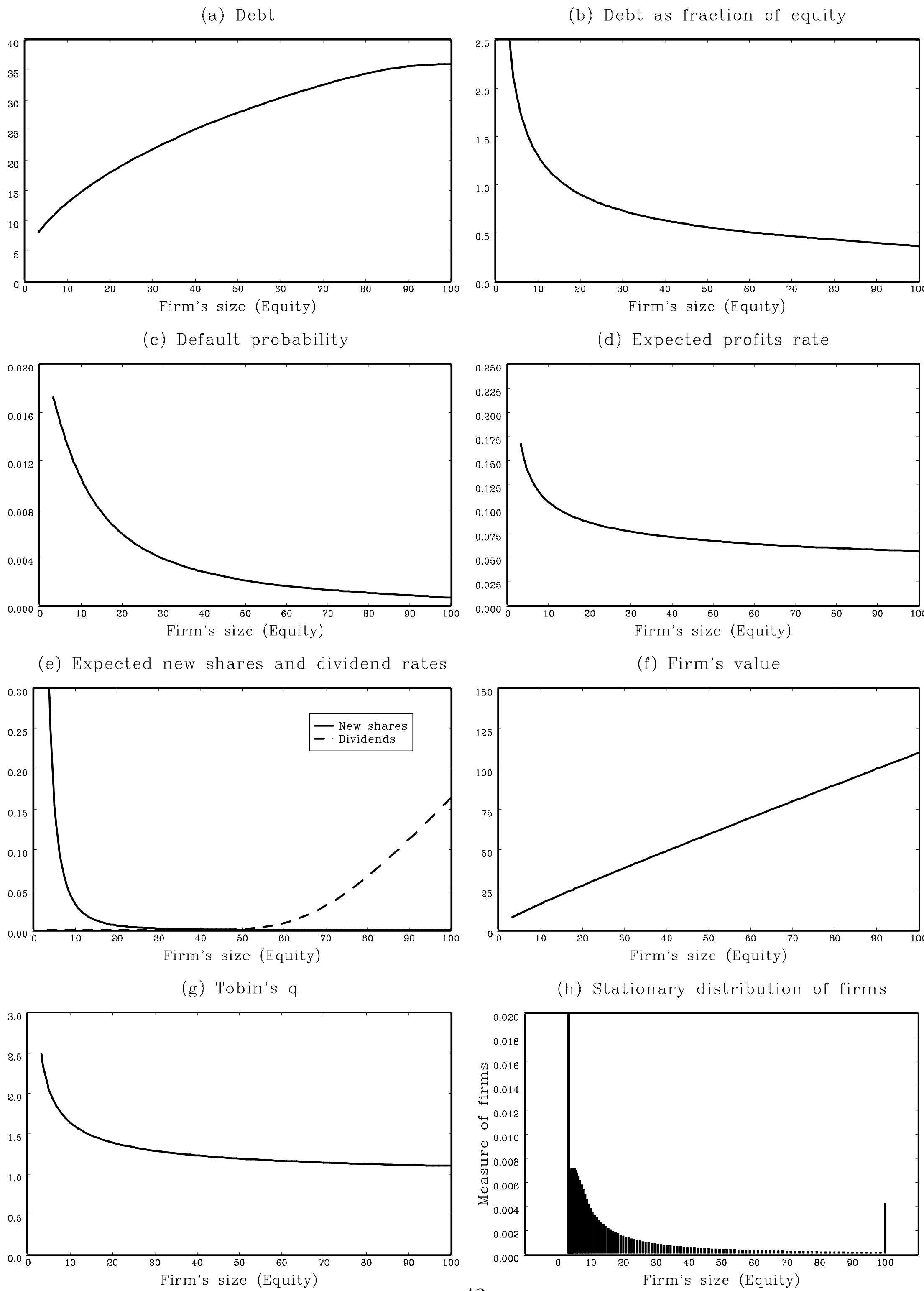

Figure 4. Age and size distribution of firms.

