The Value of Personalized Pricing

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Increased availability of high-quality customer information has fueled interest in personalized pricing strategies, i.e., strategies that predict an individual customer's valuation for a product and then offer a price tailored to that customer. While the appeal of personalized pricing is clear, it may also incur large costs in the form of market research, investment in information technology and analytics expertise, and branding risks. In light of these trade-offs, our work studies the value of personalized pricing strategies over a simple single price strategy.

We first provide closed-form lower and upper bounds on the ratio between the profits of an idealized personalized pricing strategy (first-degree price discrimination) and a single price strategy. Our bounds depend on simple statistics of the valuation distribution and shed light on the types of markets for which personalized pricing has little or significant potential value. Second, we consider a feature-based pricing model where customer valuations can be estimated from observed features. We show how to transform our aforementioned bounds into lower and upper bounds on the value of feature-based pricing over single pricing depending on the degree to which the features are informative for the valuation. Finally, we demonstrate how to obtain sharper bounds by incorporating additional information about the valuation distribution (moments or shape constraints) by solving tractable linear optimization problems.

Key words: price discrimination, personalization, market segmentation

1. Introduction

Over the last decade, increased availability of customer information has fueled interest in personalized pricing strategies. At a high-level, these strategies combine customer data with machine learning and optimization tools to predict an individual customer's willingness to pay and then customize a price for that customer. This customized price is often delivered as a discount to a universal, posted price via a mobile application or other channel.

The appeal of personalized pricing is clear – If a seller could accurately predict individual customer valuations, then it could (in principle) charge each customer exactly their valuation, increasing profits and market penetration. Given this appeal, grocery chains (Clifford 2012), department stores (D'Innocenzio 2017), airlines (Tuttle 2013), and many other industries (Obama 2016) have begun experimenting with personalized pricing. Moreover, within the operations community, there has been a surge in research on how to practically and effectively implement personalized pricing strategies (e.g., Aydin and Ziya (2009), Phillips (2013), Bernstein et al. (2015), Chen et al. (2015), Ban and Keskin (2017)).

Unfortunately, implementing any form of price discrimination, including personalized pricing, may be costly and/or difficult. A firm would need to engage in price experimentation and market research, invest in information systems to store customer data, and build analytics expertise to transform these data into a personalized pricing strategy (see Arora et al. (2008) for an extensive discussion). Moreover, price discrimination tactics involve serious branding risks and potential customer ill-will, and, in some markets, may be of questionable legality. Finally, personalized pricing may impact competitors' (Zhang 2011) and manufacturers' (Liu and Zhang 2006) behavior.

In light of these tradeoffs, in this work we complement the existing operations literature on *how* to implement personalized pricing by quantifying *when* personalized pricing offers significant value. Specifically, for a single-product monopolist, we provide various upper and lower bounds on the profit ratio between personalized pricing and a simple single price strategy. We consider two different strategies: *(i) idealized personalized pricing (PP)*, i.e., charging each customer exactly their willingness to pay, and *(ii) feature-based personalized pricing (XP)*, i.e., charging each customer a price based on their observed feature data. For both personalization strategies, we benchmark the profit against the simple *single price (SP)* strategy that offers one price uniformly to all customers. The bounds we develop on the profit ratios between personalized pricing and single pricing can guide managers in assessing the upside of personalized pricing in potential markets. For example, in settings where an upper bound is close to one, we know that *any* form of price discrimination necessarily has limited value, while in settings where a lower bound is far from one, we are guaranteed the value of personalized pricing is significant.

With full-information about the customer valuation distribution, computing the exact ratio between personalized pricing over single pricing is straightforward; there is no need for bounding. However, in our opinion, a firm not currently engaging in personalized pricing is unlikely to know the full valuation distribution. Indeed, it is not necessary to learn this distribution to price effectively (Besbes et al. 2010, Besbes and Zeevi 2015) and learning it may be difficult since real-world distributions are typically complex and irregular (see, e.g., Celis et al. (2014) for a discussion in an auction setting).

Consequently, we focus instead on parametric bounds that depend on a few statistics of the valuation distribution. On the one hand, we believe these statistics are more easily estimated by a seller not currently engaging in personalized pricing than the full valuation distribution. For example, in data-poor settings, managers may be able to estimate simple statistics such as the mean based on domain knowledge or comparable products, but may find it impossible to accurately specify an entire distribution. Even in data-rich settings, no non-parametric density estimator using n data points converges in mean-integrated squared error (MISE) at a rate faster than $O(n^{-4/5})$, while a simple sample moment converges to its true moment in mean-squared error at a rate of $O(n^{-1})$ Van der Vaart (2000, Chapt. 24). On the other hand, and perhaps more importantly, parametric bounds based on these statistics provide structural insights into the types of markets where the value of personalized pricing is potentially large or minimal. These structural insights can guide practitioners weighing the benefits of price discrimination for a particular market against the aforementioned drawbacks.

More specifically, in the first part of the paper, we prove upper and lower bounds on the profit ratio between idealized personalized pricing and single pricing. Notice that idealized personalized pricing as we define it is often called first-degree price discrimination in the economics literature, and observe that it upper bounds the profit of any other price discrimination strategy. We prove upper and lower bounds that are tight, closed-form, and depend on simple properties of the valuation distribution. Specifically, our upper bounds depend on three unit-less statistics of the valuation distribution: (i) the *scale*, which is the ratio of the upper bound of the support to the mean, (ii) the *margin*, which we define as the margin of a unit sold at a price equal to the mean valuation, and (iii) the *coefficient of deviation*, which is the mean absolute deviation over twice the mean. Knowing these three quantities is equivalent to knowing the mean, support, and mean absolute deviation of the distribution. Our upper bounds are tight in the sense that we give an explicit valuation distribution for which the value of personalized pricing over single-pricing matches the bound. The precise form of the tight distribution depends on the relevant parameters, but consists of a mixture of Pareto and two-point distributions. Perhaps surprisingly, we also find that our upper bound is maximal for intermediate values of the coefficient of deviation and approaches one as the coefficient deviation increases with all other parameters fixed.

We complement our upper bounds with lower bounds that depend on the coefficient of deviation and mild shape assumptions on the valuation distribution such as i) unimodality or ii) unimodality and symmetry. We also show that without any shape assumptions, no non-trivial lower bound is theoretically possible. To the best of our knowledge, our lower bounds yield the first provable separation between personalized pricing and single price strategies for a generic class of distributions. Indeed, our lower bounds provide precise conditions for when increased heterogeneity in the market guarantees increased value in personalized pricing. Together our bounds yield strong conditions for identifying which markets are ripe for personalized pricing and which are well-served by a single price.

Idealized personalized pricing is not implementable in practice as it assumes the monopolist can perfectly predict each customer's valuation. Hence, we also study an alternate pricing strategy that we call feature-based pricing, where the seller observes a feature vector (sometimes called a context) for each customer which the seller can use to (imperfectly) predict the customer's valuation and offer a custom price. This strategy more closely resembles price discrimination strategies implemented in practice. We prove a theorem that relates lower and upper bounds on the profit ratio of feature-based pricing over single pricing to the profit ratio of idealized personalized pricing over single pricing (discussed above). The relationship between these two ratios is driven by the degree to which the observable contexts are informative for the unknown customer valuation, as measured by the size of the residual error when predicting valuations. More specifically, our bounds depend on the mean absolute deviation of this residual error. Our bounds make precise the intuition that when the contexts are very informative, feature-based pricing performs comparably to first-degree price discrimination, but when contexts are uninformative, feature-based pricing offers little benefit over single-pricing. Moreover, our bounds show how one can decompose the value of feature-based pricing strategies into the potential benefits of perfect personalization and the losses from less than perfectly informative features.

In the last part of our paper, we then show how to generalize our work to other moments besides the coefficient of deviation. Specifically, we provide an algorithmic procedure to compute essentially tight bounds on the value of idealized personalized pricing over single pricing given any generalized moment of the valuation distribution, such as the variance or quantile information. The key ideas leverage infinite-dimensional linear optimization duality and a careful discretization argument to generate a tractable optimization formulation suitable for off-the-shelf software. We show that when using variance (coefficient of variation), our bounds have the same insights and structure as the ones derived in closed-form for the case of coefficient of deviation.

We summarize our contributions below:

1. We prove closed-form, tight upper bounds for the value of idealized personalized pricing over single-pricing when the scale, margin, and coefficient of deviation of the valuation distribution are known (cf. Theorems 1 and 2). When these upper bounds are small, this suggests the value of any personalized pricing strategy is rather limited.
2. We prove closed-form lower bounds on the value of idealized personalized pricing that rely on necessary shape assumptions such as unimodality or unimodality and symmetry (cf. Theorem 3). In the latter case, our bound is tight for any specified coefficient of deviation. Our lower bounds provide guarantees on how much increased value personalized pricing can provide as a function of the market heterogeneity.
3. We then consider the more practical feature-based pricing, and generate lower and upper bounds on its value in comparison to the ideal case and single pricing (cf. Theorems 4 and 5). These bounds make explicit the relationship between the informational value of the features, and the value of feature-based pricing in a market. The proof fundamentally utilizes the previously derived bounds from the ideal case.
4. Finally, we provide a general methodology for computing essentially tight upper and lower bounds on the value of personalized pricing over single pricing when additional or different moment information is known about the valuation distribution. Our methodology also allows for shape assumptions such as unimodality without losing computational tractability (cf. Theorems 6, 7, and 8).

In the interest of reproducibility, open-source code for computing all of our bounds and reproducing all of our plots is available at BLINDED FOR REVIEW.

1.1. Connections to Existing Literature

The study of price discrimination tactics has a long history in economics dating back at least to Robinson (1934). Historically, the economics literature has focused on how various forms of price discrimination affect social welfare (see, e.g., Narasimhan (1984), Schmalensee (1981), Varian (1985), Shih et al. (1988) or Bergemann et al. (2015), Cowan (2016), Xu and Dukes (2016) for more recent results). In contrast to these works, we take an operational perspective, focusing on the individual firms relative profits under first-degree price discrimination and other forms of pricing.

Previous authors have also studied the value of personalized pricing over single pricing under different distributional assumptions. Barlow et al. (1963) prove that if the valuation distribution has monotone hazard rates (MHR), the value of personalized pricing is at most $e \approx 2.718$. In experiments, we show this bound is generally loose even when the assumption is satisfied (c.f. Fig. 2). Tamuz (2013) shows that if the ratio of the geometric mean over the mean of the valuation distribution is at least $1-\delta$, then the value of personalized pricing is at most $(1-2^{\frac{4}{3}}\delta^{\frac{1}{3}})^{-1}$, while Medina and Vassilvitskii (2017), shows the value of personalized pricing over single pricing is at most $4.78 + 2\log(1+C^2)$, where C is the coefficient of variation of the valuation distribution. These two bounds are not tight in dependence on δ and C, respectively. By contrast, our analogous upper bounds rely on coefficient of deviation and are proven to be tight for all possible values. We also stress that these existing results all pertain to *upper* bounds on the value of personalized pricing. To the best of our knowledge, we are the first to develop lower bounds for the value of personalized pricing over single-pricing and the first to develop bounds on the value of feature-based pricing over single-pricing.

As mentioned above, idealized personalized pricing (first-degree price discrimination) is an idealized strategy. In practice, firms implement some form of third-degree price discrimination such as the feature-based pricing strategy we consider. Indeed, the operations literature contains many examples of (implicit or explicit) third-degree price discrimination strategies including intertemporal pricing (Su (2007), Besbes and Lobel (2015)), opaque selling (Jerath et al. (2010), Elmachtoub and Hamilton (2017)), rebates/promotions (Chen et al. (2005), Cohen et al. (2017)), markdown optimization (Caro and Gallien (2012), Özer and Zheng (2015)), product differentiation (Moorthy (1984), Choudhary et al. (2005)), dynamic pricing and learning (Cohen et al. (2016), Qiang and Bayati (2016), Javanmard and Nazerzadeh (2016)), and many others.

By contrast, the focus of our work is not on "how to price discriminate" but rather the value of price discrimination. Our results shed insight into on when the value of such price discrimination tactics may be high and worth pursuing, and when the value may be low and not worthwhile. Huang et al. (2019) also studies the value of personalized pricing, but in a social network. There, all customers are identical except for their position in the network, and the proven bounds are asymptotic in the size of the (random) graph.

Finally, we contrast our work to several recent works that study how to set a single-price near-optimally given limited distribution information such as the support (Cohen et al. 2015), mean and variance (Chen et al. 2017, Azar et al. 2013), or a neighborhood containing the true valuation distribution (Bergemann and Schlag 2011). Indeed, these works support our earlier claim that it is not generally necessary to learn the whole valuation distribution in order to price effectively, but are very different in perspective from our work.

2. Model and Preliminaries

We consider a profit-maximizing monopolist selling a product with per unit cost c. A random customer's valuation for the product is denoted by the non-negative random variable $V \sim F$. The mean valuation $\mathbb{E}[V]$ is denoted by μ. For convenience we shall assume V has at most countably many point masses. We shall also define $\overline{F}(p) := \mathbb{P}(V \geq p)$, which is the probability that a customer shall purchase a product if priced at p. [1] Since it is never profitable to sell to customers with valuations less than c, assume without loss of generality, that $V \geq c$ almost surely. We consider a spectrum of three pricing strategies for the monopolist:

1) Single Pricing (SP): In the single pricing strategy, the monopolist offers the product to all customers at the same price p. Thus, the probability that a customer purchases is given by $\overline{F}(p)$, and the seller's corresponding expected profit is $(p-c)\overline{F}(p)$. Let $\mathcal{R}_{SP}(F,c) := \max_p\{(p-c)\overline{F}(p)\}$ denote the seller's maximal expected profit under single-pricing.

2) Feature-Based Pricing (XP): In the feature-based pricing strategy, the monopolist observes a feature vector $\mathbf{X}$ for each customer before offering a price, but does *not* directly observe their valuation V. Based on $\mathbf{X}$, the seller offers a customized price $p(\mathbf{X})$, and the customer purchases with probability $\mathbb{P}(V \geq p(\mathbf{X}) \mid \mathbf{X})$. Given a joint distribution $F_{\mathbf{X}V}$ of $(\mathbf{X}, V)$, let $\mathcal{R}_{XP}(F_{\mathbf{X}V}, c) := \max_{p(\cdot)} \mathbb{E}\left[(p(\mathbf{X}) - c)\mathbb{I}(V \geq p(\mathbf{X}))\right]$ denote the optimal profit under feature-based pricing.

3) Idealized Personalized Pricing (PP): In the idealized personalized pricing strategy, the monopolist can potentially offer a different price to each customer and has full knowledge of each customer's valuation. Since $V \geq c$, it is optimal to offer each customer precisely

[1] It is traditional to assume that if a customer values a product exactly at the price, then a purchase is made. $\overline{F}(\cdot)$ thus includes the $\mathbb{P}(V = p)$, and is not the complementary CDF of V. Note however that since V has countable many point masses, that $\int_{x_1}^{x_2} \overline{F}(t)dt = \int_{x_1}^{x_2} \mathbb{P}(V > t)dt$ for any $x_1 < x_2$.

their valuation and, thus, the total revenue earned is $\mathbb{E}[V] = \mu$. Let $\mathcal{R}_{PP}(F, c) := \mu - c$ denote the seller's maximal expected profit under idealized personalized pricing.

By construction, $\mathcal{R}_{SP}(F, c) \leq \mathcal{R}_{XP}(F_{\mathbf{X}V}, c) \leq \mathcal{R}_{PP}(F, c)$. Given F and c, we define the *value of idealized personalized pricing over single-pricing* as $\frac{\mathcal{R}_{PP}(F,c)}{\mathcal{R}_{SP}(F,c)}$. The value of feature-based pricing over single-pricing is defined similarly. When F, $F_{\mathbf{X}V}$, and c are clear from context, we sometimes omit them and write, e.g., $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}$.

2.1. The Lambert-W Function

Many of our closed-form bounds involve $W_{-1}(\cdot)$, the negative branch of the *Lambert-W* function. Although the Lambert-W function is pervasive in mathematics, it is somewhat less common in the pricing literature. We refer the reader to Corless et al. (1993) for a thorough review of its properties and provide a brief overview in Section A.

3. The Value of Idealized Personalized Pricing over Single Pricing

In this section, we provide upper and lower bounds on the value of idealized personalized pricing over single pricing using simple statistics and/or shape assumptions of the valuation distribution F. The statistics we shall consider are *scale* (S), *margin* (M), and *coefficient of deviation* (D) defined respectively as

$$S := \frac{\inf\{k \mid F(k) = 1\}}{\mu}, \quad M := 1 - \frac{c}{\mu}, \quad D := \frac{\mathbb{E}[|V - \mu|]}{2\mu}.$$

These three statistics are unit-less and can be thought of as (rescaled) measurements of the maximal valuation, per unit cost, and mean absolute deviation. More specifically, S is the ratio of the largest valuation in the market to the average valuation. By construction, $S \geq 1$, and measures the maximal dispersion of valuations. We stress that S might be infinite when valuations are unbounded, and, indeed, all of our closed-form bounds below will still be valid in this setting. By contrast, $M = \frac{\mu - c}{\mu} \in [0, 1]$, and can be interpreted as the margin of a unit sold at a price equal to the mean valuation. Finally, by construction, $D \in [0, 1]$ since $\mathbb{E}[|V - \mu|] \leq E[|V|] + \mu = 2\mu$ by the triangle inequality. Note D is the (rescaled) mean absolute deviation of V. Mean absolute deviation (MAD) is a common measure of a random variable's dispersion, similar to standard deviation. Intuitively, D measures the overall level of heterogeneity in the market.

Next, we introduce a transformation that reduces the problem of bounding the value of personalization for a product with $c > 0$ and $\mu > 0$ to an equivalent problem with $c = 0$ and $\mu = 1$. This reduction is used repeatedly throughout the paper.

LEMMA 1 **(Reduction to Zero Costs and Unit Mean)**. *Let $V \sim F$, and let the distribution of $V_c := \frac{1}{\mu - c}(V - c)$ be denoted by F_c. Then,*

$$\frac{\mathcal{R}_{PP}(F,c)}{\mathcal{R}_{SP}(F,c)} = \frac{\mathcal{R}_{PP}(F_c,0)}{\mathcal{R}_{SP}(F_c,0)}.$$

Moreover, if the scale, margin, and coefficient of deviation of F are S, M and D, respectively, then the mean, scale, margin, and coefficient of deviation of F_c (with no marginal cost) are $\mu_c = 1$, $S_c = \frac{S+M-1}{M}$, $M_c = 1$, and $D_c = \frac{D}{M}$, respectively.

We sometimes refer to $V_c \sim F_c$ as the standardized valuation distribution.

3.1. A First Upper Bound

We begin by first providing an upper bound on $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}$ using only the scale S and margin M. The key to the bound is that $\mathcal{R}_{SP}(F,0)$ directly yields a bound on the tail behavior of F. Indeed, for any price $p > 0$, $p\overline{F}(p) \le \mathcal{R}_{SP}(F,0)$ by definition, and thus $\overline{F}(p) \le \mathcal{R}_{SP}(F,0)/p$. We use this result repeatedly below, terming it the *pricing inequality*:

$$\overline{F}(x) \le \frac{\mathcal{R}_{SP}(F,0)}{x}, \quad \forall x > 0. \qquad \text{(Pricing Inequality)}$$

This inequality drives Theorem 1 below.

THEOREM 1 **(Upper Bounding $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}$ using S and M)**. *For any F with scale $S > 1$ and margin $M > 0$, we have*

$$\frac{\mathcal{R}_{PP}(F,c)}{\mathcal{R}_{SP}(F,c)} \le -W_{-1}\left(\frac{-M}{e(S+M-1)}\right).$$

Moreover, this bound is tight.

Proof. First, suppose $c = 0$ and $\mu = 1$. Then, $\mathcal{R}_{PP} = 1$ and $M = 1$. Since $\mu = 1$, $\overline{F}(S) = 0$, i.e., $0 \le V \le S$, a.s. Using the tail integral formula for expectation, we have that

$$\begin{aligned}
\mathcal{R}_{PP} &= \int_0^S \overline{F}(x)dx && (1)\\
&\le \mathcal{R}_{SP} + \int_{\mathcal{R}_{SP}}^S \overline{F}(x)dx && (0 \le \mathcal{R}_{SP} \le S) \quad (2)\\
&\le \mathcal{R}_{SP} + \int_{\mathcal{R}_{SP}}^S \frac{\mathcal{R}_{SP}}{x}dx && \text{(Pricing Inequality)} \quad (3)\\
&= \mathcal{R}_{SP} + \mathcal{R}_{SP}\log\left(S\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}\right) && (\text{since } \mathcal{R}_{PP} = 1).
\end{aligned}$$

Rearranging this inequality yields

$$\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \leq 1+\log\left(S\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}\right). \tag{4}$$

We next use properties of $W_{-1}(\cdot)$ to simplify Eq. (4). Exponentiating both sides yields,

$$e^{\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}} \leq eS\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \quad\Longleftrightarrow\quad \frac{1}{eS} \leq \frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}e^{-\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}} \quad\Longleftrightarrow\quad \frac{-1}{eS} \geq -\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}e^{-\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}} \tag{5}$$

Since $\frac{-1}{eS} \in [-1/e, 0)$ and the function $W_{-1}(\cdot)$ is non-increasing on this range, applying it to both sides of (5) and multiplying by -1 yields

$$\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \leq -W_{-1}\left(\frac{-1}{eS}\right), \tag{6}$$

which proves the bound when $c=0$ and $\mu=1$, since $M=1$.