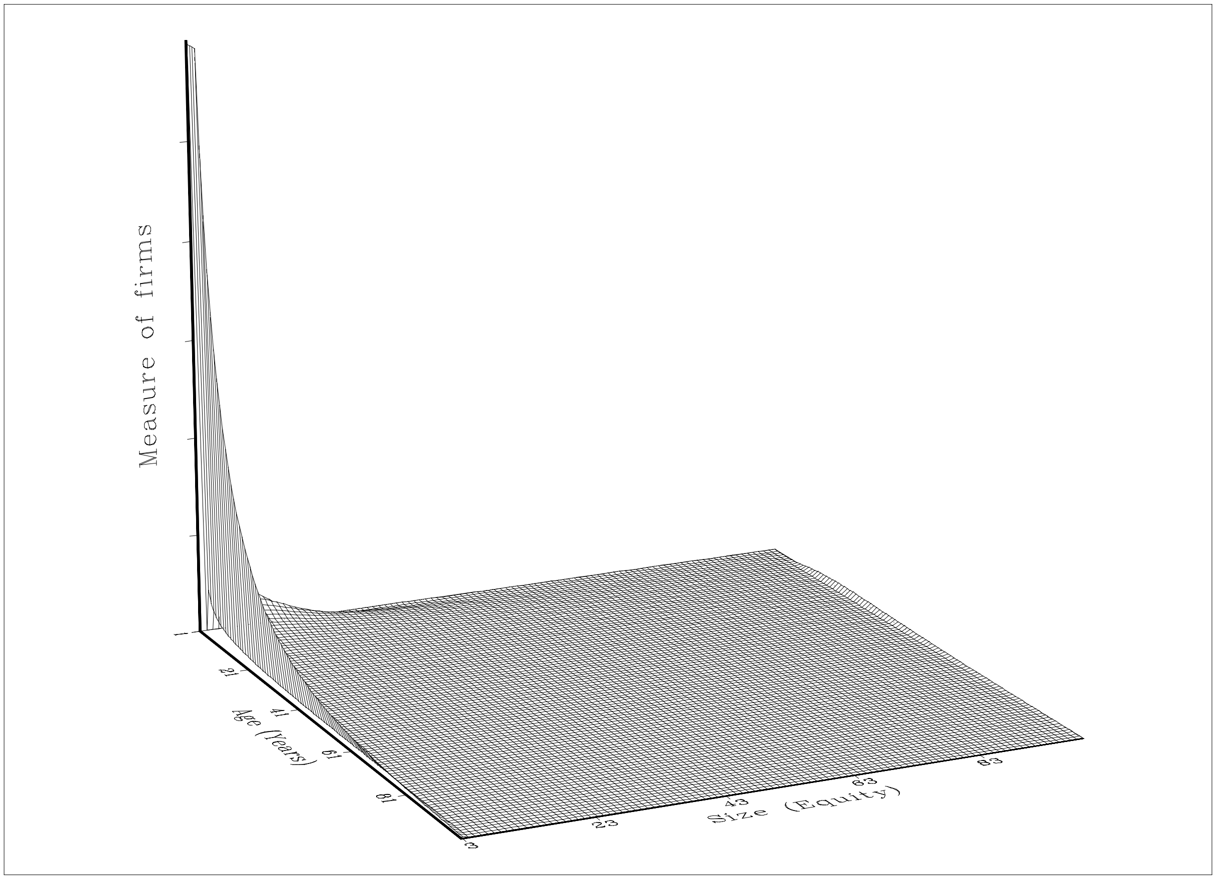

Figure 5. Industry dynamics with i.i.d. shocks.