To prove tightness, it suffices to construct a nonnegative random variable $V \sim F$ with $\mu=1$ and scale S, such that $\mathcal{R}_{SP}(F,0) = \frac{-1}{W_{-1}\left(\frac{-1}{eS}\right)}$. For convenience, define $\alpha = \frac{-1}{W_{-1}\left(\frac{-1}{eS}\right)}$, and notice, by definition of $W_{-1}(\cdot)$,

$$-\frac{1}{Se} = -\frac{1}{\alpha}e^{-\frac{1}{\alpha}} \quad\Longleftrightarrow\quad \frac{\alpha}{S} = e^{1-\frac{1}{\alpha}} \quad\Longleftrightarrow\quad \log\left(\frac{\alpha}{S}\right) = 1-\frac{1}{\alpha} \quad\Longleftrightarrow\quad \frac{1}{\alpha} = 1+\log\left(\frac{S}{\alpha}\right). \tag{7}$$

Next consider a random variable with

$$\overline{F}_S(x) = \begin{cases} 1 \text{ if } x \in (0,\alpha], \\ \frac{\alpha}{x} \text{ if } x \in (\alpha, S], \; 0 \text{ otherwise.} \end{cases}$$

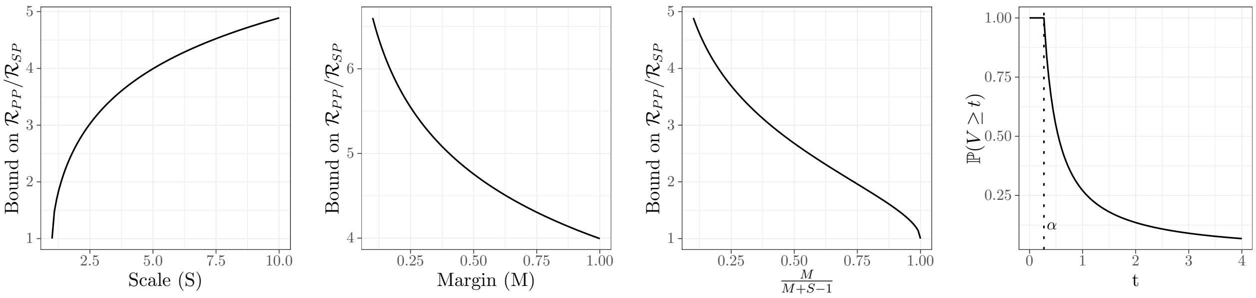
Observe that F_S has mean 1, since

$$\mu = \int_0^S \overline{F}_S(x)dx = \alpha + \alpha\log\left(\frac{S}{\alpha}\right) = \alpha\left(1+\log\left(\frac{S}{\alpha}\right)\right) = 1,$$

by Eq. (7). By inspection, F_S has scale S. Finally, for any $x \in (\alpha, S]$, $x\overline{F}_S(x) = \alpha$, and for any other x, $x\overline{F}_S(x) \leq \alpha$. Hence, $\mathcal{R}_{SP}(F,0) = \alpha$, and, thus, the bound is tight for F_S.

For a general $c>0$ and $\mu \neq 1$, use Lemma 1 to transform to a standardized valuation distribution with $c=0$, $\mu_c=1$, $M_c=1$, and $S_c = \frac{S+M-1}{M}$. Lemma 1 and Eq. (6) then imply that $\frac{\mathcal{R}_{PP}(F,c)}{\mathcal{R}_{SP}(F,c)} = \frac{\mathcal{R}_{PP}(F_c,0)}{\mathcal{R}_{SP}(F_c,0)} \leq -W_{-1}\left(\frac{-1}{eS_c}\right)$. Replacing S_c proves the upper bound. Create a tight distribution by scaling F_{S_c} (defined above) by $\mu-c$ and shifting by c. □

Figure 1 Bounds and Tight Distribution from Theorem 1.



Note. The first panel shows the bound in Thm. 1 when $M = 1$ and as S varies from 1 and 10. The second panel shows the bound in Thm. 1 when $S = 5$ and as M varies from 0.1 and 1.0. The third panel shows the bound in Thm. 1 as $\frac{M}{M+S-1}$ varies from 0.1 and 1.0. The fourth panel shows the tight distribution of Thm. 1 when $M = 1$ and $S = 5$.

The described tight distribution in the proof is a truncated Pareto distribution on $[\alpha, S]$ for some $\alpha \in [c, S]$, which satisfies $\overline{F}_S(x) \propto 1/x$ on its support (see rightmost panel of Fig. 1). In the auction literature, this distribution is sometimes called the "equal-revenue" distribution, since all prices in $[\alpha, S]$ yield the same single-pricing profit. Thus, one optimal pricing strategy for this distribution is to price at $p = \alpha$ and sell to *all* customers.

In the first three panels of Figure 1, we plot the bound of Theorem 1 as a function of S, M, and the fraction $\frac{M}{S+M-1}$, since the bound only depends on this ratio. Intuitively, as the scale increases, valuations become more dispersed and personalization offers greater potential value, as seen in the first panel. On the other hand, increasing the margin with a fixed mean is equivalent to decreasing the cost per unit. As discussed above, under the tight distribution, an optimal single-pricing strategy is to price at $p = \alpha$, which has the same market share as idealized personalized pricing. Thus, in the second panel, as margin increases, the profits of both idealized personalized pricing and single pricing increase at the same rate, and their relative ratio decreases. We stress that this behavior crucially depends on the properties of the tight distribution.

Remark 1. Many of our subsequent proofs utilize techniques similar to the proof of Theorem 1. Consequently, we highlight some of its high-level features before proceeding. First, the proof is centered around an integral representation of a moment of V (in this case μ) in terms of $\overline{F}$ (cf. Eq. (1)). The key step is to point-wise upper bound $\overline{F}(x)$ at each x. For $x \leq \mathcal{R}_{SP}$, the tightest bound possible is simply 1 (cf. Eq. (2)). For $x \geq \mathcal{R}_{SP}$, we use the Pricing Inequality (cf. Eq. (3)). The tight distribution is constructed by constructing a *valid* CDF that *simultaneously* makes each of these point-wise bounds tight. The remaining steps are simple algebraic manipulation. Thus, the three key elements are an integral representation in terms of the cCDF, point-wise bounds on the cCDF, and identifying a single distribution which simultaneously matches all point-wise bounds. □

3.2. Upper Bound Incorporating the Coefficient of Deviation

A drawback of Theorem 1 is that the bound becomes vacuous as the scale $S \to \infty$. The issue is that S, alone, cannot distinguish between markets where most customers have relatively similar valuations (which may be relatively low or high) and markets where customer valuations vary widely. We next provide sharper upper bounds on the value of idealized personalized pricing by incorporating a measure of the market's heterogeneity, i.e., the coefficient of deviation D.

Intuitively, when D is small, we expect most valuations to be close to μ, and, hence, the value of personalization to be small. By contrast, when D is large, we expect larger dispersion in valuations, and, hence, the potential value of personalization to be larger.

This intuition is not entirely correct as we shall see below. In fact, when D is *very* large and S is finite, there is a boundary effect; F is approximately a two-point distribution concentrated near c and μS, and single-pricing strategies are very effective. A single price can be used to capture the high valuation customers, while the low valuation customers are simply ignored since their potential profitability is near zero. Consequently, for very large D, the value of personalization is, in fact, low.

This qualitative description is formalized in Theorem 2 which upper bounds the value of personalization in terms of S, M, and D. The theorem partitions the space of markets into three distinct regimes depending on the magnitude of D and provides distinct bounds for each regime. Specifically, we define the three regimes by

(L) Low Heterogeneity: $0 \le D \le \delta_L$

(M) Medium Heterogeneity: $\delta_L \le D \le \delta_M$

(H) High Heterogeneity: $\delta_M \le D \le \delta_H$,

where $\delta_L, \delta_M, \delta_H$ are constants that depend on M and S:

$$\delta_L := -\frac{M \log\left(\frac{S+M-1}{M}\right)}{W_{-1}\left(\frac{-1}{e^{\frac{S+M-1}{M}}}\right)}, \quad \delta_M := \frac{M \log\left(\frac{S+M-1}{M}\right)}{1+\log\left(\frac{S+M-1}{M}\right)}, \quad \delta_H := \frac{M(S-1)}{S+M-1}.$$

The following lemma states that these regimes form a true partition and is proved in Section B.3 of the appendix.

Lemma 2 **(Partitioning the Range of D).** *Given F with scale S and margin M, the coefficient of deviation of F satisfies $0 \le D \le \delta_H$. Moreover, $0 \le \delta_L \le \delta_M \le \delta_H$.*

Equipped with Lemma 2, we can state Theorem 2, the main upper bound of this section.

Theorem 2 **(Upper Bounding $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}$ using S, M, and D).** *For any F with scale $S > 1$, margin $M > 0$, and coefficient of deviation D, we have the following:*

a) If $0 \le D \le \delta_L$, then

$$\frac{\mathcal{R}_{PP}(F,c)}{\mathcal{R}_{SP}(F,c)} \le \frac{-W_{-1}\left(\frac{\frac{D}{M}-1}{e}\right)}{1-\frac{D}{M}}. \qquad \text{(Low Heterogeneity)}$$

b) If $\delta_L \le D \le \delta_M$, then

$$\frac{\mathcal{R}_{PP}(F,c)}{\mathcal{R}_{SP}(F,c)} \le \frac{M \log\left(\frac{S+M-1}{M}\right)}{D}. \qquad \text{(Medium Heterogeneity)}$$

c) If $\delta_M \le D \le \delta_H$, then

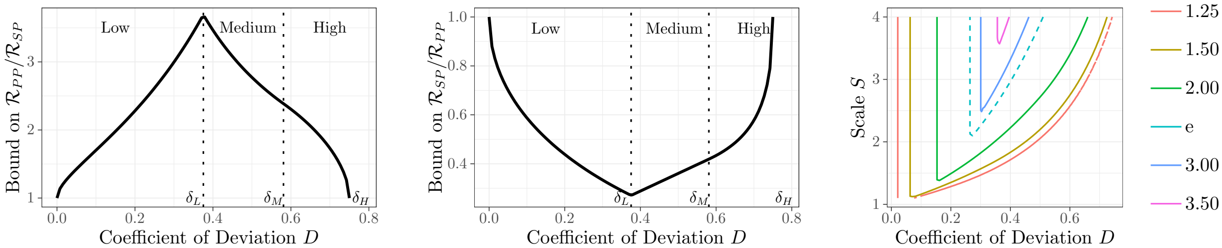
$$\frac{\mathcal{R}_{PP}(F,c)}{\mathcal{R}_{SP}(F,c)} \le -W_{-1}\left(\frac{-1}{e(\frac{S+M-1}{M})(1-\frac{D}{M})}\right). \qquad \text{(High Heterogeneity)}$$

Moreover, for any S, M, D there exists a valuation distribution F with scale S, margin M and coefficient of deviation D such that the corresponding bound is tight.

Theorem 2 gives a complete, closed-form upper bound on the value of personalized pricing for any distribution in terms of its scale, margin, and coefficient of deviation. The bound is defined piecewise, but is continuous (cf. Fig. 2). Note that the bound captures the intuition that the value of personalization increases as D increases for small to moderate D, but also captures the boundary behavior as D becomes very large. Recall that since $\mathcal{R}_{PP}$ upper bounds the value of *any* price-discrimination strategy, when D is either very small or very large and the bound is close to 1, Theorem 2 suggests that there is limited benefits to *any* price-discrimination strategy.

The maximal point in Fig. 2, at the transition between the low and medium regimes, corresponds to the bound in Theorem 1. Moreover, when S is infinity, $\delta_L = \delta_M = \delta_H = 1$ and the low heterogeneity bound (which does not depend on S) always pertains. Like Theorem 1, Theorem 2 is a tight bound. The distributions which achieve the bounds depends on the regime but is not unique. See Fig. EC.2 for typical examples and Lemma EC.3 in the appendix for explicit formulas.

Figure 2 Value of Idealized Personalized Pricing vs. the Coefficient of Deviation



Note. The left panel plots the bound from Theorem 2 as a function of D with $S=4$ and $M=1$. The middle panel plots the inverse of this bound, which we note is convex. The right panel shows Theorem 2 as a surface plot, where D ranges over $[0,1]$, and S ranges over $[1.1,4]$. The dashed contour is the uniform bound for MHR distributions, $e \approx 2.718$, from Barlow et al. (1963) and Hartline et al. (2008).

We also observe that our bound in the right panel of Figure 2 can be significantly above or below e, the uniform bound proven for monotone hazard rate (MHR) distributions in Barlow et al. (1963) and Hartline et al. (2008). In summary, although the value of personalized pricing can be large in some settings, our refined analysis based on D characterizes precisely markets which necessarily have a low value of personalized pricing.

Sketch of Proof of Theorem 2: The proof of Theorem 2 utilizes the same basic technique as in Theorem 1 and outlined in Remark 1, however instead of being centered around an integral representation of the mean, the proof is centered around two convenient representations of the coefficient of deviation. To that end, we now establish two integral representations of D in terms of $\overline{F}(x)$.

Lemma 3 **(Integral Representations of D).** *For any F with scale S and margin M, the coefficient of deviation D satisfies*

$$D = \int_{M}^{S+M-1} \overline{F}(\mu x + c)dx = \int_{0}^{M} 1 - \overline{F}(\mu x + c)dx. \tag{8}$$

The proof then proceeds separately for each regime. In the Low (Medium) Heterogeneity regime we start with the second (first) identity of Lemma 3 and proceed similarly to Remark 1. The High Heterogeneity bound is also derived in this way, starting with the second identity of Lemma 3 but using a different bounding of the cCDF which is tighter when D is large. For the full details of the proof see Section B.2.

Single-Pricing Guarantee: An alternative interpretation of Theorem 2 is that the reciprocal of the bound is a tight guarantee on the performance of single-pricing relative to idealized personalized-pricing. In other words, the single-pricing strategy is guaranteed to earn at least the given percentage of the idealized personalized pricing profits. This perspective, i.e., interpreting single-pricing as an approximation to idealized personalized pricing, is common in the approximation algorithm literature.

We plot this guarantee, i.e., the reciprocal of the bound in Theorem 2, in the middle panel of Fig. 2. Perhaps surprisingly, the reciprocal is convex as a function of D (our original function was neither convex nor concave). We prove this formally in Lemma EC.5.

Asymptotics: Finally, from a theoretical point of view, one might seek to characterize the value of personalized pricing as D approaches its extreme values $D \to 0$ or $D \to \delta_H$. In particular, we will see in Section 4.1 that the first limit also provides insight into the performance of certain third-degree price discrimination tactics. These limits are below:

Corollary 1 **(Asymptotic Behavior)**. *For any S, M, D, let $1/\alpha(D,M,S)$ denote the bound from Theorem 2. Then,*

a) As $D \to 0$,

$$\frac{1}{\alpha(S,M,D)} = 1 + \sqrt{2\frac{D}{M}} + O\left(\frac{D}{M}\right).$$

b) As $D \to \delta_H$,

$$\frac{1}{\alpha(S,M,D)} = 1 + \sqrt{2\frac{S+M-1}{M}} \cdot \sqrt{\delta_H - \frac{D}{M}} + O\left(\delta_H - \frac{D}{M}\right).$$

3.3. Lower Bounds on the Value of Personalized Pricing

In this subsection, we complement our upper bounds on the value of personalized pricing with closed-form lower bounds. Such lower bounds are helpful in identifying when personalized pricing strategies can guarantee increased revenues. Unfortunately, when only S, M, and D are given, only a vacuous lower bound exists, i.e., no lower bound strictly greater than 1 can be proven. Consider the following two-point distribution in Example 1 where $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} = 1$ for any S, M, and D.

Example 1 ($\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} = 1$ for a Two-Point Distribution). Given S, M, and D, recall that $D \le \frac{M(S-1)}{S+M-1}$ by Lemma 2. Define the two point distribution

$$V = \begin{cases} 1 - M & \text{with probability } \frac{D}{M} \\ \frac{D(1-M)-M}{D-M} & \text{with probability } 1 - \frac{D}{M} \end{cases}$$

and let F be the corresponding cdf. One can confirm directly that $\mathbb{E}[V] = 1$ and the coefficient of deviation of V is D. Furthermore, $D \le \frac{M(S-1)}{S+M-1}$ implies $S \ge \frac{D(1-M)-M}{D-M}$ so that the scale of V is at most S. Finally, one can confirm that pricing at $\frac{D(1-M)-M}{D-M}$ earns a profit of $M = 1 - c = \mathcal{R}_{PP}(F,c)$. Hence, $\frac{\mathcal{R}_{PP}(F,c)}{\mathcal{R}_{SP}(F,c)} = 1$. $\square$

To avoid these pathological two-point distributions, we require additional assumptions about the distribution's *shape*. We study two such assumptions below.

DEFINITION 1. A random variable V is *unimodal* with mode m if $\overline{F}(t) := \mathbb{P}(V \geq t)$ is a concave function on $(-\infty, m]$ and convex on (m, ∞).

DEFINITION 2. A random variable V is *symmetric* about point m if $\mathbb{P}(V \in [m-x, m]) = \mathbb{P}(V \in [m, m+x])$ for all $x \geq 0$.

These two definitions generalize the usual notions definitions of unimodality and symmetry for random variables that admit densities to allow for point masses.

We utilize these shape assumptions to prove non-trivial, parametric lower bounds on the value of personalized pricing over single-pricing in Theorem 3. These bounds yield strict separation between the revenue of idealized personalized pricing and a single price strategy for a general class of distributions based on the level of heterogeneity in the market. The bounds describe markets where one is guaranteed that personalized pricing improves upon single-pricing.

THEOREM 3 **(Lower Bounding $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}$ using D).** *Consider a valuation distribution $V \sim F$, with margin $M > 0$ and coefficient of deviation D.*

a) If V is unimodal and symmetric, then $\frac{\mathcal{R}_{PP}(F,c)}{\mathcal{R}_{SP}(F,c)} \geq \frac{1}{1-2\frac{D}{M}}$. Moreover, for every value of $\frac{D}{M}$ there exists a unimodal and symmetric distribution such that this bound is tight.

b) If V is unimodal and

- $0 \leq \frac{D}{M} \leq \frac{1}{3}$, *then* $\frac{\mathcal{R}_{PP}(F,c)}{\mathcal{R}_{SP}(F,c)} \geq \frac{1}{1-\frac{D}{M}}$.
- $\frac{1}{3} \leq \frac{D}{M} \leq 1$, *then* $\frac{\mathcal{R}_{PP}(F,c)}{\mathcal{R}_{SP}(F,c)} \geq \frac{8\frac{D}{M}}{\left(1+\frac{D}{M}\right)^2}$.

Moreover, if $\frac{D}{M} = 0$, this bound is tight, and, as $\frac{D}{M}$ tends to 1, there exists a family of unimodal valuation distributions such that this bound is asymptotically tight.

Theorem 3 gives optimal (near-optimal), closed-form lower bound on the value of personalized pricing for any symmetric & unimodal (unimodal) distribution in terms of its margin and coefficient of deviation. We prove part (a) of Theorem 3 below. The proof of part (b) is similar, and we relegate it to Appendix B.4 for brevity.

Proof of Theorem 3(a). First suppose $c = 0$ and $\mu = 1$. Symmetry implies that the mode equals μ (which equals 1) and $1 \leq S \leq 2$. Moreover, by Lemma EC.4 in Appendix B.4, we have $D \leq 0.25$ for any symmetric, unimodal distribution. Now, consider two cases based on the optimal single-price p^*.

Case 1: $p^* > 1$**.** Define the function $\overline{G}(x) = \overline{F}(x)$ for all $x \in (1,2]$, and $\overline{G}(1) := \lim_{t\downarrow 1} \overline{F}(t)$. These functions agree everywhere except perhaps at 1 if V has a point mass at 1. Moreover, by unimodality, $\overline{G}(x)$ is convex on $[1,2]$.

Next, by symmetry about 1, $G(1) = \lim_{t\downarrow 1} \overline{F}(t) \leq \frac{1}{2}$ and since $S \leq 2$, $\overline{G}(2) = \overline{F}(2) = 0$. In particular, this implies $p^* \leq 2$. Thus, writing p^* as a convex combination,

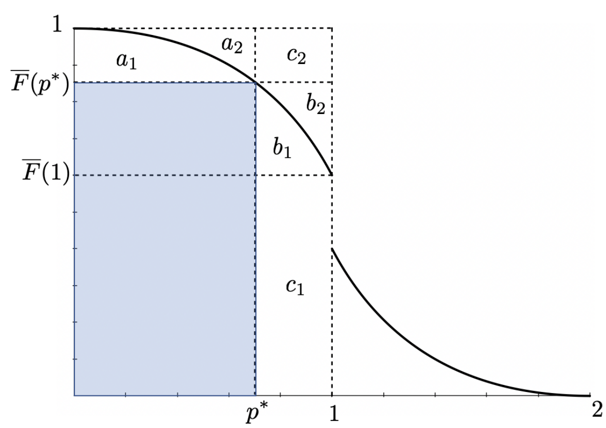
$$\overline{G}(p^*) \ = \ \overline{G}\big((2-p^*)\cdot 1 + (p^*-1)\cdot 2\big) \ \leq \ (2-p^*)\overline{G}(1) + (p^*-1)\overline{G}(2) \ \leq \ (2-p^*)\cdot\frac{1}{2}.$$

Hence,

$$\mathcal{R}_{SP} \ = \ p^*\overline{F}(p^*) \ = \ p^*\overline{G}(p^*) \ \leq \ p^*(2-p^*)/2 \ \leq \ \max_x \ x(2-x)/2 \ = \ \frac{1}{2}.$$

Finally, since $D \leq .25$, $\mathcal{R}_{SP} \leq 1/2 \leq 1-2D$, and $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \geq \frac{1}{1-2D}$.

Figure 3 Geometric Proof of Theorem 3(a).



Note. The revenue of a single pricing using price $p^* < 1$ (shaded rectangle) is depicted relative to the area under a symmetric unimodal cCDF (solid line). The proof relates this rectangle to the area of regions a_1, a_2, b_1, b_2, c_1, and c_2.

Case 2: $p^* \leq 1$**.** Referring to Fig. 3 note that $\mathcal{R}_{SP} = p^*\overline{F}(p^*)$ is the area of the shaded rectangle. Re-express this quantity as the area of the unit-square (dashed rectangle in figure) minus the area of the regions a_1, a_2, b_1, b_2, c_1, and c_2. Formally,

$$\mathcal{R}_{SP} = 1 - \text{Area}(a_1 \cup b_1 \cup c_1) - \text{Area}(a_2 \cup b_2 \cup c_2),$$

because the regions are disjoint.

Next, by unimodality, $\overline{F}$ is concave on $[0,1]$, hence $\text{Area}(a_1) \geq \text{Area}(a_2)$ and $\text{Area}(b_1) \geq \text{Area}(b_2)$. Moreover, by symmetry, $\lim_{t\uparrow 1} \overline{F}(t) \geq \frac{1}{2}$, hence $\text{Area}(c_1) \geq \text{Area}(c_2)$, and, in sum, $\text{Area}(a_1 \cup b_1 \cup c_1) \geq \text{Area}(a_2 \cup b_2 \cup c_2)$. Substituting above shows $\mathcal{R}_{SP} \leq 1 - 2\text{Area}(a_2 \cup$

$b_2 \cup c_2$). Finally, by Lemma 3, $D = 1 - \int_0^1 \overline{F}(x)dx$. Referring to Fig. 3, this shows that $D = \text{Area}(a_2 \cup b_2 \cup c_3)$. Substituting above shows $\mathcal{R}_{SP} \le 1 - 2D$, which implies $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \ge \frac{1}{1-2D}$.

To show the bound is tight we construct a distribution that is a mixture of a point mass at 1 and a uniform random variable on $[0, 2]$, namely,

$$V_0 = \begin{cases} 1 & \text{with probability } 1 - 4D \\ \text{Unif}[0,2] & \text{with probability } 4D. \end{cases}$$

By inspection, V_0 is symmetric, unimodal, satisfies $\mathbb{E}[V_0] = 1$ and $\frac{\mathbb{E}[|V_0 - 1|]}{2} = D$ and pricing at 1 earns revenue $1 - 2D$. Hence, for V_0, $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} = \frac{1}{1-2D}$. This completes the proof for standardized valuation distributions.

For general c and μ, apply Lemma 1 to transform to a standardized distribution $V_c \sim F_c$. From above, the value of personalized pricing for V_c is at least $\frac{1}{1-2D_c}$. Replace $D_c = D/M$ to prove the bound, and scale V_0 by $(\mu - c)$ and shift by c to form a tight distribution. $\square$

4. From First-Degree to Third-Degree Price Discrimination

As mentioned in the introduction, idealized personalized pricing is unachievable in practice. Here we study a more realistic form of personalized pricing termed feature-based pricing.

4.1. Feature-Based Pricing

In feature-based pricing, the seller predicts the customer valuation V from a set of observed customer features, $\mathbf{X}$. From a practical point of view, feature-based pricing approximates a host of third-degree price discrimination strategies in common use. For example, student discounts are a form of feature-based pricing where $\mathbf{X}$ is binary and indicates if the customer is a student. More generally, in online retailing settings, sellers often have access to rich contextual information for each customer from their cookies such as demographics, browsing history, etc., that can be used to personalize the offered price via a custom coupon. Clearly, if one can perfectly predict V from $\mathbf{X}$, feature-based pricing is equivalent to idealized personalized pricing. Typically, however, $\mathbf{X}$ is not rich enough to predict V perfectly, entailing some loss in profits.

Formally, let the random variable $\mu(\mathbf{X}) := \mathbb{E}[V \mid \mathbf{X}]$ and define the residual $\epsilon := V - \mathbb{E}[V \mid \mathbf{X}]$. By construction, $\mathbb{E}[\epsilon \mid \mathbf{X}] = 0$ almost surely, i.e., the noise term always has conditional mean 0. More importantly, when $\mathbf{X}$ is very informative for V, we expect ϵ to be "small".

In this sense, the size of ϵ measures the degree to which $\boldsymbol{X}$ can be used to predict V. Intuitively, one might think of ϵ as the residual in a non-parametric regression of V on $\boldsymbol{X}$.

A first, perhaps obvious, observation is that given $\mathbf{X}$, it is not optimal to price at $\mathbb{E}[V \big| \mathbf{X}]$. To the contrary, one should price at the optimal price for the conditional distribution $F_{V|\mathbf{X}}$. Thus, for any joint distribution $F_{\mathbf{X}V}$, we have

$$\mathcal{R}_{XP}(F_{\mathbf{X}V}, c) = \mathbb{E}[\mathcal{R}_{SP}(F_{V|\mathbf{X}}, c)]. \tag{9}$$

The main results of this section are bounds on the ratio between feature-based pricing ($\mathcal{R}_{XP}$) and a single pricing strategy ($\mathcal{R}_{SP}$) that depend explicitly on the degree to which $\boldsymbol{X}$ is informative for V as measured by the size of the residual ϵ (more specifically, $\frac{\mathbb{E}[|\epsilon|]}{2\mu}$). To this end, we first bound the ratio between $\mathcal{R}_{XP}$ and $\mathcal{R}_{PP}$ in terms of the magnitude of the residual noise ϵ. For convenience, we define $D_\epsilon := \frac{\mathbb{E}[|\epsilon|]}{2\mu}$.

Theorem 4 **(Feature-Based Pricing vs. Idealized Personalized Pricing)**. *Suppose that $V = \mu(\mathbf{X}) + \epsilon$ where the residual ϵ is independent of $\mathbf{X}$ and let $D_\epsilon = \frac{\mathbb{E}[|\epsilon|]}{2\mu}$.*

a) Then,

$$\frac{\mathcal{R}_{PP}(F, c)}{\mathcal{R}_{XP}(F_{\mathbf{X}V}, c)} \leq \frac{1}{\alpha\,(S, M, D_\epsilon)},$$

where $\alpha(S, M, D)$ denotes the reciprocal of the bound in Theorem 2.

b) If, additionally, ϵ is unimodal and symmetric, then,

$$\frac{\mathcal{R}_{PP}(F, c)}{\mathcal{R}_{XP}(F_{\mathbf{X}V}, c)} \geq \frac{1}{1 - \frac{2D_\epsilon}{M}}.$$

Notice that when $\boldsymbol{X}$ is very informative for V, D_ϵ is small, and thus the first part of Theorem 4 implies $\mathcal{R}_{PP}$ offers limited benefits over $\mathcal{R}_{XP}$. Correspondingly, when $\boldsymbol{X}$ does not contain much information about V, the second part guarantees (idealized) personalized pricing earns significantly more than feature-based pricing under some additional assumptions. As an example, we note that Gaussian noise is unimodal and symmetric, so that the second part of the theorem applies.

We leverage Theorem 4 to bound $\frac{\mathcal{R}_{XP}}{\mathcal{R}_{SP}} = \frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \cdot \frac{\mathcal{R}_{XP}}{\mathcal{R}_{PP}}$ by bounding the second term.

Theorem 5 **(Feature-Based Pricing vs. Single Pricing)**. *Suppose $V = \mu(\mathbf{X}) + \epsilon$ with ϵ independent of $\mathbf{X}$. Let $D_\epsilon = \frac{\mathbb{E}[|\epsilon|]}{2\mu}$.*

a) Then,

$$\frac{\mathcal{R}_{XP}(F, c)}{\mathcal{R}_{SP}(F, c)} \geq \frac{1 - \frac{D_\epsilon}{M}}{-W_{-1}\left(\frac{\frac{D_\epsilon}{M} - 1}{e}\right)} \cdot \frac{\mathcal{R}_{PP}(F, c)}{\mathcal{R}_{SP}(F, c)}.$$

b) If, additionally, ϵ is unimodal and symmetric, then

$$\frac{\mathcal{R}_{XP}(F,c)}{\mathcal{R}_{SP}(F,c)} \leq \left(1-\frac{2D_\epsilon}{M}\right)\cdot \frac{\mathcal{R}_{PP}(F,c)}{\mathcal{R}_{SP}(F,c)}.$$

The proof for Theorem 5 is immediate. Note we have used the (looser) low-heterogeneity bound of Theorem 2 in place of $\alpha(S,M,D_\epsilon)$. As noted in the proof of Theorem 2, this bound pertains to all D and is strongest when D is small. Since we expect one to be interested in feature-based pricing mostly in settings with relatively informative features $\mathbf{X}$, we state the bound with this simpler constant. Moreover, we have used Theorem 3(a) to form the upper bound which requires symmetry of ϵ. With minor modifications, one can instead use Theorem 3(b) which does not require symmetry but increases the constant beyond $\frac{1}{1-\frac{2D_\epsilon}{M}}$.

Intuitively, Theorem 5 decomposes the benefits of feature-based pricing into those stemming from pure price discrimination and those (losses) stemming from prediction error. From a theoretical point of view, this result highlights that the value of personalized pricing ($\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}$) is *the* fundamental mathematical quantity for study. Indeed, using Theorem 5, we can plug-in *any* bounds on $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}$ and obtain corresponding bounds on $\frac{\mathcal{R}_{XP}}{\mathcal{R}_{SP}}$. These include the bounds developed in Section 3 above and the bounds developed in Section 5 below. Although we focus on feature-based pricing in this paper, we also suspect that $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}$ may be a primitive "building block" when studying other forms price discrimination.

From a more practical point of view, Theorem 5 allows a seller who is currently using a single-pricing strategy and considering switching to a feature-based pricing strategy to assess the potential benefits of the switch. The key issue is the informativeness (as measured by D_ϵ) of the features $\boldsymbol{X}$ that the seller currently has or hopes to obtain. If these features are not sufficiently informative, the second part of the theorem shows there is little value to the switch. On the other hand, if one intends to collect additional features on the customers, Theorem 5 also indicates how informative those features must be to guarantee a desired fraction of idealized personalized-pricing profits. From Theorem 5(a), we see that to be guaranteed to halve the relative performance gap between personalized pricing and feature-based pricing, one needs to reduce the size of ϵ by a factor of 4. Loosely, this corresponds to collecting features $\boldsymbol{X}$ which allow one to predict V four times more accurately. [2]

[2] More specifically, using Corollary 1, we can rewrite Theorem 5(a)) as $\frac{\mathcal{R}_{XP}}{\mathcal{R}_{SP}} \geq \left(1-\sqrt{2D_\epsilon/M}+o(\sqrt{D_\epsilon/M})\right)\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}$, where we have used the fact that $(1+\sqrt{2D_\epsilon/M}+o(\sqrt{D_\epsilon/M}))^{-1} = 1-\sqrt{2D_\epsilon/M}+o(\sqrt{D_\epsilon/M})$. Rearranging shows $\frac{\mathcal{R}_{PP}-\mathcal{R}_{XP}}{\mathcal{R}_{PP}} \leq \sqrt{2D_\epsilon/M}+o(\sqrt{D_\epsilon})$. Hence reducing D_ϵ by a factor of 4 halves the relative performance gap.

5. Bounds Based upon General Moments

In Section 3 we developed upper and lower bounds for $\frac{\mathcal{R}_{PP}(F,c)}{\mathcal{R}_{SP}(F,c)}$ based upon the coefficient of deviation. Although the coefficient of deviation is amenable to closed-form analysis, bounds using other statistics are of interest. In this section we compute upper and lower bounds on the value of personalized pricing over single-pricing for other statistics while possibly imposing shape constraints (such as unimodality) on F. Via Theorem 5, these bounds can be transformed into bounds on the value of feature-based pricing over single-pricing.

Specifically, throughout the section we assume F satisfies a single moment constraint of the form $\mathbb{E}[h(V)] = \mu_h$ for some known, fixed function $h(\cdot)$ and constant μ_h. Examples include:

- **Coefficient of Deviation:** When $h(v) = \frac{|v-\mu|}{2\mu}$ and $\mu_h = D$, this constraint ensures the coefficient of deviation of F is D, generalizing our analysis from Section 2.
- **Coefficient of Variation:** When $h(v) = \frac{(v-\mu)^2}{\mu^2}$ and $\mu_h = C^2$, this constraint ensures the coefficient of variation of F is C.
- **Geometric Mean:** When $h(v) = -\log(v/\mu)$ and $\mu_h = -\log(B/\mu)$, this constraint ensures the geometric mean of F is B i.e., $\exp(\mathbb{E}[\log(V)]) = B$. As mentioned, the value of personalized pricing given the geometric mean has previously been studied (in a different context) by Tamuz (2013).
- **Incumbent Price:** When $h(v) = \mathbb{I}(v \geq \hat{p}\mu)$ and $\mu_h = q$, this constraint ensures that a fraction q of the market purchases at price $\hat{p}\mu$. Here, $\hat{p}\mu$ might represent an incumbent price that has been used historically.

The key idea of our approach is to formulate an optimization problem over probability measures to explicitly compute the value of personalized pricing. Similar ideas have been used to develop generalized Chebyshev inequalities (Bertsimas and Popescu (2005), Popescu (2005)).

Before delving into the details of our formulations, we summarize the main insights via a numerical example in Fig. 4. In each panel we compare upper (solid lines) and lower bounds (dashed lines) on the value of personalized pricing assuming no shape constraints (red lines), unimodality (green-dashed line), and unimodality with a mode at $m = 1$ (blue lines). The four panels correspond to the four examples of moment functions $h(\cdot)$ described above. In all panels, we take $S = 2$, $M = .9$, and $\mu = 1$. Since $S = 2$, the maximal deviation achievable by any unimodal distribution is only .25 (achieved by a uniform distribution),

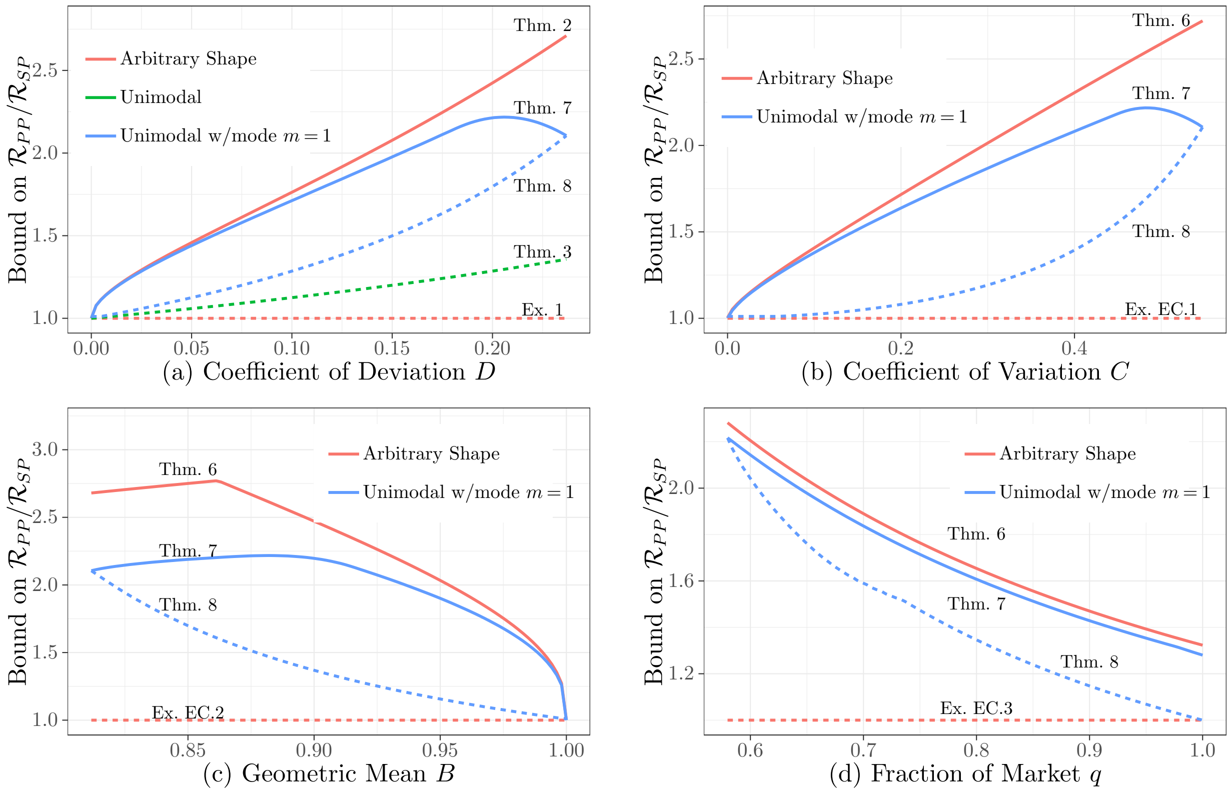


Figure 4 **Upper and Lower Bounds on $\mathcal{R}_{PP}/\mathcal{R}_{SP}$ Given Various Shape Constraints.** In all panels $\mu=1$, $S=2$ and $M=.9$. The incumbent price $\hat{p}\mu=.8$ in panel (d). We plot upper (solid lines) and lower (dashed lines) bounds assuming no shape constraints (red), unimodality (green) and unimodality with mode $m=1$ (blue). Bounds are annotated by their corresponding theorem. Panels plot bounds in terms of different possible moments of the valuation distribution.

not 1. Thus, we restrict the plot in first panel to this range. Similar restrictions apply to the other moment functions and other three panels. Panel (d) uses an incumbent price of $\hat{p}\mu=.8$.

Overall, as was seen in Section 3.3, enforcing shape constraints significantly strengthens the bounds, especially for intermediate values of heterogeneity. In the first panel, we have added the bound from Theorem 3 (b) for comparison. The gap between the "Unimodal" curve (green dashed line, computed from Theorem 3 (b)) and the "Unimodal ($m=1$)" (blue dashed line, computed from Theorem 8) arises because Theorem 3 holds for all scales S and possible locations of the mode, m while Theorem 8 is parameterized by S and m. All four panels show similar qualitative behavior.

We stress that these are only 4 examples of moment functions $h(\cdot)$. In Sections 5.2 to 5.4 below, we formulate generic optimization problems to compute bounds for *any* $h(\cdot)$. We believe these formulations provide a general framework for managers to assess the value of personalization under different a priori assumptions on valuations. Naturally, the computational complexity of these optimization problems hinges on the particular moment function $h(\cdot)$ and shape constraints. To streamline exposition, we defer all discussion of tractability until Section 5.5 where we also argue that the 4 examples above are tractable.

5.1. Reduction to Standardized Valuations

Notice that if $\mathbb{E}[h(V)] = \mu_h$, then $\mathbb{E}[\overline{h}(V_c)] = 0$ where $\overline{h}(t) := h(\mu M(t-1)+\mu) - \mu_h$ and V_c is the standardized valuation distribution of Lemma 1. Hence, to bound the value of personalized pricing with a moment constraint defined by h, it suffices to bound the value of personalized pricing for a standardized valuation distribution satisfying a moment constraint defined by a standardized function $\overline{h}$. For example, the corresponding standardized functions for our four examples above are: i) $\overline{h}(t) = M\,|t-1|\,/2 - D$ for the coefficient of deviation ii) $\overline{h}(t) = M^2(t-1)^2 - C^2$ for the coefficient of variation iii) $\overline{h}(t) = -\log(M(t-1)+1) + \log(B/\mu)$ for the geometric mean and iv) $\overline{h}(t) = \mathbb{I}\{M(t-1)+1 \geq \hat{p}\} - q$. We use this reduction to the standardized distribution V_c and standardized moment function $\overline{h}(\cdot)$ repeatedly in what follows.

Finally, with some loss of generality, we assume throughout this section that $S < \infty$ as it simplifies many of our formulations.[3]

5.2. Upper Bounds Based upon General Moments

Consider the optimization problem

$$
\begin{aligned}
z^* := \inf_{y, d\mathbb{P}_v} \quad & y \qquad\qquad (10)\\
\text{s.t.} \quad & \int_0^{S_c} d\mathbb{P}_v = 1, \quad d\mathbb{P}_v \geq 0, \quad \forall v \in [0, S_c]\\
& \int_0^{S_c} v\, d\mathbb{P}_v = 1, \quad \int_0^{S_c} \overline{h}(v)\, d\mathbb{P}_v = 0, \quad y \geq p \int_0^{S_c} \mathbb{I}(v \geq p)\, d\mathbb{P}_v, \quad \forall p \in [0, S_c].
\end{aligned}
$$

The decision variables here are $\mathbb{P}_v$, which represents the distribution of V_c (a standardized valuation distribution), and y, which represents the single-pricing profit. The first two constraints ensure that $\mathbb{P}_v$ is a valid probability measure. The next two constraints ensure $\mathbb{P}_v$ has mean 1, and $\mathbb{P}_v$ satisfies the moment constraint. Finally, the last (infinite) family of constraints ensures that y is at least the revenue achieved by pricing at p for any $p \in [0, S]$.

[3] The case of $S = \infty$ can be handled with similar techniques, albeit somewhat more tedious calculations.