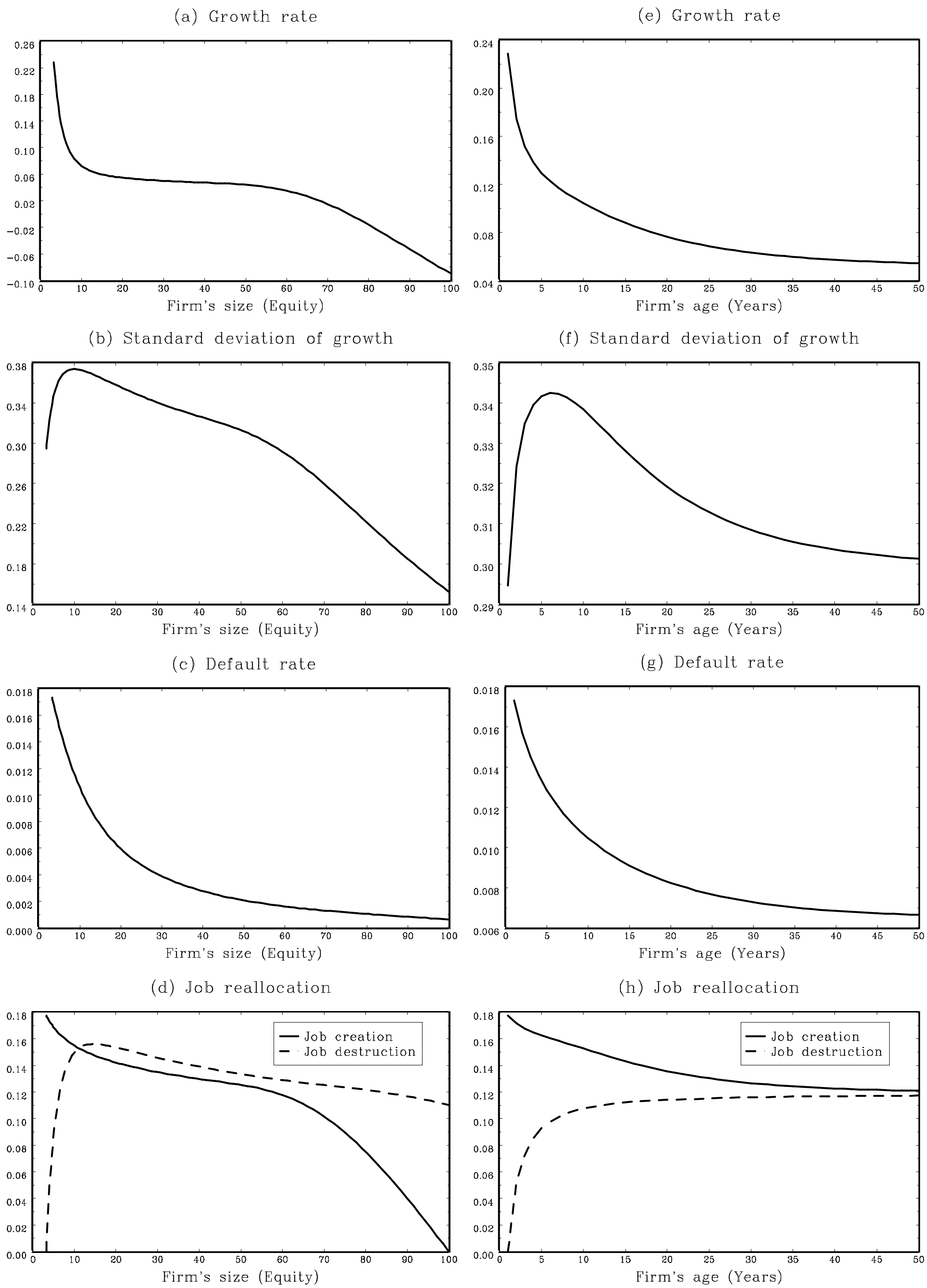

Figure 6. Financial behavior and industry dynamics for low and high productivity firms.