At optimality, y will equal the optimal single price revenue. Therefore, $1/z^*$ is a tight upper bound on the value of personalized pricing for a standardized valuation distribution satisfying $\mathbb{E}[\overline{h}(V_c)] = 0$. From our remarks in Section 5.1, $1/z^*$ can then be used to bound the value of personalized pricing for a general valuation distribution.

Unfortunately, since Problem (10) has both an infinite number of variables $\mathbb{P}_v$ for $v \in [0, S_c]$ and an infinite number of constraints (indexed by $p \in [0, S_c]$), it is not clear how to solve it. A first thought might be to discretize Eq. (10) by restricting $\mathbb{P}_v$ to have (fixed) finite, discrete support and only enforcing the semi-infinite constraint on some grid. The resulting value, however, is *not* a valid lower bound on $\mathcal{R}_{SP}$, and, hence, its reciprocal does not upper bound the value of personalized-pricing.

Theorem 6 below provides an alternate approach by discretizing the dual of (10) which does yield a valid bound. See Appendix B.7 for details.

Theorem 6 **(Upper Bounding VoPP for General Moments)**. *Let F be any valuation distribution with scale S, margin M and mean μ that satisfies $\mathbb{E}[h(v)] = \mu_h$ for a fixed, known $h(\cdot)$ and constant μ_h. Let $0 = p_0 < p_1 < \ldots < p_{N-1} < p_N = \frac{S+M-1}{M}$ be a discretization of the interval $[0, \frac{S+M-1}{M}]$ and define*

$$\begin{aligned} z_N^* \;:=\; & \max_{\theta, \boldsymbol{\lambda}, \mathbf{Q}} \quad \theta + \lambda_1 \qquad\qquad (11)\\ & s.t. \quad \sum_{j=0}^{N} Q_j = 1, \quad \boldsymbol{Q} \geq \mathbf{0}, \quad \theta + \lambda_1 \frac{S+M-1}{M} + \lambda_2 \left(h(S\mu) - \mu_h\right) \leq \sum_{j=0}^{N} p_j Q_j \;, \\ & \qquad \theta + \lambda_1 v + \lambda_2 \Big(h(\mu M(v-1) + \mu) - \mu_h \Big) \leq \sum_{j=0}^{k-1} p_j Q_j, \qquad \forall v \in [p_{k-1}, p_k), \quad k = 1, \ldots, N. \end{aligned}$$

Then, $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \leq 1/z^* \leq 1/z_N^*$.

5.3. Upper Bounds Based upon General Moments under Unimodality

We next compute upper bounds on the value of personalized pricing under a general moment constraint *and* assuming F is unimodal with mode m (we call such a distribution *m-unimodal*). We focus on the case of unimodality as it seems most relevant for pricing applications, however, our techniques can be applied to other shape constraints that describe a convex class of distributions, e.g., symmetric distributions, by leveraging the appropriate representation theorems from Popescu (2005).

We adapt our argument in Theorem 6 by leveraging Lemma 4.2 of Popescu (2005). The key idea is that any m-unimodal distribution can be represented as a mixture of uniform distributions supported on $[t,m]$ for $t<m$, uniform distributions supported on $[m,t]$ for $t>m$, and a Dirac distribution at m. More formally, let $\text{Unif}[t,m]$ denote the uniform distribution on $[t,m]$ if $t \le m$ and the uniform distribution on $[m,t]$ otherwise. Then, if V_c is standardized valuation distribution that is m_c-unimodal, then there exists random variable $\tau \sim \mathbb{M}$ supported on $[0,S_c]$ such that $V_c \sim_d W$ where $W|\tau \sim \text{Unif}[\tau, m_c]$.

Using this representation of m-unimodal distributions, we can formulate our optimization problem by reparameterizing in terms of the mixing distribution $\mathbb{M}$. Specifically, observe that if $Y_t \sim \text{Unif}[t,m]$, then $\mathbb{E}[Y_t] = (t+m)/2$, and

$$\mathbb{P}(Y_t \ge p) := G(p,m,t) := \begin{cases} 0 & \text{if } p > \max(m,t) \\ 1 & \text{if } p < \min(m,t) \\ \mathbb{I}(m \ge p) & \text{if } m = t, \\ \frac{\max(m,t)-p}{|m-t|} & \text{otherwise.} \end{cases} \qquad \mathbb{E}[\overline{h}(Y_t)] := H(t,m) := \begin{cases} \frac{1}{m-t}\int_t^m \overline{h} & \text{if } m \ne t, \\ \overline{h}(m) & \text{otherwise.} \end{cases} \tag{12}$$

Consequently, using our representation of V_c as a mixture distribution, $\mathbb{E}[V_c] = \mathbb{E}[W] = \mathbb{E}[(\tau + m_c)/2]$, and $\mathbb{E}[\overline{h}(V_c)] = \mathbb{E}[\overline{h}(W)] = \mathbb{E}[H(\tau, m_c)]$ by conditioning on τ.

We can then write an analogue of Eq. (10) when V_c is m_c-unimodal as

$$\begin{aligned} z^{*,m_c} := \inf_{y,\mathbb{M}} \quad & y \\ \text{s.t.} \quad & \int_0^{S_c} d\mathbb{M}_t = 1, \quad d\mathbb{M}_t \ge 0, \quad \int_0^{S_c} \frac{t+m_c}{2} d\mathbb{M}_t = 1, \quad \int_0^{S_c} H(t,m_c) d\mathbb{M}_t = 0, \\ & y \ge p \int_0^{S_c} G(p,m_c,t) d\mathbb{M}_t, \quad \forall p \in [0,S_c]. \end{aligned} \tag{13}$$

Here $\mathbb{M}_t$ is the distribution of τ, i.e., the mixing distribution over the requisite uniform distributions, and the constraints ensure the mixture distribution satisfies the moment constraints, similar to Problem 10. Using the dual to this optimization problem, we prove:

THEOREM 7 **(Upper Bounding VoPP under Unimodality)**. *Let F be any m-unimodal valuation distribution with scale S, margin M, and mean μ that satisfies $\mathbb{E}[h(v)] = \mu_h$ for a fixed, known $h(\cdot)$ and constant μ_h.*

Let $m_c := \frac{m-\mu+\mu M}{\mu M}$*, and* $0 = p_0 < p_1 < \ldots < p_N = \frac{S+M-1}{M}$*, be a discretization of* $[0, \frac{S+M-1}{M}]$ *such that* $p_{j^*} = m_c$ *for some* j^**. Let* z_N^{*,m_c} *denote the optimal value of*

$$\begin{aligned}
\sup_{\theta,\boldsymbol{\lambda},\boldsymbol{Q}} \quad & \theta + \lambda_1(2-m_c) \qquad (14)\\
s.t. \quad & \boldsymbol{Q} \geq 0, \quad \sum_{j=0}^{N} Q_j = 1,\\
& \theta + \lambda_1 m_c + \lambda_2\left(h(m) - \mu_h\right) \leq \sum_{j=0}^{j^*} p_j Q_j,\\
& \theta(m_c - t) + \lambda_1 t(m_c - t) + \lambda_2 \int_t^{m_c} \left(h(\mu M(s-1)+\mu) - \mu_h\right) ds\\
& \qquad \leq \sum_{j=0}^{k} p_j Q_j (m_c - t) + \sum_{j=k+1}^{j^*} p_j Q_j (m_c - p_j), \quad \forall t \in [p_k, p_{k+1}), \quad k = 0, \ldots, j^*-1,\\
& \theta(t - m_c) + \lambda_1 t(t - m_c) - \lambda_2 \int_t^{m_c} \left(h(\mu M(s-1)+\mu) - \mu_h\right) ds\\
& \qquad \leq \sum_{j=0}^{j^*} p_j Q_j (t - m_c) + \sum_{j=j^*+1}^{k} p_j Q_j (t - p_j), \quad \forall t \in (p_k, p_{k+1}], \quad k = j^*+1, \ldots, N-1.
\end{aligned}$$

Then, $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} = 1/z^{*,m_c} \leq 1/z_N^{*,m_c}$.

5.4. Lower Bounds Based upon General Moments under Unimodality

We next complement the upper bounds of the previous section by lower bounds. For many moment functions $h(\cdot)$, we can adapt the argument underlying Example 1 to construct a two-point distribution satisfying the given moment constraint for which the value of personalized pricing over single pricing is 1 (see Section B.8 for discussion of our four examples). Consequently, we focus below on the cases where V is m-unimodal to derive more informative bounds.

Using the same mixture distribution representation of a unimodal distribution from the previous section, we claim that for a standardized, m_c-unimodal valuation distribution pricing at p earns at most

$$\begin{aligned}
r^{m_c}(p) := \sup_{d\mathbb{M}_t} \quad & \int_0^{S_c} pG(p, m_c, t) d\mathbb{M}_t\\
\text{s.t.} \quad & \int_0^{S_c} d\mathbb{M}_t = 1, \quad d\mathbb{M}_t \geq 0, \quad \int_0^{S_c} \frac{t+m_c}{2} d\mathbb{M}_t = 1, \quad \int_0^{S_c} H(t, m_c) d\mathbb{M}_t = 0,
\end{aligned}$$

where $G(\cdot)$ and $H(\cdot)$ were defined in Eq. (12). As in Problem (13), $\mathbb{M}_t$ describes the relevant mixing distribution. Unlike in the previous section, the objective here maximizes the single pricing profit for pricing at p. Thus, the value of personalized pricing over single pricing satisfies $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} = \frac{1}{\max_{p\in[0,S_c]} r^{m_c}(p)}$.

By combining a duality argument with a careful discretization of the prices, we can lower bound the value of personalization. Since the techniques and results are quite similar to those in the previous section, we simply summarize the main result and relegate the precise formulations and proofs to Appendix B.9.

THEOREM 8 **(Lower Bounding VoPP under Unimodality)**. *Let F be any m-unimodal valuation distribution with scale S, margin M, and mean μ that satisfies $\mathbb{E}[h(v)] = 0$ for a fixed, known $h(\cdot)$. Let $m_c := \frac{m-\mu+\mu M}{\mu M}$, and fix any $0 < \delta < 1$. Then, $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \geq r_\delta^{*,m}$ where r_δ^{*,m_c} is non-increasing in δ and tight in the limit $\delta \to 0$. Moreover, r_δ^{*,m_c} can be evaluated by solving $N := \lceil 1 + \frac{\log\left(\frac{S+M-1}{M\delta}\right)}{\log(1+\delta)} \rceil$ optimization problems. Each of these N problems has three decision variables θ, λ_1, λ_2, and at most 2 semi-infinite constraints of the form*

$$a_1 H(t, m_c) + a_2 t \geq a_3 \quad \forall t \in [l, u], \quad \text{and} \quad a_4 t^2 + a_5 t + a_6 \int_t^{m_c} \left(h(\mu M(s-1)+\mu) - \mu_h\right) ds \geq a_7 \quad \forall t \in [l, u], \tag{15}$$

where a_i $i = 0, \ldots 7$ are (known) affine functions of $\theta, \boldsymbol{\lambda}$ and $[l, u] \subseteq [0, \frac{S+M-1}{M}]$.

5.5. Computational Tractability

Thus far we have not discussed the computational tractability of problems described in Theorems 6 to 8. Each of these problems has a small number of variables and simple constraints, and, additionally, a small number of semi-infinite constraints. For example, Problem (11) has the constraint (indexed by v)

$$\theta + \lambda_1 v + \lambda_2 \left(h(\mu M(v-1)+\mu) - \mu_h\right) \leq \sum_{j=0}^{k-1} p_j Q_j, \qquad \forall v \in [p_{k-1}, p_k).$$

Semi-infinite constraints are well-studied in the robust optimization literature (Ben-Tal and Nemirovski 2000, Ben-Tal et al. 2015). For many classes of $h(\cdot)$, they are both theoretically and practically tractable. In some cases, classical results yield explicit, convex reformulation of these semi-infinite constraints in terms of a finite number of variables and constraints. These reformulations can then be passed directly to off-the-shelf solvers.

For general $h(\cdot)$ that might not admit a simple reformulation, such constraints are still computationally tractable if one can separate efficiently over the constraint. In the example above, this amounts to finding an optimizer of

$$\max_{v \in [p_{k-1}, p_k]} \lambda_1 v + \lambda_2 \left(h(\mu M(v-1)+\mu) - \mu_h\right) \tag{16}$$

for a given k, λ_1, and λ_2. Such a subroutine can be used with constraint-generation to solve the optimization problems in Theorems 6 to 8 as a linear optimization problems efficiently (see Bertsimas et al. (2016) for details). Fortunately, for many $h(\cdot)$, an optimizer is often available in closed-form.

To illustrate these ideas, Propositions EC.1 to EC.4 in Appendix B.10 show that each of the above optimization problems is tractable for the four cases considered either by i) using techniques from the robust optimization literature to reformulate the relevant semi-infinite constraints or ii) by showing we can separate over the constraint in closed-form or via bisection search.

6. Conclusions

Increasingly rich consumer profiles enable retailers to price discriminate among customers at finer and finer granularity for increased profits. However, such price discrimination strategies entail upfront investment costs in the form of information technology, analytics expertise, and market research. Motivated by this trade-off, we provide a framework to quantify the benefits of personalized pricing in terms of the features of the underlying market. In particular, we exactly characterized the value of personalized pricing over posting a single price for all customers in terms of the scale, coefficient of deviation, and margin of the valuation distribution in closed-form.

Using our closed-form bounds, we are also able to bound the value of certain third-degree price discrimination tactics that more closely mirror current practice. Specifically, we show how to transform our previous bounds on idealized personalized pricing into more practical bounds on the value of feature-based pricing over single price strategies. We also show how to incorporate alternative moment information for sharper bounds by solving tractable optimization problems.

Overall, we believe that our results provide a rigorous foundation for analyzing pricing strategies in the context of personalization. Our results can be used both by researchers attempting to design algorithms for personalized pricing, as well as by managers seeking to implement or improve their pricing strategies. Future research directions might include computing the value of personalized pricing directly from data, especially in the presence of censoring or competition.

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Online Appendix: The Value of Personalized Pricing

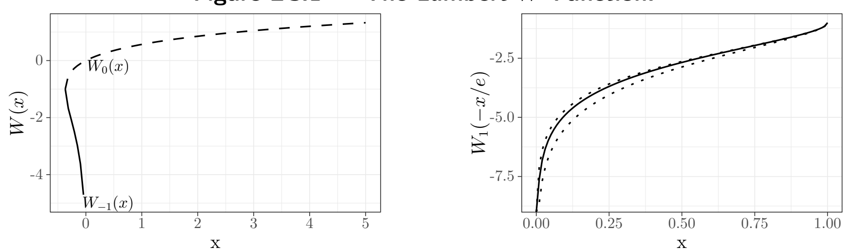
Appendix A: A Primer on the Lambert-W Function

The general (multi-valued) Lambert-W function $W(x)$, is defined as a solution to

$$W(x)e^{W(x)} = x.$$

When $x \in [-1/e, 0)$, this equation has two distinct real solutions. The branch $W_{-1}(\cdot)$ gives the solution that lies in $(-\infty, -1]$. The other branch $W_0(\cdot)$ gives the solution in $[-1, \infty)$, but is not be needed in our work. Both branches are illustrated in the left panel of Fig. EC.1.

Figure EC.1 The Lambert-W Function.



Note. The left panel shows the two real branches of the Lambert-W function, $W_0(\cdot)$ (dashed black), and $W_{-1}(\cdot)$ (solid). Our bounds depend upon the $W_{-1}(\cdot)$ branch (rescaled), as shown in right panel, and which can be upper and lower bounded via Chatzigeorgiou (2013) (dotted).

To build intuition, we encourage the reader to think of $W_{-1}(\cdot)$ as analogous to the natural logarithm, $\log(\cdot)$. Indeed, like $W_{-1}(x)$, $\log(x)$ is defined as a solution to an equation, namely, $e^{\log(x)} = x$. For a handful of values, both $W_{-1}(\cdot)$ and $\log(\cdot)$ can be evaluated exactly. For example, $W_{-1}(-1/e) = -1$, $\log(1) = 0$, and $\lim_{x\to 0} W_{-1}(x) = \lim_{x\to 0}\log(x) = -\infty$. For most values, however, both functions must be evaluated numerically. Fortunately, numerically evaluating $W_{-1}(\cdot)$ is no more difficult than evaluating $\log(\cdot)$.

Moreover, the natural logarithm provides simple bounds on $W_{-1}(\cdot)$. Indeed, Chatzigeorgiou (2013) proves that for $0 < x \leq 1$,

$$-1 - \sqrt{2\log(1/x)} - \log(1/x) \quad \leq \quad W_{-1}\left(-\frac{x}{e}\right) \quad \leq \quad -1 - \sqrt{2\log(1/x)} - \frac{2}{3}\log(1/x). \tag{EC.1}$$

(Recall $W_{-1}(\cdot)$ is defined on $[-1/e, 0)$, so that this inequality spans its domain.) The right panel in Fig. EC.1 illustrates these bounds and shows they are quite tight.

Appendix B: Omitted Proofs

B.1. Proof of Lemma 1

Proof. First note the profit from personalized pricing under valuation distribution F is $\mathcal{R}_{PP}(F,c) = \mathbb{E}[V] - c = \mu - c$ and under F_c is $\mathcal{R}_{PP}(F_c, 0) = \mathbb{E}[\frac{1}{\mu - c}(V-c)] - 0 = 1$. Hence, it suffices to show that $\mathcal{R}_{SP}(F,c) = (\mu - c)\mathcal{R}_{SP}(F_c, 0)$ to prove the first statement. Observe that

$$\begin{aligned}
\mathcal{R}_{SP}(F,c) &= \max_p (p-c)\mathbb{P}(V \geq p) \\
&= \max_p (p-c)\mathbb{P}\left(\frac{V-c}{\mu-c} \geq \frac{p-c}{\mu-c}\right) \\
&= \max_q (\mu-c)q\mathbb{P}\left(\frac{V-c}{\mu-c} \geq q\right) && \left(\text{Making the substitution } \frac{p-c}{\mu-c} \to q\right) \\
&= (\mu-c)\mathcal{R}_{SP}(F_c, 0).
\end{aligned}$$

For the last statement of the theorem, note that $\mu_c = \mathbb{E}[\frac{1}{\mu - c}(V-c)] = 1$, $M_c = 1 - 0/\mu_c = 1$,

$$S_c = \frac{\inf\{k \mid F_c(k) = 1\}}{\mu_c} = \frac{\frac{1}{\mu-c}(\inf\{k \mid F(k)=1\} - c)}{1} = \frac{\mu}{\mu - c}\left(\frac{\inf\{k \mid F(k)=1\}}{\mu} - \frac{c}{\mu}\right) = \frac{S-1+M}{M},$$

and

$$D_c = \frac{\mathbb{E}[|V_c - \mu_c|]}{2\mu_c} = \frac{\mathbb{E}\left[\left|\frac{V-c}{\mu-c} - 1\right|\right]}{2} = \frac{\mathbb{E}\left[|V-c-(\mu-c)|\right]/\mu}{2(\mu-c)/\mu} = \frac{D}{M}.$$

This completes the proof. □

B.2. Proof of Theorem 2

Proof. For simplicity, we first consider the special case when $c=0$ and $\mu=1$ and treat each regime of D separately. In this setting $\mathcal{R}_{PP} = \mu = 1$ and $M=1$. We follow the general technique of Theorem 1. Starting with the second identity of Lemma 3,

$$D = \int_0^1 1 - \overline{F}(x)dx \geq \int_0^{\frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}}} 0 \; dx + \int_{\frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}}}^1 1 - \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}}\frac{1}{x}\; dx, \tag{EC.2}$$

where we have pointwise upper bounded $\overline{F}(x)$ by 1 for $x \in [0, \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}}]$ and used the Pricing Inequality for $x \in [\frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}}, 1]$. Evaluating the integrals yields,

$$D \geq \left(1 - \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}}\right) + \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}} \log\left(\frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}}\right). \tag{EC.3}$$

We next use properties of $W_{-1}(\cdot)$ to rewrite the inequality. For brevity, let $\alpha = \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}}$. Then,

$$\begin{aligned} D \geq 1 - \alpha + \alpha\log(\alpha) &\iff D - 1 \geq \alpha(\log(\alpha) - 1) \\ &\iff \frac{D-1}{e} \geq e^{\log(\alpha)-1}(\log(\alpha)-1) \qquad \text{(using } \alpha = e \cdot e^{\log(\alpha)-1}\text{)}. \end{aligned}$$

Since $D \in [0,1]$, the left hand side is between $-1/e$ and 0, and since $\alpha > 0$ the right hand side is greater than -1/e. Applying $W_{-1}(\cdot)$ to both sides (and recalling this function is non-increasing) yields

$$W_{-1}\left(\frac{D-1}{e}\right) \leq \log(\alpha) - 1 \iff e \cdot e^{W_{-1}\left(\frac{D-1}{e}\right)} \leq \alpha \iff \frac{W_{-1}\left(\frac{D-1}{e}\right)}{D-1} \geq \frac{1}{\alpha} \tag{EC.4}$$

$$\iff \frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \leq \frac{W_{-1}\left(\frac{D-1}{e}\right)}{D-1}, \tag{EC.5}$$

where the penultimate implication follows from the definition of $W_{-1}(\cdot)$, and the last line follows from the definition of α. We stress Eq. (EC.5) holds for all D and coincides with the Low Heterogeneity bound when $c=0$, $\mu=1$.

Similarly, we can bound the cCDF in the first identity in Lemma 3 to yield an alternate bound. Specifically,

$$D = \int_1^S \overline{F}(x)dx \leq \int_1^S \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}}\frac{dx}{x} = \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}}\log(S).$$

Rearranging yields,

$$\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \leq \frac{\log(S)}{D}. \tag{EC.6}$$

Again, Eq. (EC.6) holds for all D and coincides with the Medium Heterogeneity bound.

The High Heterogeneity bound can be derived similarly, using a different bounding of the cCDF which is tighter when D is large. We defer the details to the next subsection and only state the result in Lemma EC.1 below.

Lemma EC.1 **(High Heterogeneity Bound when $c = 0$ and $\mu = 1$).** *If $D > \delta_M$, then,*

$$\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \leq -W_{-1}\left(\frac{-1}{eS(1-D)}\right). \tag{EC.7}$$

To summarize, Eqs. (EC.5) and (EC.6) hold for all $0 \leq D \leq \delta_H$ and Eq. (EC.7) holds for all $\delta_M \leq D \leq \delta_H$. These results are sufficient to prove that the bounds from the theorem are valid. For completeness, however, the next lemma further proves that in each regime, the bound for that regime is the strongest of the applicable bounds.

Lemma EC.2 **(Strongest Bound by Regime).**

a) The function

$$D \mapsto \frac{-W_{-1}\left(-\frac{1-D}{e}\right)}{1-D} - \frac{\log(S)}{D},$$

is negative for $D \in (0, \delta_L)$, is positive for $D \in (\delta_L, \delta_H]$, and has a unique root at $D = \delta_L$.

b) The function

$$D \mapsto \frac{\log(S)}{D} + W_{-1}\left(\frac{-1}{eS(1-D)}\right),$$

has a unique root at $D = \delta_M$ and is non-negative for all $D \in [0, \delta_H]$.

A consequence of Lemma EC.2 is

- When $D \in [0, \delta_L]$, Eq. (EC.5) dominates Eq. (EC.6).
- When $D \in (\delta_L, \delta_M]$, Eq. (EC.6) dominates Eq. (EC.5).
- When $D \in (\delta_M, \delta_H]$, Eq. (EC.7) dominates Eqs. (EC.5) and (EC.6).

This concludes the proof that the bounds are valid when $c = 0$ and $\mu = 1$.

For a general $c > 0$ and $\mu > 0$, we transform the problem to one in which $c = 0$ and $\mu = 1$ using Lemma 1 and apply the results from Eqs. (EC.5) to (EC.7) using the new S_c, M_c and D_c. Simplifying proves that the bounds are valid for general c and μ.

It only remains to establish that the bounds are tight. We use the same technique as in Theorem 1. Namely, in each regime, given S, M, D, and μ, we construct a cCDF that makes all pointwise bounds on the cCDF simultaneously. A difference from Theorem 1 is that the integral representations of D in the proof of Theorem 2 do not determine $\overline{F}$ over its whole domain $[0, S\mu]$; they only span $[0, \mu]$, or $[\mu, S]$ depending on the regime. This introduces some freedom in constructing the cCDF on the remaining segment and causes the tight distributions to be non-unique. We defer the details to Lemma EC.3 in the section subsection for brevity. □

B.3. Omitted Details, Proofs, and Lemmas for Theorem 2

We now provide proofs for the lemmas necessary to complete the proof of Theorem 2.

Proof of Lemma 2. Consider the case when $c = 0$ and $\mu = 1$, which implies that $M = 1$. We first prove that $D \leq \delta_H$ and that there exists an F whose coefficient of deviation is exactly δ_H. To this end, consider an arbitrary random variable V, and define the new random variable $\overline{V}$ with two-point support

$$\overline{V} = \begin{cases} \mathbb{E}[V | V \leq 1] & \text{with probability } \mathbb{P}(V \leq 1) \\ \mathbb{E}[V | V > 1] & \text{with probability } \mathbb{P}(V > 1). \end{cases}$$

By construction, $\mathbb{E}[\overline{V}] = \mathbb{E}[V] = 1$. Furthermore,

$$\begin{aligned}
\mathbb{E}\Big[|V-1|\Big] &= \mathbb{E}\Big[\ |V-1| \mid V \le 1\Big]\mathbb{P}(V\le 1) + \mathbb{E}\Big[|V-1| \mid V>1\Big]\mathbb{P}(V>1) \\
&= \mathbb{E}\Big[1-V \mid V\le 1\Big]\mathbb{P}(V\le 1) + \mathbb{E}\Big[V-1 \mid V>1\Big]\mathbb{P}(V>1) \\
&= \Big(1-\mathbb{E}\Big[V \mid V\le 1\Big]\Big)\mathbb{P}(V\le 1) + \Big(\mathbb{E}\Big[V \mid V>1\Big]-1\Big)\mathbb{P}(V>1) \\
&= \mathbb{E}\Big[|\overline{V}-1|\Big],
\end{aligned}$$

i.e., both V and $\overline{V}$ have the same coefficient of deviation. Thus, to find a distribution with maximal coefficient of deviation, it suffices to consider two-point distributions.

We compute such a distribution explicitly via the following optimization problem:

$$\begin{aligned}
\frac{1}{2}\max_{x,y,q} \quad & q(1-x)+(1-q)(y-1) \\
\text{s.t.} \quad & qx+(1-q)y = 1 \\
& 0\le x\le 1\le y\le S, \quad 0\le q\le 1,
\end{aligned}$$

where the objective is the coefficient of deviation of a distribution with mass q at $x<1$ and mass $1-q$ at $y>1$. The constraint ensures that the mean is 1. In particular, this constraint implies $q=\frac{y-1}{y-x}$ for any feasible solution, whereby the objective simplifies to $\frac{(1-x)(2y-1)}{y-x}$. This function is decreasing in x, whereby the optimal solution is $x^*=0$, $y^*=S$ and $q^*=\frac{S-1}{S}$ with optimal value $\frac{S-1}{S}$. Note $\frac{S-1}{S}=\delta_H$ since $M=1$.

Next we show $0\le\delta_L\le\delta_M\le\delta_H$. Notice that $\delta_L=\frac{\log(S)}{-W_{-1}(-\frac{1}{eS})}$ is the ratio of two positive terms. Thus, it is positive. To show $\delta_L\le\delta_M$, note that, since $S\ge 1$,

$$1+\log(S) \ \ge\ 1 \ =\ \frac{e^{1+\log(S)}}{eS},$$

which, after rearranging, implies

$$-\left(1+\log\left(S\right)\right)e^{-(1+\log(S))} \le \frac{-1}{eS}.$$

Applying $W_{-1}(\cdot)$ to both sides and noting this function is decreasing shows

$$-\left(1+\log\left(S\right)\right) \ge W_{-1}\left(\frac{-1}{eS}\right),$$

which implies

$$\delta_L = \frac{\log(S)}{-W_{-1}\left(\frac{-1}{eS}\right)} \le \frac{\log(S)}{1+\log(S)} = \delta_M,$$

as was to be shown.

To show $\delta_M\le\delta_H$, observe that since $S\ge 1$, $0\le\log(S)\le S-1$, which implies that

$$\delta_M = \frac{\log(S)}{1+\log(S)} \le \frac{S-1}{1+(S-1)} = \delta_H,$$

since $x\mapsto\frac{x}{1+x}$ is an increasing function for $x\ge 0$. This completes the proof in the case $c=0$ and $\mu=1$.

For general $c>0$ and $\mu>0$, first apply Lemma 1 to obtain an instance with zero cost and unit mean with corresponding parameters D_c, S_c, and M_c. From the previous arguments, we have that $0\le D_c\le\frac{S_c-1}{S_c}$ and $0\le\frac{\log(S_c)}{W_{-1}\left(\frac{-1}{eS_c}\right)}\le\frac{\log(S_c)}{1+\log(S_c)}\le\frac{S_c-1}{S_c}$. Transform back to the original parameters to prove the lemma, noting that $D_c=\frac{D}{M}$ and $S_c=\frac{S+M-1}{M}$. □

Proof of Lemma 3. Let $V \sim F$ and note,

$$0 = \mathbb{E}[V-\mu] = \mathbb{E}[(V-\mu)^+] - \mathbb{E}[(\mu - V)^+] \implies \mathbb{E}[(V-\mu)^+] = \mathbb{E}[(\mu - V)^+].$$

Moreover, $\mathbb{E}[|V-\mu|] = \mathbb{E}[(V-\mu)^+] + \mathbb{E}[(\mu - V)^+]$, hence, combining with the above yields $\mathbb{E}[|V-\mu|] = 2\mathbb{E}[(V-\mu)^+] = 2\mathbb{E}[(\mu - V)^+]$. We use these two identities to re-express D. From the first equality and the tail integral formula for expectation,

$$D = \frac{1}{\mu}\mathbb{E}[(V-\mu)^+] = \frac{1}{\mu}\int_0^\infty \mathbb{P}\left((V-\mu)^+ \geq t\right) dt = \frac{1}{\mu}\int_0^{\mu(S-1)} \mathbb{P}\left(V \geq \mu + t\right) dt = \int_M^{S+M-1} \overline{F}(\mu x + c) dx,$$

where the last line follows from the change of variables $\mu + t \to \mu x + c$. Similarly, using second equality and the tail integral formula for expectation,

$$D = \frac{1}{\mu}\mathbb{E}[(\mu - V)^+] = \frac{1}{\mu}\int_0^\infty \mathbb{P}((\mu - V)^+ > t) dt = \frac{1}{\mu}\int_0^{\mu - c} \mathbb{P}(V \leq \mu - t) dt = \int_0^M F(\mu x + c) dx,$$

where the last line follows from the change of variables $\mu - t \to \mu x + c$. □

Proof of Lemma EC.1. We follow the same strategy as previous two regimes bounds. Note that when the coefficient of deviation is high, the probability that V is "close" to 1 is low, since $\mu = 1$. Formally, we claim that

$$\mathbb{P}(V \geq t) \leq 1 - D \quad \forall t \in (1, S). \tag{EC.8}$$

To prove the claim, note that $D = E[(1-V)^+] \leq \mathbb{P}(V \leq 1)$, where the equality is Lemma 3 and the inequality uses $(1-V)^+ \leq 1$. Rearranging proves $\mathbb{P}(V \geq 1) \leq 1 - D$, which in turn implies Eq. (EC.8).

We use this inequality when pointwise bounding our integral representation. Specifically, for any $1 \leq t_0 \leq S$, we have

$$\begin{aligned}
D &= \int_1^S \mathbb{P}(V > t) dt && \text{(Lemma 3)} \\
&= \int_1^{t_0} \mathbb{P}(V > t) + \int_{t_0}^S \mathbb{P}(V > t) dt \\
&\leq \int_1^{t_0} (1 - D) dt + \int_{t_0}^S \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}} \frac{dt}{t} && \text{(Eq. (EC.8) and Pricing Inequality)} \\
&= (t_0 - 1)(1 - D) + \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}} \log\left(\frac{S}{t_0}\right). && \text{(EC.9)}
\end{aligned}$$

Minimizing over t_0 yields $t_0 = \max\left\{1, \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}} \frac{1}{(1-D)}\right\}$. We next argue that $D \geq \delta_M$ implies $1 \leq \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}} \frac{1}{(1-D)}$, so that the unique minimizer is $t_0 = \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}} \frac{1}{(1-D)}$.

Recall by Eq. (EC.6) $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \leq \frac{\log(S)}{D}$ for all values of D and, in particular, we have that for $D \in [\delta_M, \delta_H]$,

$$\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \leq \frac{\log(S)}{D} \leq \frac{\log(S)}{\delta_M} = 1 + \log(S).$$

Further $D \geq \delta_M = \frac{\log(S)}{1+\log(S)}$ implies that $1 + \log(S) \leq \frac{1}{1-D}$. Combining shows

$$\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \leq \frac{1}{1-D} \iff 1 \leq \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}} \frac{1}{(1-D)},$$

which confirms that $t_0 = \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}} \frac{1}{(1-D)}$ is the unique minimizer.

Plugging in this value $t_0 = \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}} \frac{1}{(1-D)}$ into Eq. (EC.9) yields:

$$1 \le \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}} + \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}} \log \left(\frac{S(1-D)}{\frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}}} \right).$$

We next use properties of the Lambert-W function to simplify this equation. For notational convenience define $\alpha = \frac{\mathcal{R}_{SP}}{\mathcal{R}_{PP}}$. Then,

$$\begin{aligned} 1 \le \alpha + \alpha \log \left(\frac{S(1-D)}{\alpha} \right) &\iff 1 \le \alpha(1 + \log(S(1-D)) - \log(\alpha)) \\ &\iff -1 \ge \alpha(\log(\alpha) - \log(eS(1-D))). \end{aligned} \tag{EC.10}$$

Note $\alpha = e^{\log(\alpha)} = e^{\log(\alpha) - \log(eS(1-D))} \cdot e \cdot S(1-D)$. Substituting above proves

$$\frac{-1}{eS(1-D)} \ge e^{\log(\alpha) - \log(eS(1-D))} (\log(\alpha) - \log(eS(1-D))).$$

The left hand side is between $-1/e$ and 0 by inspection. The function $W_{-1}(\cdot)$ is non-increasing on this range, so that applying $W_{-1}(\cdot)$ to both sides yields

$$\begin{aligned} W_{-1}\left(\frac{-1}{eS(1-D)} \right) \le \log(\alpha) - \log(eS(1-D)) &\iff \alpha \ge eS(1-D) \cdot e^{W_{-1}\left(\frac{-1}{eS(1-D)}\right)} \\ &\iff \frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \le -\frac{-1}{eS(1-D)} e^{-W_{-1}\left(\frac{-1}{eS(1-D)}\right)}. \end{aligned} \tag{EC.11}$$

Finally, from the definition of W_{-1},

$$\frac{-1}{eS(1-D)} = W_{-1}\left(\frac{-1}{eS(1-D)} \right) e^{W_{-1}\left(\frac{-1}{eS(1-D)}\right)},$$

which we use to simplify the last inequality to obtain $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \le -W_{-1}\left(\frac{-1}{eS(1-D)} \right)$. □

Proof of Lemma EC.2. First consider part a). Recalling that $-W_{-1}(-1/e) = 1$, we confirm directly that the given function is negative as $D \downarrow 0$ since it is continuous. Notice further that $-W_{-1}(\cdot)$ is an increasing function (cf. Fig. EC.1), whereby $\frac{-W_{-1}\left(-\frac{1-D}{e}\right))}{1-D}$ is an increasing function, while $\log(S)/D$ is a decreasing function. It follows that the given function has a unique root, and it suffices to show this root is δ_L to complete the proof. To this end, write,

$$\begin{aligned} -\frac{W_{-1}\left(-\frac{1-D}{e}\right)}{1-D} = \frac{\log(S)}{D} &\iff W_{-1}\left(-\frac{1-D}{e}\right) = \log\left(S^{\frac{D-1}{D}}\right) \\ &\iff -\frac{1-D}{e} = \log\left(S^{\frac{D-1}{D}}\right) \cdot \exp\left(\log\left(S^{\frac{D-1}{D}}\right)\right) \quad \text{(definition of Lambert-}W\text{)} \\ &\iff -\frac{1}{eS} = S^{\frac{-1}{D}} \cdot \frac{-\log(S)}{D} \quad \text{(simplifying)} \\ &\iff -\frac{1}{eS} = \exp\left(-\frac{\log(S)}{D}\right) \cdot \frac{-\log(S)}{D} \quad \left(\text{using } S^{\frac{-1}{D}} = \exp\left(-\frac{\log(S)}{D}\right)\right) \\ &\iff W_{-1}\left(-\frac{1}{eS}\right) = -\frac{\log(S)}{D} \quad \text{(Applying } W_{-1}(\cdot)\text{)} \\ &\iff D = -\frac{\log(S)}{W_{-1}\left(-\frac{1}{eS}\right)} = \delta_L. \end{aligned}$$

This completes the proof of part a).

To prove part b), first observe that

$$W_{-1}\left(-\frac{1}{eS(1-D)}\right) \geq -\frac{\log(S)}{D} \iff -\frac{1}{eS(1-D)} \leq -\frac{\log(S)}{D}\exp\left(-\frac{\log(S)}{D}\right),$$

because the function $y \mapsto ye^y$ is the inverse of $W_{-1}(\cdot)$ and is non-increasing on the domain of $W_{-1}(\cdot)$, i.e., $[-1/e, 0)$. Simplifying the righthand inequality yields,

$$\frac{-1}{e} \leq \log\left(S^{\frac{D-1}{D}}\right) \cdot S^{\frac{D-1}{D}}.$$

Now make the substitution $\log\left(S^{\frac{D-1}{D}}\right) \to y$ so this last inequality is equivalent to $\frac{-1}{e} \leq ye^y$. One can confirm by differentiation that $y \mapsto ye^y$ has a unique minimizer at $y = -1$, and, thus, this last inequality holds for all y. This proves the function defined in part b) is nonnegative everywhere. Moreover, it has a root at $y = 1$ which corresponds to $\log\left(S^{\frac{D-1}{D}}\right) = -1$. Simplifying shows this condition is equivalent to $D = \log(S)/(1+\log(S)) = \delta_M$, as was to be proven. $\square$

We next explicitly describe the distributions which make Theorem 2 tight. By Lemma 1, it suffices to consider the case where $c = 0$ and $\mu = 1$. The general case can be handled by scaling and shifting the below tight distributions:

Lemma EC.3 **(Tight distributions)**.

a) Suppose $D \in [0, \delta_L]$, and let $\alpha_L = \left(\frac{W_{-1}\left(\frac{D-1}{e}\right)}{D-1}\right)^{-1}$. Then, there is a random variable V with cCDF

$$\overline{F}_L(x) = \begin{cases} 1 & \textit{if } 0 \leq x < \alpha_L \\ \frac{\alpha_L}{x} & \textit{if } \alpha_L \leq x \leq 1 \\ \frac{D}{\log(S)x} & \textit{if } 1 < x \leq S \\ 0 & \textit{otherwise}, \end{cases} \qquad \text{(Tight cCDF, Low Heterogeneity)}$$

and this random variable has scale S, coefficient of deviation D, and mean 1 and satisfies Eq. (EC.5) *with equality.*

b) Suppose $D \in [\delta_L, \delta_M]$, and let $\alpha_M = \frac{D}{\log(S)}$. Then, there is a random variable V with cCDF

$$\overline{F}_M(x) = \begin{cases} 1 & \textit{if } x = 0, \\ \frac{\alpha_M}{e} S^{\frac{1}{D}-1} & \textit{if } x \in (0,\ eS^{1-\frac{1}{D}}) \\ \frac{\alpha_M}{x} & \textit{if } x \in [eS^{1-\frac{1}{D}}, S] \\ 0 & \textit{otherwise}, \end{cases} \qquad \text{(Tight cCDF, Medium Heterogeneity)}$$

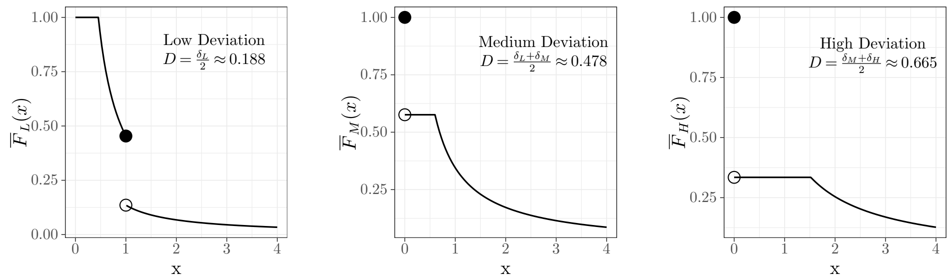
and this random variable has scale S, coefficient of deviation D, and mean 1 and satisfies Eq. (EC.6) *with equality.*

c) Suppose $D \in [\delta_M, \delta_H]$, and let $\alpha_H := \left(-W_{-1}\left(\frac{-1}{eS(1-D)}\right)\right)^{-1}$. Then, there is a random variable V with cCDF

$$\overline{F}_H(x) = \begin{cases} 1 & \textit{if } x = 0, \\ 1-D & \textit{if } x \in (0, \frac{\alpha_H}{1-D}] \\ \frac{\alpha_H}{x} & \textit{if } x \in (\frac{\alpha_H}{1-D}, S) \\ 0 & \textit{otherwise}, \end{cases} \qquad \text{(Tight cCDF, High Heterogeneity)}$$

and this random variable has scale S, coefficient of deviation D, and mean 1 and satisfies Eq. (EC.7) *with equality.*

Figure EC.2 Tight distributions for Theorem 2



Note. $S = 4$, $\mu = 1$ and $M = 1$. In all three regimes, a worst-case distribution can be constructed from a mixture of a two-point distribution and truncated Pareto distributions; what differs between the regimes is the placement and sizes of these components. We show in the course of proving Theorem 2 that any price along the truncated Pareto section is an optimal price for the single-pricing strategy. These results generalize a folklore result that the Pareto distribution represents the worst-case valuation distribution when S and D are unrestricted to the case where these values are known. Note that the distribution varies by regime and is non-unique. See Lemma EC.3 for explicit formulas.

Proof of Lemma EC.3. Intuitively, $\overline{F}_L$, $\overline{F}_M$, and $\overline{F}_H$ each make all the pointwise bounds on the cCDF the integral representation of D used in the proofs of Eqs. (EC.5) to (EC.7) tight, simultaneously. Thus, they will make the overall bound tight. See Figure EC.2 for examples of these tight distributions.