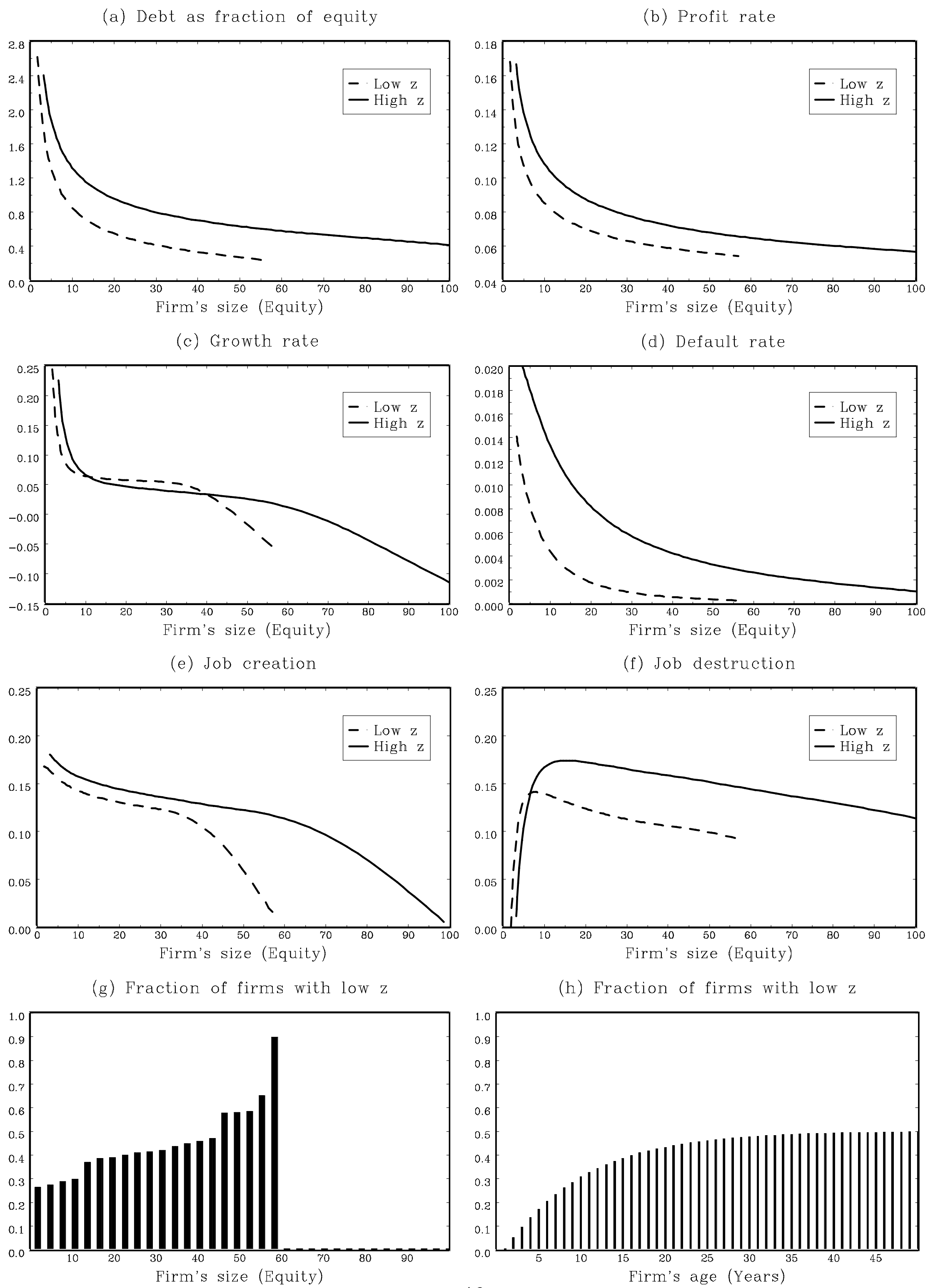

Figure 7. Industry dynamics conditional on age and size.