To prove the lemma formally, we will prove that $\overline{F}_L$, $\overline{F}_M$ and $\overline{F}_H$ are valid cCDFs, each with mean 1, scale S, and coefficient of deviation D, and that $\mathcal{R}_{SP}(F_L, 0) = \alpha_L$, $\mathcal{R}_{SP}(F_M, 0) = \alpha_M$ and $\mathcal{R}_{PP}(F_H, 0) = \alpha_H$, respectively. The lemma then follows directly from the definition of α_L, α_M and α_H since $\mathcal{R}_{PP}(F_L, 0) = \mathcal{R}_{PP}(F_M, 0) = \mathcal{R}_{PP}(F_H, 0) = \mu = 1$.

a) (Low Heterogeneity) Note that replacing α by α_L and the inequality by equality in Eq. (EC.4) and then following the implications backwards proves that α_L satisfies

$$D = 1 - \alpha_L + \alpha_L \log(\alpha_L).$$

We next prove $\overline{F}_L$ is a valid cCDF. By inspection, we need only prove $\overline{F}_L$ is non-increasing, i.e., that $\alpha_L \geq D/\log(S) \iff 1/\alpha_L \leq \log(S)/D$. This inequality follows directly from Lemma EC.2 since $D \in [0, \delta_L]$, and the left-hand side is low-heterogeneity bound while the right side is the medium heterogeneity bound. This proves $\overline{F}_L$ is valid.

Next, write

$$\int_0^\infty \overline{F}_L(x)dx = \int_0^1 \overline{F}_L(x)dx + \int_1^S \overline{F}_L(x)dx = \alpha_L - \alpha_L \log(\alpha_L) + D = 1,$$

where the last equality uses the identity proven above for α_L. Thus, $\overline{F}_L$ has mean 1. By Lemma 3, its coefficient of deviation is

$$\int_0^1 1 - \overline{F}_L(x)dx = \int_0^{\alpha_L} 0dx + \int_{\alpha_L}^1 1 - \frac{\alpha_L}{x}\ dx = 1 - \alpha_L + \alpha_L \log(\alpha_L) = D, \tag{EC.12}$$

again using the identify for α_L. By inspection, it has scale S.

Finally, any price $x \in [\alpha_L, 1]$ earns profit α_L, while any price $x \in [0, \alpha_L)$ earns profit strictly less than α_L. Any price $x \in (1, S]$ earns profit $D/\log(S)$ which is at most α_L as we noted when proving that $\overline{F}_L$ is valid. Thus, $\mathcal{R}_{SP}(F_L, 0) = \alpha_L$, which proves that a random variable V with cCDF $\overline{F}_L$ will satisfy Eq. (EC.5) with equality.

b) (Medium Heterogeneity) To prove that $\overline{F}_M$ is a valid cCDF, it suffices to show that $eS^{1-\frac{1}{D}} \le S$, which is equivalent to $1 \ge \frac{D}{\log(S)}$. Rewrite this last inequality as $\frac{1}{\alpha_M} \ge 1$, and recall from Step 1 of the proof of Theorem 2 that $\frac{1}{\alpha_M}$ is an upper bound on the value of personalization and, thus, must be at least 1.

Next, write

$$\int_0^\infty \overline{F}_M(x)dx = \int_0^{eS^{1-\frac{1}{D}}} \overline{F}_M(x)dx + \int_{eS^{1-\frac{1}{D}}}^S \overline{F}_M(x)dx = \alpha_M + \alpha_M \log\left(\frac{S}{eS^{1-\frac{1}{D}}}\right) = 1,$$

where the last equality uses the definition of α_M. It follows that $\overline{F}_M$ has mean 1, and, by inspection, scale S. Write,

$$\int_1^S \overline{F}_M(x)dx = \alpha_M \log S = D,$$

to conclude from Lemma 3 that $\overline{F}_M$ has coefficient of deviation D. Finally, observe that any price $x \in [eS^{1-\frac{1}{D}}, S]$ earns profit α_M, while any other price earns strictly less profit. Thus, $\mathcal{R}_{SP}(F_M, 0) = \alpha_M$, completing this part of the lemma.

c) (High Heterogeneity) To prove $\overline{F}_H$ is a valid cCDF, it suffices to show that $\alpha_H/(1-D) \le S$. Note that by Lemma EC.1, $1/\alpha_H$ is an upper-bound on the value of personalization, whereby α_H is necessarily at most 1. Moreover, for the Lambert-W function defining α_H to be well-defined, we must have that $\frac{1}{S(1-D)} \le 1$ which implies $S(1-D) \ge 1$. Thus, $\alpha_H \le 1 \le S(1-D)$ which implies that $\alpha_H/(1-D) \le S$ and that $\overline{F}_H$ is a valid cCDF.

Next write,

$$\begin{aligned}\int_0^S \overline{F}(x)dx &= \int_0^{\frac{\alpha_H}{1-D}} (1-D)dx + \int_{\frac{\alpha_H}{1-D}}^S \frac{\alpha_H}{x}dx \\ &= \alpha_H + \alpha_H \log\left(\frac{S}{\alpha_H}(1-D)\right).\end{aligned} \tag{EC.13}$$

We claim this last quantity equals 1. Indeed, from the definition of $W_{-1}(\cdot)$, $\alpha_H = eS(1-D)\cdot e^{W_{-1}\left(\frac{-1}{eS(1-D)}\right)}$. Then, replace α by α_H and the inequality by equality in Eq. (EC.11) and follow the implications backwards to Eq. (EC.10), proving the claim. Thus, $\overline{F}_H$ has mean 1, and, by inspection, has scale S.

To compute its coefficient of deviation, we first claim that $\alpha_H/(1-D) \ge 1$. Indeed, recall that

$$D \ge \delta_M = \frac{\log(S)}{1+\log(S)} \iff \log(S) \le \frac{D}{1-D} \iff \frac{\log(S)}{D} \le \frac{1}{1-D}.$$

It follows that

$$\frac{\alpha_H}{1-D} \ge \alpha_H \frac{\log(S)}{D} = \frac{\alpha_H}{\alpha_M} \ge 1,$$

where the last inequality follows from Lemma EC.2. Now compute

$$\int_0^1 1 - \overline{F}_H(x)dx = D,$$

whereby $\overline{F}_H$ has coefficient of deviation D by Lemma 3.

$$\int_0^1 1 - \overline{F}(x)dx = D.$$

It remains to check that $\mathcal{R}_{SP}(F, 0) = \alpha_H$, which we verify directly by observing that any price $x \in [\frac{\alpha_H}{1-D}, S]$ obtains profit α_H any any other price obtains profit no more than α_H. $\square$

B.4. Proof of Theorem 3

Part (a) of the theorem was proven in the main text, except for the following lemma:

Lemma EC.4 **(Maximum Deviation for Symmetric, Unimodal Distributions)**. *Suppose $V \sim F$ is symmetric, unimodal and supported on $[0,S]$ with mean $\mu = 1$. Then the mean absolute deviation of V is at most $\frac{1}{4}$. Moreover, this bound is tight for uniform random variable on $[0,2]$.*

Proof. Note by unimodality, V may have at most one point mass, located at 1. Define the function $G(x) = F(x)$ for $x \in [0,1)$ and $G(1) \equiv \lim_{t\uparrow 1} F(x)$. Note, since V is unimodal, $F(x)$ and $G(x)$ are convex on $[0,1]$.

Now, by Lemma 3,

$$D = \int_0^1 F(x)dx = \int_0^1 G(x)dx,$$

since the two functions differ only at one point. Then, by convexity

$$D \le \int_0^1 xG(0) + (1-x)G(1)dx = G(1)\int_0^1 (1-x)dx = \frac{1}{2}G(1),$$

where the first equality uses $G(0) = F(0) = 0$. Finally, by symmetry, $G(1) := \lim_{t\uparrow 1} F(x) \le \frac{1}{2}$, whereby $D \le .25$. The tightness for the uniform is immediate. □

Next we prove part (b).

Proof of Theorem 3(b). First consider a standardized valuation distribution where $c = 0$ and $\mu = 1$. Fix F, D, let m be the mode of F, and suppose p^* is the revenue maximizing single price. The proof will proceed in four cases depending on the sizes of m and p^*.

(Case 1: $m \ge 1$, $p^* \le 1$) By Lemma 3, $1 - D = \int_0^1 \overline{F}(x)dx$. Thus,

$$\mathcal{R}_{SP} = p^*\overline{F}(p^*) \le \int_0^{p^*} \overline{F}(x)dx \le \int_0^1 \overline{F}(x)dx = 1 - D,$$

where the first inequality follows since $\overline{F}$ is decreasing and the second inequality follows because $p^* \le 1$. This implies $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \ge \frac{1}{1-D}$.

(Case 2: $m \ge 1$, $p^* > 1$) Since $m \ge 1$, $\overline{F}(x)$ is concave on $[0,1]$. Hence, for any $x \in [0,1)$, $\overline{F}(x) \ge (1-x)\overline{F}(0) + x\overline{F}(1) = (1-x) + x\overline{F}(1) = 1 - x(1-\overline{F}(1))$.

Thus, by Lemma 3,

$$D = 1 - \int_0^1 \overline{F}(x)dx \le 1 - \int_0^1 \left(1 - (1-\overline{F}(1))x\right)dx = \frac{1-\overline{F}(1)}{2},$$

and, hence,

$$\overline{F}(1) \le 1 - 2D. \tag{EC.14}$$

Now since $p^* > 1$,

$$\begin{aligned}\mathcal{R}_{SP} &= \overline{F}(p^*) + (p^*-1)\overline{F}(p^*) \\ &= \overline{F}(p^*) + (p^*-1)\overline{F}(p^*)\end{aligned}$$

$$
\begin{aligned}
&\le \overline{F}(p^*) + \int_1^{p^*} \overline{F}(x)dx && (\overline{F}(x) \text{ is decreasing}) \\
&\le \overline{F}(1) + D && (p^* > 1 \text{ and Lemma 3 }) \\
&\le (1-2D) + D && (Eq.\ (\text{EC.14})) \\
&= 1 - D.
\end{aligned}
$$

Thus in this case $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \ge \frac{1}{1-D}$.

(Case 3: $m \le 1$, $p^* \le m$) Much like in Case 1, since $p^* \le m \le 1$ it follows that

$$\mathcal{R}_{SP}(F) \;=\; p^*\overline{F}(p^*) \;\le\; \int_0^{p^*} \overline{F}(x)dx \;\le\; \int_0^1 \overline{F}(x) = 1 - D,$$

which implies $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \ge \frac{1}{1-D}$.

(Case 4: $m \le 1$, $p^* > m$) Let $l(x) = \overline{F}(p^*) - f(p^*)\,(x - p^*)$ be the tangent line of $\overline{F}(x)$ at p^*. This line equals 0 at $p^* + \frac{\overline{F}(p^*)}{f(p^*)}$. Since p^* is an optimal price, it satisfies the first order condition $\frac{d}{dp} p\overline{F}(p) = \overline{F}(p) - pf(p) = 0$. Thus $\frac{\overline{F}(p^*)}{f(p^*)} = p^*$ and the root of $l(x)$ is actually $p^* + \frac{\overline{F}(p^*)}{f(p^*)} = 2p^*$.

Thus, $l(x)$ passes through the points $\{(m, l(m)), (2p^*, 0)\}$, and we may equivalently rewrite $l(x) = \frac{2l(m)p^* - l(m)x}{2p^* - m}$. Hence, we also have the identity $\overline{F}(p^*) = l(p^*) = \frac{l(m)p^*}{2p^*-m}$.

Now define the parameter $\lambda := \int_0^m \overline{F}(x)dx$. The proof will proceed in two additional sub-cases depending on the size of λ.

(Sub-case 4(a): $\lambda \ge \frac{2}{3}$) Notice, because $p^* > m$, we have $p^* < 2p^* - m$, which implies that $\frac{(p^*)^2}{2p^*-m} < 2p^* - m$. Thus, we can upper bound $\mathcal{R}_{SP}$ by

$$
\begin{aligned}
\mathcal{R}_{SP} \;=\; p^*\overline{F}(p^*) \;&=\; \frac{l(m)}{2p^*-m}(p^*)^2 \\
&\le\; l(m)(2p^* - m) \\
&=\; 2\int_m^{2p^*} l(x)dx && \left(\int_m^{2p^*} l(x)dx = l(m)\frac{2p^*-m}{2}\right) \\
&\le\; 2\int_m^{\infty} \overline{F}(x)dx && (l(x) \le \overline{F}(x) \text{ for } x \in (m, \infty)) \\
&=\; 2(1-\lambda).
\end{aligned}
$$

Finally, since $\lambda \ge 2/3$, it follows that $\mathcal{R}_{SP} \le 2(1-\lambda) \le \frac{2}{3} \le \lambda$. Thus, in this subcase $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \ge \frac{1}{\lambda}$. Since $m \le 1$, $\lambda \le \int_0^1 \overline{F}(x)dx = 1 - D$, and it follows that $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \ge \frac{1}{1-D}$.

(Sub-case 4(b): $\lambda \le \frac{2}{3}$)

Write $\mathcal{R}_{SP}(F)$ as the sum before the mode and after the mode

$$\mathcal{R}_{SP}(F) = m\overline{F}(p^*) + (p^* - m)\,\overline{F}(p^*). \tag{EC.15}$$

The first term on the right hand side of Eq. (EC.15) is bounded by

$$
\begin{aligned}
m\overline{F}(p^*) &= m\overline{F}(m)\frac{m\overline{F}(p^*)}{m\overline{F}(m)} \\
&\le \lambda\frac{m\overline{F}(p^*)}{m\overline{F}(m)} && \left(\text{since } \overline{F}(x) \text{ is decreasing } \implies m\overline{F}(m) \le \int_0^m \overline{F}(x)dx\right) \\
&\le \lambda\frac{l(p^*)}{l(m)} && \left(\text{ since } l(m) \le \overline{F}(m) \text{ and } l(p^*) = \overline{F}(p^*)\right) \\
&= \lambda\frac{p^*}{2p^*-m},
\end{aligned}
\tag{EC.16}
$$

The second term on the right hand side of Eq. (EC.15) is bounded by

$$\begin{aligned}
(p^*-m)\,\overline{F}(p^*) &= l(m)\left(p^*-\frac{m}{2}\right)\frac{(p^*-m)\,\overline{F}(p^*)}{l(m)(p^*-\frac{m}{2})} \\
&= \left(\int_m^{2p^*} l(x)dx\right)\frac{(p^*-m)\,\overline{F}(p^*)}{l(m)(p^*-\frac{m}{2})} && \left(l(m)(p^*-\frac{m}{2}) = \int_m^{2p^*} l(x)dx\right) \\
&\le (1-\lambda)\frac{(p^*-m)\,\overline{F}(p^*)}{l(m)(p^*-\frac{m}{2})} && \left(\int_m^{2p^*} l(x) \le \int_m^{\infty}\overline{F}(x)dx = 1-\lambda\right) \\
&= (1-\lambda)\frac{2p^*\,(p^*-m)}{(2p^*-m)^2}. && \left(\overline{F}(p^*) = \frac{l(m)p^*}{2p^*-m}\right) \qquad \text{(EC.17)}
\end{aligned}$$

Thus we can upper bound $\mathcal{R}_{SP}$ by combining Eqs. (EC.16) and (EC.17)

$$\mathcal{R}_{SP}(F) \;\le\; \lambda\frac{p^*}{2p^*-m} + (1-\lambda)\frac{2p^*\,(p^*-m)}{(2p^*-m)^2} \;=\; \frac{p^*(2p^*+m(\lambda-2))}{(2p^*-m)^2} \;\le\; \max_{p\ge m}\frac{p(2p+m(\lambda-2))}{(2p-m)^2}.$$

This last optimization problem is differentiable in p. As $p\to\infty$, the objective tends to $\frac{1}{2}$. At $p=m$, the objective becomes λ. There is one critical point obtained by differentiation at $p=\frac{m(2-\lambda)}{2\lambda} > m$ since $\lambda \le 2/3$. At the critical point, the objective is $\frac{(2-\lambda)^2}{8(1-\lambda)}$. For $0\le\lambda\le 2/3$, this value always exceeds λ and $\frac{1}{2}$, and hence this is the optimum.

Thus in this subcase $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \ge \frac{8(1-\lambda)}{(2-\lambda)^2}$. Further, since $m\le 1$, $\lambda \le \int_0^1 \overline{F}(x)dx = 1-D$ and it follows that $\frac{8D}{(1+D)^2}$.

Combining all cases and sub-cases gives $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \ge \min\{\frac{1}{1-D}, \frac{8D}{(1+D)^2}\}$ which yields the desired bound.

The tightness at $D=0$ is immediate since the only feasible distribution is a point mass at m and $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} = 1$.

To prove the asymptotic tightness as $D\to 1$, we construct a family of unimodal distributions $\{V_\delta\}$ indexed by δ such that as $\delta\to 0$, the coefficient of deviation of V_δ tends to 1, and the value of personalized pricing tends to 2. Namely,

$$V_\delta = \begin{cases} \text{Unif}[0,\delta] & \text{with probability } 1-\delta \\ \text{Unif}[\delta, \frac{2}{\delta}-1] & \text{with probability } \delta. \end{cases}$$

By inspection, each distribution in this family is unimodal with mode δ and $\mathbb{E}[V]=1$. To see the deviation tends to 1 as δ tends to 0 consider the lower uniform component of V_δ i.e. $\lim_{\delta\to 0^+}\mathbb{E}[|V_\delta - 1|] \ge \lim_{\delta\to 0^+}(1-\delta)(1-\delta/2) = 1$. Furthermore, pricing at $\frac{1}{\delta}-\frac{1-\delta}{2}$ earns revenue $(\frac{1}{\delta}-\frac{1-\delta}{2})\mathbb{P}\left(V_\delta \ge \frac{1}{\delta}-\frac{1-\delta}{2}\right) = (\frac{1}{\delta}-\frac{1-\delta}{2})\frac{\delta}{2}$. Taking the limit as δ tends to 0 yields revenue 1/2 and thus value of personalized pricing of 2 matching the above lower bound. $\square$

B.5. Other Omitted Results and Proofs from Section 3

Lemma EC.5 **(Convexity of the Single-Pricing Guarantee)**. *For any S, M, and D, let $\alpha(S,M,D)$ denote the reciprocal of the bound on the value of personalized pricing in Theorem 2. Then $\alpha(S,M,D)$ is a convex function in D.*

Proof of Lemma EC.5. Let us fix S and M, and define $\alpha(D) := \alpha(S,M,D)$. Fix any D_1, D_2, with $0\le D_1 \le D_2 \le \delta_H$, and any $t\in[0,1]$. We will show that $\alpha(tD_1+(1-t)D_2) \le t\alpha(D_1) + (1-t)\alpha(D_2)$ to prove the theorem.

By Theorem 2, there exists random variables $V_1 \sim F_1$ and $V_2 \sim F_2$ each with scale S and margin M such that the coefficient of deviation of F_1 is D_1, the coefficient of deviation of F_2 is D_2, $\alpha(D_1) = \frac{\mathcal{R}_{SP}(F_1,c)}{\mathcal{R}_{PP}(F_1,c)}$ and $\alpha(D_2) = \frac{\mathcal{R}_{SP}(F_2,c)}{\mathcal{R}_{PP}(F_2,c)}$.

Since both V_1 and V_2 have the same margin and cost, they also have the same mean $\mu = \frac{c}{1-M}$. Take X to be a Bernoulli random variable with parameter t, and let $\tilde{V} \equiv XV_1 + (1-X)V_2$ where X, V_1, V_2 are sampled independently. Note that $\tilde{V}$ has mean μ, margin M, and scale S. Furthermore, the coefficient of deviation of $\tilde{V}$ is

$$\begin{aligned}
\tilde{D} &= \frac{1}{2\mu}\left(\mathbb{E}\Big[|XV_1 + (1-X)V_2 - \mu|\Big]\right) \\
&= \mathbb{P}(X=1)\cdot\frac{1}{2\mu}\mathbb{E}\Big[|V_1 - \mu|\Big] + \mathbb{P}(X=0)\cdot\frac{1}{2\mu}\mathbb{E}\Big[|V_2 - \mu|\Big] \\
&= tD_1 + (1-t)D_2.
\end{aligned} \tag{EC.18}$$

To conclude the proof, write

$$\begin{aligned}
t\alpha(D_1) + (1-t)\alpha(D_2) &= t\frac{\mathcal{R}_{SP}(F_1,c)}{\mathcal{R}_{PP}(F_1,c)} + (1-t)\frac{\mathcal{R}_{SP}(F_2,c)}{\mathcal{R}_{PP}(F_2,c)} \\
&= \frac{t\mathcal{R}_{SP}(F_1,c) + (1-t)\mathcal{R}_{SP}(F_2,c)}{\mathcal{R}_{PP}(\tilde{F},c)} \\
&\geq \frac{\mathcal{R}_{SP}(\tilde{F},c)}{\mathcal{R}_{PP}(\tilde{F},c)} \\
&\geq \alpha(\tilde{D}) \\
&= \alpha(tD_1 + (1-t)D_2).
\end{aligned}$$

The first equation follows from the definitions of F_1 and F_2. The second equation follows from the fact that the personalized pricing strategy yields $\mu - c$ for F_1, F_2, and $\tilde{F}$. The first inequality follows from the fact that the optimal single price for $\tilde{V}$ yields revenue of at most $\mathcal{R}_{SP}(F_1,c)$ for the market corresponding to V_1 and at most $\mathcal{R}_{SP}(F_2,c)$ for the market corresponding to V_2. The second inequality follows Theorem 2. The last equality follows from Eq. (EC.18). □

Proof of Corollary 1. Note that Eq. (EC.1) shows that

$$W_{-1}\left(-\frac{x}{e}\right) = 1 + \sqrt{2\log(1/x)} + O(\log(1/x)) \quad \text{as } x \to 1.$$

Substituting this expression into the bounds in the low heterogeneity and high heterogeneity regimes proves the result. □

B.6. Omitted Proofs from Section 4

Proof of Theorem 4.

Part a) Using Eq. (9), write $\mathcal{R}_{XP} = \max_{p(\cdot)} \mathbb{E}[p(\mathbf{X})\mathbb{I}(\mu(\mathbf{X}) + \epsilon \geq p(\mathbf{X})]$, where the maximization is taken over all (measurable) functions of the features representing the pricing policy. We lower bound this quantity by constructing a feasible pricing policy. Let $p_0 \in \arg\max_{p\geq 0}\ p\mathbb{P}(\mu+\epsilon \geq p)$, where $\mu = \mathbb{E}[V]$. We consider the feasible pricing policy that offers price $p_0 + \mu(\mathbf{X}) - \mu$ to a customer with features $\mathbf{X}$. Then,

$$\begin{aligned}\mathcal{R}_{XP} &\geq \mathbb{E}\left[(p_0+\mu(\mathbf{X})-\mu)\mathbb{I}(\mu(\mathbf{X})+\epsilon \geq p_0+\mu(\mathbf{X})-\mu)\right] \\ &= \mathbb{E}\left[(p_0+\mu(\mathbf{X})-\mu)\mathbb{I}(\mu+\epsilon \geq p_0)\right] \\ &= \mathbb{E}\left[p_0\mathbb{I}(\mu+\epsilon\geq p_0)\right] + \mathbb{E}\left[\mu(\mathbf{X})-\mu)\mathbb{I}(\mu+\epsilon \geq p_0)\right].\end{aligned}$$

The first expectation equals $\mathcal{R}_{SP}(F_{\mu+\epsilon}, c)$ by choice of p_0. By independence, the second expectation is $\mathbb{E}\left[\mu(\mathbf{X})-\mu)\right]\mathbb{P}(\mu+\epsilon\geq p_0) = 0$ since $\mathbb{E}[\mu(\mathbf{X})] = \mu$. Thus, we have shown $\mathcal{R}_{XP}(F,c) \geq \mathcal{R}_{SP}(F_{\mu+\epsilon}, c)$.

Finally, applying Theorem 2 to the random variable $\mu+\epsilon$, we can bound $\mathcal{R}_{SP}(F_{\mu+\epsilon}, c) \geq (\mu - c)\cdot \alpha(S_{\mu+\epsilon}, M, D_\epsilon)$, where $S_{\mu+\epsilon}$ is the scale of $\mu+\epsilon$. Notice, by independence of ϵ and $\mathbf{X}$, $S_{\mu+\epsilon} \leq S$. Hence we further lower bound this quantity by $(\mu-c)\alpha(S, M, D_\epsilon)$ to complete the first part.

Part b) Write

$$\mathcal{R}_{XP}(F,c) = \mathbb{E}[\mathcal{R}_{SP}(F_{V|\mathbf{X}}, c)] = \int_c^\infty \mathcal{R}_{SP}(F_{t+\epsilon}, c) f_{\mu(\mathbf{X})}(t)dt, \tag{EC.19}$$

where we have used the fact that $F_{V|\mathbf{X}} = F_{\mu(X)+\epsilon}$ because $\mathbf{X}$ and ϵ are independent. Now applying Theorem 3(a) to the random variable $t+\epsilon$ yields,

$$\mathcal{R}_{SP}(F_{t+\epsilon}, c) \ \leq\ (t-c)\left(1 - 2\frac{\frac{\mathbb{E}[|\epsilon|]}{2t}}{\frac{t-c}{t}}\right) \ =\ (t-c)\left(1-\frac{\mathbb{E}[|\epsilon|]}{(t-c)}\right) \ =\ t-c-\mathbb{E}[|\epsilon|].$$

Substituting into the integral above shows

$$\mathcal{R}_{XP}(F,c) \ \leq\ \int_c^\infty (t-c-\mathbb{E}[|\epsilon|])\, f_{\mu(\mathbf{X})}(t)dt \ =\ (\mu-c) - \mathbb{E}[|\epsilon|] \ =\ (\mu-c)\cdot\left(1-\frac{D_\epsilon}{M}\right).$$

Noting $\mathcal{R}_{PP} = \mu - c$ and rearranging completes the proof. □

B.7. Omitted Proofs from Sections 5.2 and 5.3.

Proof of Theorem 6. Based on the reduction described in Section 5.1, it suffices to upper bound the value of personalized pricing for a standardized distribution with standardized moment function $\overline{h}(\cdot)$. We do this by providing a lower-bound on Eq. (10). Following Shapiro (2001), the dual to Eq. (10) is

$$\begin{aligned}\sup_{\theta,\boldsymbol{\lambda},d\mathbb{Q}_p} \quad & \theta+\lambda_1 \\ \text{s.t.} \quad & \int_0^{S_c} d\mathbb{Q}_p = 1, \quad d\mathbb{Q}_p \geq 0, \\ & \theta + \lambda_1 v + \lambda_2\overline{h}(v) - \int_0^{S_c} p\mathbb{I}(v\geq p)d\mathbb{Q}_p \leq 0, \quad \forall v\in[0,S_c].\end{aligned} \tag{EC.20}$$

Here, $\mathbb{Q}_p$ is a probability measure defined on $p\in[0,S_c]$. By weak-duality, any feasible solution to problem (EC.20) yields a valid lower bound to $\mathcal{R}_{SP}$. To form such a feasible solution to (EC.20), we constrain $\mathbb{Q}_p$ to be supported only on $\{p_0,\ldots,p_N\}$ (noting $p_N = S_c$) and denote the corresponding point masses as $Q_0, Q_1, \ldots, Q_N$. Then, the value of (EC.20) is at least

$$\begin{aligned} z_N^* := \max_{\theta,\boldsymbol{\lambda},\mathbf{Q}} \quad & \theta+\lambda_1 \\ \text{s.t.} \quad & \sum_{j=0}^N Q_j = 1, \qquad \boldsymbol{Q}\geq \mathbf{0} \\ & \theta+\lambda_1 v + \lambda_2\overline{h}(v) - \sum_{j=0}^N p_j\mathbb{I}(v\geq p_j)Q_j \leq 0, \qquad \forall v\in[0,S_c]\ .\end{aligned} \tag{EC.21}$$

Notice that the sum of indicators in Eq. (EC.21) is constant over $v \in [p_{k-1}, p_k)$. Thus, we can rewrite this constraint of Eq. (EC.21) as $N+1$ separate sets of constraints:

$$\theta + \lambda_1 v + \lambda_2 \overline{h}(v) \leq \sum_{j=0}^{k-1} p_j Q_j, \quad \forall v \in [p_{k-1}, p_k),\ k = 1, \ldots, N,$$

$$\theta + \lambda_1 S_c + c + \lambda_2 \overline{h}(S_c) \leq \sum_{j=0}^{N} p_j Q_j\ .$$

Replacing Eq. (EC.21) with these $N+1$ sets of constraints, and then substituting in the definitions of S_c and $\overline{h}(\cdot)$ provides a lower bound on the personalized-pricing revenue. Taking a reciprocal completes the proof. □

Proof of Theorem 7. Based on the reduction described in Section 5.1, it suffices to upper bound the value of personalized pricing for a standardized distribution with standardized moment function $\overline{h}(\cdot)$. We do this by lower bounding Eq. (13). Note that if V is m-unimodal, then the standardized distribution V_c is m_c-unimodal.

Using the fact that $\int_0^{S_c} d\mathbb{M}_t = 1$, we can replace the constraint $\int_0^{S_c} \frac{t+m_c}{2} d\mathbb{M}_t = 1$ by the constraint $\int_0^{S_c} t d\mathbb{M}_t = 2 - m_c$. Then, following Shapiro (2001) the dual to Eq. (13) with this constraint replaced is

$$\sup_{\theta, \boldsymbol{\lambda}, \mathbb{Q}} \quad \theta + \lambda_1 (2 - m_c) \tag{EC.22}$$

$$\text{s.t.} \quad \theta + \lambda_1 t + \lambda_2 H(t, m_c) \leq \int_0^{S_c} p G(p, m_c, t) d\mathbb{Q}_p \quad \forall t \in [0, S_c], \tag{EC.23}$$

$$d\mathbb{Q}_p \geq 0, \quad \int_0^{S_c} d\mathbb{Q}_p = 1.$$

Again, by weak duality, any feasible solution to this problem lower-bounds z^{*,m_c}. We restrict $\mathbb{Q}$ to discrete distributions supported on the given discretization over p (noting $p_N = S_c$), and denote the corresponding point masses as $Q_0, Q_1, \ldots, Q_N$. The last two constraints then become $\boldsymbol{Q} \geq \boldsymbol{0}$ and $\sum_{j=0}^{N} Q_j = 1$.

Constraint (EC.23) can also be written as the following three (families) of constraints

$$\theta + \lambda_1 t + \lambda_2 H(t, m_c) \leq \sum_{j=0}^{N} p_j G(p_j, m_c, t) Q_j \quad \forall t \in [p_k, p_{k+1}) \quad k = 0, \ldots, j^* - 1, \tag{EC.24a}$$

$$\theta + \lambda_1 m + \lambda_2 H(m_c, m_c) \leq \sum_{j=0}^{N} p_j G(p_j, m_c, m_c) Q_j, \tag{EC.24b}$$

$$\theta + \lambda_1 t + \lambda_2 H(t, , m_c) \leq \sum_{j=0}^{N} p_j G(p_j, m_c, t) Q_j \quad \forall t \in (p_k, p_{k+1}] \quad k = j^* + 1, \ldots, N - 1. \tag{EC.24c}$$

These three cases correspond to whether t is less than, equal to, or greater than the mode. We further simplify these constraints.

Consider Eq. (EC.24a), fix some k and note that necessarily $p_k \leq t < p_{k+1} < m$. Split the sum as

$$\sum_{j=0}^{k} p_j G(p_j, m_c, t) Q_j + \sum_{j=k+1}^{j^*} p_j G(p_j, m_c, t) Q_j + \sum_{j=j^*+1}^{N} p_j Q_j G(p_j, m_c, t).$$

In the first sum, $p_j \le t$, in the second sum, $t < p_j \le m_c$, and in the third sum, $m_c < p_j$. Consequently, by the definition of $G(\cdot)$, we can rewrite these three sums as

$$\sum_{j=0}^{k} p_j Q_j + \sum_{j=k+1}^{j^*} p_j \frac{m_c - p_j}{m_c - t} Q_j.$$

Plugging this expression back into Eq. (EC.24a) and multiplying through by $(m_c - t)$ yields,

$$\theta(m_c - t) + \lambda_1 t(m_c - t) + \lambda_2 \int_t^{m_c} \overline{h}(s)ds \le \sum_{j=0}^{k} p_j Q_j (m_c - t) + \sum_{j=k+1}^{j^*} p_j Q_j (m_c - p_j), \quad \forall t \in [p_k, p_{k+1}), \tag{EC.25}$$

for all $k = 0, \dots, j^* - 1$.

Next consider Eq. (EC.24b) and use that $H(m_c, m_c) = \overline{h}(m_c)$ and $G(p, m_c, m_c) = \mathbb{I}(m_c \ge p)$ to rewrite it as

$$\theta + \lambda_1 m_c + \lambda_2 \overline{h}(m_c) \le \sum_{j=0}^{j^*} p_j Q_j. \tag{EC.26}$$

Finally consider Eq. (EC.24c), fix some k and note that necessarily $m_c < p_k < t \le p_{k+1}$. Split the sum as

$$\sum_{j=0}^{j^*} p_j G(p_j, m_c, t) Q_j + \sum_{j=j^*+1}^{k} p_j G(p_j, m_c, t) Q_j + \sum_{j=k+1}^{N} p_j G(p_j, m_c, t) Q_j.$$

In the first sum, $p_j \le m_c$, in the second sum, $m_c < p_j < t$, and in the third sum, $t \le p_j$. Hence, by the definition of $G(\cdot)$, we can rewrite these three sums as

$$\sum_{j=0}^{j^*} p_j Q_j + \sum_{j=j^*+1}^{k} p_j \frac{t - p_j}{t - m_c} Q_j.$$

Plugging this expression back into Eq. (EC.24c) and multiplying through by $(t - m_c)$yields

$$\theta(t - m_c) + \lambda_1 t(t - m_c) - \lambda_2 \int_t^{m_c} \overline{h}(s)ds \le \sum_{j=0}^{j^*} p_j Q_j (t - m_c) + \sum_{j=j^*+1}^{k} p_j Q_j (t - p_j), \quad \forall t \in (p_k, p_{k+1}], \tag{EC.27}$$

for all $k = j^* + 1, \dots, N - 1$. Combining these three families of constraints and substituting in the definitions of S_c and $\overline{h}(\cdot)$ completes the proof. □

B.8. Omitted Examples from Section 5.4

In this subsection, we show that in the absence of shape constraints on the distribution, only vacuous lower bounds on the value of personalized pricing exist for the four moments described in Section 5. Recall that when $h(v)$ corresponds to the coefficient of deviation, in Example 1 we explicitly constructed a distribution with margin M, scale at most S, coefficient of deviation D, and value of personalized pricing equal to 1. We next discuss the other three cases.

Example EC.1 (Vacuous Lower Bound for Coefficient of Variation). Let $h(v) = \frac{(v-\mu)^2}{\mu^2}$ and fix $S > 1$, $M \le 1$, and C. We shall construct a random variable V with margin M, $\mu = 1$, and scale at most S such that $\mathbb{E}[h(V)] = C^2$, and $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} = 1$. Specifically, let V be the two-point distribution

$$V = \begin{cases} 1 - M & \text{with probability } \frac{\frac{C^2}{M^2}}{1 + \frac{C^2}{M^2}} \\ 1 + \frac{C^2}{M} & \text{with probability } \frac{1}{1 + \frac{C^2}{M^2}} \end{cases}$$

and let F be the corresponding cdf. One can confirm that $\mathbb{E}[V]=1$ and $\mathbb{E}[h(V)]=C^2$. By Theorem 1 of Bhatia and Davis (2000), any random variable V with mean 1 and supported on $[c,S]$ satisfies:

$$C^2=\mathbb{E}[(V-1)^2]\le(S-1)M.$$

Thus, the scale of V satisfies $1+\frac{C^2}{M}\le 1+S-1=S$. Finally, observe that a single price at $1+\frac{C^2}{M}$ earns a profit of $1-c$ and hence, $\frac{\mathcal{R}_{PP}(F,c)}{\mathcal{R}_{SP}(F,c)}=1$. □

Example EC.2 (Vacuous Lower Bound for Geometric Mean). Let $h(v)=-\log\left(\frac{v}{\mu}\right)$ and fix $M<1$ and B. We shall construct a random variable V with margin M such that $\mathbb{E}[h(V)]=-\log\left(\frac{B}{\mu}\right)$ and $\mu=1$. We let V be the two-point distribution

$$V=\begin{cases}1-M & \text{with probability } \epsilon\\ 1+\frac{\epsilon M}{1-\epsilon} & \text{with probability } 1-\epsilon\end{cases}$$

and let F be the corresponding cdf, where $\epsilon\in(0,1)$ shall be determined later. One can confirm that $\mathbb{E}[V]=1$ and that single pricing at $1+\frac{\epsilon M}{1-\epsilon}$ yields a profit of $1-c$ and hence $\frac{\mathcal{R}_{PP}(F,c)}{\mathcal{R}_{SP}(F,c)}=1$.

What remains is to show that there exists an ϵ such that $\mathbb{E}[h(V)]=-\log\left(\frac{B}{\mu}\right)$, which reduces to

$$B=(1-M)^{\epsilon}\left(\frac{1-\epsilon+\epsilon M}{1-\epsilon}\right)^{1-\epsilon}. \tag{EC.28}$$

Note that the RHS of (EC.28) is a continuous function which equals 1 when $\epsilon=0$ and equals $1-M$ when $\epsilon=1$. Moreover, by Jensen's inequality and $V\ge 1-M$ almost surely, it must be that $B\in[1-M,1]$. Thus, there exists an ϵ that solves (EC.28). □

Example EC.3 (Vacuous Lower Bound for Incumbent Price). Let $h(v)=\mathbb{I}\{v\ge\hat{p}\mu\}$ and fix $M<1$, $\hat{p}>1-M$, and $q\in[0,1]$. We shall construct a random variable V with margin M such that $\mathbb{E}[h(V)]=q$ and $\mu=1$. We let V be the two-point distribution

$$V=\begin{cases}1-M & \text{with probability } 1-q\\ 1-M+\frac{M}{q} & \text{with probability } q\end{cases}$$

and let F be the corresponding cdf. One can confirm that $\mathbb{E}[V]=1$. In order for this distribution to be valid with margin M, we must satisfy the fact that pricing at $\hat{p}$ yields a profit of at most $\mathcal{R}_{PP}(F,c)=1-c=M$, i.e. $(\hat{p}-c)q=(\hat{p}+M-1)q\le M$. From this inequality, it follows that $\hat{p}\in(1-M,1-M+\frac{M}{q}]$ and $\mathbb{E}[h(V)]=q$. Finally, one can confirm that single pricing at $1-M+\frac{M}{q}$ yields a profit of M and thus $\frac{\mathcal{R}_{PP}(F,c)}{\mathcal{R}_{SP}(F,c)}=1$. □

B.9. Omitted Proofs from Section 5.4

Recall we have shown in the main text that when V_c is m_c unimodal, $\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}}=\frac{1}{\max_{p\in[0,S_c]} r^{m_c}(p)}$. Unfortunately, this maximization is not concave. Hence, to form a bound, we discretize the price space. The next lemma quantifies the error induced from such a procedure.

Lemma EC.6 **(Error from Geometric Price Ladder)**. *Let F_c be a standardized valuation distribution with scale S_c. Fix $0<\delta<1$ and let $N=\lceil 1+\frac{\log(S_c/\delta)}{\log(1+\delta)}\rceil$. Let $p_0=0$, and $p_j=\delta(1+\delta)^{j-1}$ for $j=1,\dots,N$, so that $\{p_j\}_{j=0}^N$ discretize the interval $[0,S_S]$. Define $r_\delta^*:=\max_{j:0\le j\le N} p_j\mathbb{P}(V_c\ge p_j)$. Then,*

$$r_\delta^*\ \le\ \mathcal{R}_{SP}(F_c,0)\ \le\ \max\left(\delta,\ (1+\delta)r_\delta^*\right).$$

Proof of Lemma EC.6. The first inequality follows because the price ladder restricts the feasible region and hence reduces the possible single-pricing revenue. For the second, let p^* be the optimal single price and let k be such that $p_k \le p^* \le p_{k+1}$. We consider two cases: If $k=0$, then $\mathcal{R}_{SP}(F_c,0) = p^*\mathbb{P}(V_c \ge p^*) \le p_1 = \delta$. Alternatively, if $k \ge 1$, then,

$$\mathcal{R}_{SP}(F_c,0) \;=\; p^*\mathbb{P}(V_c \ge p^*) \;\le\; p_{k+1}\mathbb{P}(V_c \ge p_k) \;\le\; (1+\delta)p_k\mathbb{P}(V_c \ge p_k) \;\le\; (1+\delta)r^*_\delta.$$

Combining yields the lemma. $\square$

Notice by inspection, the error from this discretization decreases as $\delta \to 0$ and is tight in the limit. We next leverage Lemma EC.6 together with duality to prove Theorem 8.

Proof of Theorem 8. Based on the reduction described in Section 5.1, it suffices to lower bound the value of personalized pricing for a standardized distribution with standardized moment function $\overline{h}(\cdot)$. Note that if V is m-unimodal, then the standardized distribution V_c is m_c-unimodal.

Now consider the geometric price ladder described in Lemma EC.6. By that lemma, we have

$$\frac{\mathcal{R}_{PP}}{\mathcal{R}_{SP}} \ge \frac{1}{\max\left(\delta,\ (1+\delta)\max_{j:0\le j\le N} r^{m_c}(p_j)\right)} \equiv r^{*,m_c}_\delta.$$

This bound clearly improves as $\delta \to 0$ and is tight in the limit. Thus it only remains to prove that $r^{m_c}(p_j)$ can be evaluated as an optimization problem for each j.

Since $\int_0^{S_c} d\mathbb{M}_t = 1$, we can replace the constraint $\int_0^{S_c} \frac{t+m_c}{2} d\mathbb{M}_t = 1$ by $\int_0^{S_c} t d\mathbb{M}_t = 2-m_c$ in the definition of $r^{m_c}(p_j)$. Then, by duality, we have

$$\begin{aligned} r^{m_c}(p_j) = \inf_{\theta,\boldsymbol{\lambda}} \quad & \theta + \lambda_2(2-m_c) \\ \text{s.t.} \quad & \theta + \lambda_1 H(t,m_c) + \lambda_2 t \;\ge\; p_j G(p_j,m_c,t) \quad t \in [0,S_c]. \end{aligned}$$

We consider three cases based on the value of p_j:

Case i) $p_j < m_c$. Separate the semi-infinite constraint into two constraints depending on whether $t \in [0,p_j]$, $t \in (p_j,S_c]$ and use the definition of $G(p_j,m_c,t)$ to write it as

$$\begin{aligned} \theta + \lambda_1 H(t,m_c) + \lambda_2 t \;&\ge\; p_j\left(\frac{m_c-p_j}{m_c-t}\right) \quad t \in [0,p_j]. \\ \theta + \lambda_1 H(t,m_c) + \lambda_2 t \;&\ge\; p_j \qquad t \in [p_j,S_c]. \end{aligned}$$

Multiply the first of these constraints through by $m_c - t > 0$ and combine to obtain the optimization problem

$$\begin{aligned} r^{m_c}(p_j) = \inf_{\theta,\boldsymbol{\lambda}} \quad & \theta + \lambda_2(2-m_c) \\ \text{s.t.} \quad & \theta(m_c-t) + \lambda_1 \int_t^{m_c} \overline{h}(s)ds + \lambda_2 t(m_c-t) \;\ge\; p_j(m_c-p_j) \quad \forall t \in [0,p_j], \\ & \theta + \lambda_1 H(t,m_c) + \lambda_2 t \;\ge\; p_j \quad \forall t \in [p_j,S_c]. \end{aligned}$$

Both constraints are special cases of Eq. (15).

Case ii) $p_j = m_c$. In this case we separate the semi-infinite constraint into two constraints depending on whether $t \in [0,m_c)$ or $t \in [m_c,S_c]$, and use the definition of $G(m_c,m_c,t)$ to write

$$\theta + \lambda_1 H(t,m_c) + \lambda_2 t \ge 0 \quad \forall t \in [0,m_c]$$

$$\theta + \lambda_1 H(t,m_c) + \lambda_2 t \ge m_c \quad \forall t \in [m_c,S_c],$$

where we have used continuity to close the half-open interval. Substituting above yields the optimization problem

$$
\begin{aligned}
r^{m_c}(m_c) = \inf_{\theta,\boldsymbol{\lambda}} \quad & \theta + \lambda_2(2 - m_c) \\
\text{s.t.} \quad & \theta + \lambda_1 H(t, m_c) + \lambda_2 t \ge 0 \quad \forall t \in [0, m_c] \\
& \theta + \lambda_1 H(t, m_c) + \lambda_2 t \ge m_c \quad \forall t \in [m_c, S_c].
\end{aligned}
$$

Both constraints are special cases of Eq. (15).

Case iii) $p_j > m_c$ We now consider two cases depending on whether $t \in [0, p_j]$ or $t \in (p_j, S_c]$. Again, split the semi-infinite constraint and use the definition of $G(p_j, m_c, t)$ to write

$$
\begin{aligned}
\theta + \lambda_1 H(t, m_c) + \lambda_2 t \ &\ge 0 \quad \forall t \in [0, p_j] \\
\theta + \lambda_1 H(t, m_c) + \lambda_2 t &\ge p_j \left(\frac{t - p_j}{t - m_c} \right) \quad \forall t \in [p_j, S_c].
\end{aligned}
$$

Multiply the second constraint through by $t - m_c > 0$, and combine to show that

$$
\begin{aligned}
r^{m_c}(p_j) = \inf_{\theta,\boldsymbol{\lambda}} \quad & \theta + \lambda_2(2 - m_c) \\
\text{s.t.} \quad & \theta + \lambda_1 H(t, m_c) + \lambda_2 t \ \ge 0 \quad \forall t \in [0, p_j] \\
& \theta(t - m_c) - \lambda_1 \int_t^{m_c} \overline{h}(s) ds + \lambda_2 t(t - m_c) \ge p_j(t - p_j) \quad \forall t \in [p_j, S_c].
\end{aligned}
$$

Both constraints are special cases of Eq. (15).

These three cases thus complete the proof. □

B.10. Omitted Proofs from Section 5.5

In this section, we prove that each of our mathematical programming bounds on the value of personalized pricing is computationally tractable for the four cases considered in the main text.

For clarity, recall the standardized moment function $\overline{h}(t) := h(\mu M(t-1) + \mu) - \mu_h$. Using this function to simplify notation, we see that the optimization problem in Theorem 6 can be solved as a linear optimization problem with constraint generation if we can identify an optimizer of

$$
\max_{v \in [p_k, p_{k+1})} a_1 v + a_2 \overline{h}(v) \tag{EC.29}
$$

for every $\boldsymbol{a} \in \mathbb{R}^2$ and k.

Moreover, the optimization problem in Theorem 7 can be solved as a linear optimization problem with constraint generation if we can identify an optimizer of

$$
\max_{t \in [p_k, p_{k+1}]} a_1 t + a_2 t^2 + a_3 \int_t^{m_c} \overline{h}(s) ds \tag{EC.30}
$$

for every $\boldsymbol{a} \in \mathbb{R}^3$ and k.

Finally, the optimization problem in Theorem 8 can be solved as a linear optimization problem with constraint generation if we can identify an optimizer for each of

$$
\min_{t \in [l,u]} \ a_1 H(t, m_c) + a_2 t \quad \text{and} \quad \min_{t \in [l,u]} \ a_1 t^2 + a_2 t + a_3 \int_t^{m_c} \overline{h}(s) ds, \tag{EC.31}
$$

for any $\boldsymbol{a} \in \mathbb{R}^3$ and $[l, u] \subseteq [0, S]$. Notice this second optimization is of the same form as Eq. (EC.30).

Thus to prove that our mathematical programming bounds are computationally tractable for our four previous examples, it suffices to give optimization procedures for each of these problems for the corresponding standardized moment functions $\overline{h}(\cdot)$.

Proposition EC.1 **(Tractability of VoPP Optimizations for Coefficient of Deviation)**. *Suppose $\overline{h}(t) = M\,|t-1|\,/2 - D$. Then,*

a) For each $\boldsymbol{a} \in \mathbb{R}^3$ and k, an optimizer to Eq. (EC.30) *can be found in closed-form.*

b) For any $\boldsymbol{a} \in \mathbb{R}^3$ and $[l,u]$ with $-\infty < l < u < \infty$, optimizers to the two problems in Eq. (EC.31) *can be found by bisection search and in closed-form, respectively.*

In other words, the problems in Theorems 7 and 8 can each be solved efficiently as a linear optimization with constraint generation.

Remark EC.1. Note that for the special case of $h(t) = M\,|t-1|\,/2 - D$, Theorem 6 is superceded by the closed-form bound Theorem 2, and, hence, omitted above. □

Proof of Proposition EC.1.

Part a): An optimizer to Eq. (EC.30) occurs either at an endpoint p_k, p_{k-1}, or else at a critical point, i.e., solutions to $\frac{a_1}{a_3} + \frac{2a_2}{a_3}t = \frac{M}{2}\,|t-1| - D$. We first seek roots where $t \le 1$. There is at most one such root, given by $t_{\le 1} \equiv \frac{-2a_1 - 2D + a_3 M}{4a_2 + a_3 M}$, but only if this value is less than equal to 1. Otherwise, there is no root less than 1. We next seek roots for $t \ge 1$. Again, there is at most one such root, given by $t_{\ge 1} \equiv \frac{-2a_1 - 2D - a_3 M}{4a_2 - a_3 M}$, but only if this value is great than or equal to 1. Otherwise there is no root greater than one.

In summary, an optimizer is one of p_k, p_{k-1}, $t_{\le 1}$ (if $t_{\le 1} \le 1$) or $t_{\ge 1}$ (if $t_{\ge 1} \ge 1$), and can be identified by simply checking the feasibility and comparing these (at most) 4 values.

Part b): Consider the first of the two optimization problems. Notice that if $Y_t \sim \text{Unif}[t, m_c]$, we can write $Y = t + (m_c - t)\xi$ with $\xi \sim \text{Uniform}[0,1]$. Hence, we can rewrite $H(t, m_c) = \mathbb{E}[\overline{h}(t + (m_c - t)\xi)]$. Since $\overline{h}(\cdot)$ is convex, it follows that $H(t, m_c)$ is convex in t; $\overline{h}(t + (m_c - t)\xi)$ is the composition of a convex and affine function, and expectations preserve convexity.

We conclude that if $a \le 0$, the first optimization problem is the minimization of a concave function, and the optimum occurs at an end point $\{l, u\}$. If $a > 0$, then it is the minimization of a convex function. The optimum occurs either at an end point $\{l, u\}$, or else at t^* solving $\partial_t H(t, m_c) + b/a = 0$. Such a t^* can be found by bisection search.

A procedure for solving the second problem was given in Part a). □

Proposition EC.2 **(Tractability of VoPP Optimizations for Coefficient of Variation)**. *Suppose $\overline{h}(t) = M^2(t-1)^2 - C^2$. Then,*

a) Problem (11) *can be solved explicitly as a (finite) convex second order cone problem.*

b) For each $\boldsymbol{a} \in \mathbb{R}^3$ and k, an optimizer to Eq. (EC.30) *can be found in closed-form.*

c) For any $\boldsymbol{a} \in \mathbb{R}^3$ and $l, u \in \mathbb{R}$, optimizers to the two problems in Eq. (EC.31) *can be found by bisection search and in closed-form, respectively.*

In other words, the problems in Theorems 6 to 8 are each computationally tractable.

Remark EC.2. Notice in Part a), we do not use separation. The problem is an explicit second order cone problem that can be passed to off-the-shelf software. □

Proof of Proposition EC.2.

Part a): Since $\overline{h}(\cdot)$ is continuous, it suffices to reformulate the semi-infinite constraint

$$\lambda_1 v + \lambda_2 \overline{h}(v) \leq \sum_{j=0}^{k-1} p_j Q_j - \theta, \quad \forall v \in [p_{k-1}, p_k],$$

Since $v \in [p_{k-1}, p_k] \iff (v - p_{k-1})(v - p_k) \leq 0$, we can use the definition of $\overline{h}(\cdot)$ to rewrite the k^{th} constraint as

$$\theta - \sum_{j=0}^{k} p_j Q_j - \lambda_2 C^2 \leq \min_{v:(v-p_{k-1})(v-p_k)\leq 0} -\lambda_1 v - \lambda_2 M^2 (v-1)^2.$$

The (possibly non-convex) minimization on the right is an example of a quadratic optimization problem in which quadratic forms in the objective and in the constraint are simultaneously diagonalizable. Such problems were studied in Ben-Tal and Den Hertog (2014) which shows they can be equivalently written as convex, second order cone problems. Indeed, applying the results of that paper shows the k^{th} constraint is equivalent to the constraints

$$\begin{aligned}
(y_k + \lambda_2 M^2) p_{k-1} p_k - x &\geq \theta - \sum_{j=0}^{k} p_j Q_j + \lambda_2 (M^2 - C^2) \\
4 y_k x_k &\geq z_k^2 \\
z_k &= 2\lambda_2 M^2 - \lambda_1 - (y_k + \lambda_2 M^2)(p_{k-1} + p_k) \\
x_k, y_k &\geq 0, \quad y + \lambda_2 M^2 \geq 0,
\end{aligned} \tag{EC.32}$$

with the auxiliary variables x_k, y_k, z_k. This formulation is always convex (Constraint (EC.32) is a rotated second-order cone constraint; see Boyd and Vandenberghe (2004)). Performing this transformation for each of the semi-infinite constraints yields a (convex) second order cone representation, proving the theorem.

Part b): Again, an optimizer of Eq. (EC.30) occurs either at endpoint p_k, p_{k-1}, or else at a critical point, i.e., a solution to $\frac{a_1}{a_3} + \frac{2a_2}{a_3} t = M^2(t-1)^2 - C^2$. This equation has two roots, given by $\frac{a_2 + a_3 M^2 \pm \sqrt{a_2^2 + a_3(a_1 + 2a_2 + a_3 C^2) M^2}}{a_3 M^2}$. These roots can only be optimizers of Eq. (EC.30) if they lie within $[p_k, p_{k+1}]$. Hence there are at most 4 possible optimizers, and we can identify an optimizer in closed form by comparing their objective values.

Part c): Consider the first of the two optimization problems. The same convexity argument that applied in the case of Proposition EC.1 applies here unchanged. Hence, when $a_1 \leq 0$, an optimum occurs at an end point of $\{l, u\}$. If $a_1 > 0$, then an optimum occurs either at this end point or else the solution t^* to $\partial_t H(t^*) + a_2/a_1 = 0$ which can be obtained by bisection.

A procedure for solving the second problem was given in Part b). $\square$

PROPOSITION EC.3 **(Tractability of VoPP Optimization with Geometric Mean)**. *Suppose* $\overline{h}(t) = -\log(M(t-1)+1) + \log(B/\mu)$. *Then,*

a) Problem (11) *can be solved by solving two explicit (finite) convex optimization problems. Alternatively, for any* k, λ_1 *and* λ_2, *an optimizer of Eq.* (EC.29) *can be found in closed-form. Hence, Problem* (11) *can also be solved efficiently by constraint generation as a linear optimization problem.*

b) For each $\boldsymbol{a} \in \mathbb{R}^3$ *and* k*, an optimizer to Eq.* (EC.30) *can be found in closed-form, and hence Problem* (14) *can be solved efficiently by constraint generation as a linear optimization problem.*

c) For any $\boldsymbol{a} \in \mathbb{R}^3$ *and* $l, u \in \mathbb{R}$*, optimizers to the two problems in Eq.* (EC.31) *can be found by bisection search and in closed-form, respectively.*

Proof of Proposition EC.3.

Part a): We formulate two separate convex optimization problems corresponding to the cases where the optimal $\lambda_2 > 0$ and $\lambda_2 \leq 0$ and note that the solution to Problem (11) is the better of these two objective values.

To formulate an optimization problem when $\lambda_2 \geq 0$, notice that when $\lambda_2 > 0$, $\max_{v \in [p_{k-1}, p_k]} \ \lambda_1 v + \lambda_2 \overline{h}(v)$ is the maximization of a convex function, and hence the maximum occurs at one of the two endpoints. Hence, we can simply replace the k^{th} semi-infinite constraint by two linear constraints, namely,

$$\lambda_1 p_{k-1} + \lambda_2 \overline{h}(p_{k-1}) \leq \sum_{j=0}^{k-1} p_j Q_j, \qquad \lambda_1 p_k + \lambda_2 \overline{h}(p_k) \leq \sum_{j=0}^{k-1} p_j Q_j.$$

Applying this transformation for each k and adding the constraint $\lambda_2 \geq 0$ yields our first convex optimization problem (in fact a linear optimization problem).

To formulate an optimization problem when $\lambda_2 \leq 0$, use the definition of $\overline{h}(\cdot)$ to write

$$\max_{v \in [p_{k-1}, p_k]} \ \lambda_1 v + \lambda_2 \overline{h}(v) \iff -|\lambda_2| \log(B/\mu) \ + \ \max_{v,w} \quad \lambda_1 v + |\lambda_2| \log\big(Mv + (1-M)\big) \quad \text{s.t.} \quad v = w, \quad w \in [p_{k-1}, p_k].$$

By Lagrangian duality, we relax the equality constraint yielding

$$\begin{array}{ll} \max\limits_{v} & (\lambda_1 - \beta_k) v + |\lambda_2| \log\big(Mv + (1-M)\big) \\ \text{s.t.} & v \in \mathbb{R} \end{array} \ + \ \begin{array}{ll} \max\limits_{w} & \beta_k w \\ \text{s.t.} & w \in [p_{k-1}, p_k] \end{array}$$

The second optimization can be solved in closed form yielding $\max(\beta_k p_{k-1}, \beta_k p_k)$, which is convex in β_k. The first optimization can also be solved explicitly by looking at the first-order condition yielding

$$|\lambda_2| \left(\log\left(\frac{M\,|\lambda_2|}{\beta_k - \lambda_1} \right) - 1 \right) + (\beta_k - \lambda_1) \frac{1-M}{M} \quad \text{if} \quad \beta_k > \lambda_1,$$

and infinity otherwise. Substituting back, shows we can equivalently write the k^{th} semi-infinite constraint when $\lambda_2 \leq 0$ as

$$\begin{aligned} & \exists \beta_k \in \mathbb{R} \ \text{ s.t.} \\ & \quad \beta_k \geq \lambda_1, \quad \lambda_2 \leq 0, \\ & \quad \left(\log\left(\frac{M\,|\lambda_2|}{\beta_k - \lambda_1} \right) - 1 \right) + (\beta_k - \lambda_1) \frac{1-M}{M} + \max(\beta_k p_{k-1}, \beta_k p_k) \leq \sum_{j=0}^{k-1} p_j Q_j. \end{aligned}$$

These constraints are convex. Making this transformation for each k yields the convex optimization problem for the case $\lambda_2 \leq 0$. Taking the better of these two optimization problems yields a solution to Problem (11).

To prove the last statement about Problem **??**, notice that an optimum must occur either at a critical point or an endpoint $\{p_{k-1}, p_k\}$. Differentiating, we see the only critical point is $1 - 1/M + a_2/a_1$, if this value is in $[p_{k-1}, p_k]$. Comparing these (at most) three values yields an optimizer.

Part b): Again, an optimizer occurs either at an endpoint $\{p_{k-1}, p_k\}$ or a critical point. Using the definition of $\overline{h}(t)$, critical points satisfy $a_1 + 2a_2 t = -a_3 \log(M(t-1)+1) + a_3 \log(B/\mu)$. When $a_3 = 0$, this yields the unique critical point $-a_1/2a_2$ if this value is in $[p_{k-1}, p_k]$.

When $c \neq 0$, we rewrite this equation as

$$\frac{2a_2}{a_3} t + \log(Mt + 1 - M) = \log(B/\mu) - \frac{a_1}{a_3}.$$

Make the substitution $y \leftarrow Mt + 1 - M$ yielding,

$$\frac{2a_2 y}{a_3 M} + \log(y) = \log(B/\mu) - \frac{a_1}{a_3} + \frac{2a_2}{a_3}\left(\frac{1}{M} - 1\right) \iff y \exp\left(\frac{2a_2 y}{a_3 M}\right) = \frac{B}{\mu} \exp\left(\frac{2a_2}{a_3}\left(\frac{1}{M} - 1\right) - \frac{a_1}{a_3}\right),$$

where the implication follows by exponentiating both sides. Finally multiplying both sides by $\frac{2a_2}{a_3 M}$ yields

$$\frac{2a_2 y}{a_3 M} \exp\left(\frac{2a_2 y}{a_3 M}\right) = \frac{2a_2 B}{a_3 M \mu} \exp\left(\frac{2a_2}{a_3}\left(\frac{1}{M} - 1\right) - \frac{a_1}{a_3}\right).$$

This equation implies that

$$\frac{2a_2 y}{a_3 M} = W_*\left(\frac{2a_2 B}{a_3 M \mu} \exp\left(\frac{2a_2}{a_3}\left(\frac{1}{M} - 1\right) - \frac{a_1}{a_3}\right)\right),$$

where W_* denotes *any* branch of the Lambert-W function. It follows that y only admits a real-valued solution if $\frac{2a_2 B}{a_3 M \mu} \exp\left(\frac{2a_2}{a_3}\left(\frac{1}{M} - 1\right) - \frac{a_1}{a_3}\right) \geq -\frac{1}{e}$. Furthermore, if this value lies within $[-\frac{1}{e}, 0)$, y admits two solutions, corresponding to the -1 and 0 branches of the function. If this value is non-negative, y admits only one solution, corresponding to the 0 branch. Transforming back to t yields at most two critical points:

$$\begin{aligned} t_1 &= 1 - 1/M + \frac{a_3}{2a_2} W_{-1}\left(\frac{2a_2 B}{a_3 M \mu} \exp\left(\frac{2a_2}{a_3}\left(\frac{1}{M} - 1\right) - \frac{a_1}{a_3}\right)\right) && \text{if } -\frac{1}{e} \leq \frac{2a_2 B}{a_3 M \mu} \exp\left(\frac{2a_2}{a_3}\left(\frac{1}{M} - 1\right) - \frac{a_1}{a_3}\right) < 0, \\ t_2 &= 1 - 1/M + \frac{a_3}{2a_2} W_0\left(\frac{2a_2 B}{a_3 M \mu} \exp\left(\frac{2a_2}{a_3}\left(\frac{1}{M} - 1\right) - \frac{a_1}{a_3}\right)\right) && \text{if } \frac{2a_2 B}{a_3 M \mu} \exp\left(\frac{2a_2}{a_3}\left(\frac{1}{M} - 1\right) - \frac{a_1}{a_3}\right) > 0, \end{aligned}$$

where we have indicated when the critical point is defined. Checking these at most 4 points yields an optimizer.

Part c): Consider the first of the two optimization problems. The same convexity argument that applied in the case of Proposition EC.1 applies here unchanged. Hence, when $a_1 \leq 0$, an optimum occurs at an end point of $\{l, u\}$. If $a_1 > 0$, then an optimum occurs either at this end point or else the unique solution t^* to $\partial_t H(t^*, m_c) + a_2/a_1 = 0$ which can be obtained by bisection.

A procedure for solving the second optimization problem was given in Part b). □

Proposition EC.4 **(Tractability of VoPP Optimizations for Incumbent Price)**. *Suppose* $\overline{h}(t) = \mathbb{I}\{M(t-1)+1 \geq \hat{p}\} - q$ *for some* $\hat{p} \in [0, S]$ *and* $q \in [0, 1]$. *Then,*

a) Problem (11) *can be solved as an explicit linear optimization problem.*

b) For each $\boldsymbol{a} \in \mathbb{R}^3$ *and* k, *an optimizer to Eq.* (EC.30) *can be found in closed-form, and, hence, Problem* (14) *can be solved efficiently by constraint generation as a linear optimization problem.*

c) For any $\boldsymbol{a} \in \mathbb{R}^3$ *and* $l, u \in \mathbb{R}$, *optimizers to the two problems in Eq.* (EC.31) *can be found in closed-form, respectively.*

In other words, the problems in Theorems 7 and 8 can each be solved efficiently as a linear optimization with constraint generation.

REMARK EC.3. Notice in Part a), we do not use separation. The problem is an explicit linear optimization problem that can be passed to an off-the-shelf software.

Proof of Proposition EC.4. Throughout, let $v_0 \equiv \frac{\hat{p}-1}{M}+1$, so that $\overline{h}(y) = \mathbb{I}\{M(y-1) \geq \hat{p}-1\} - q = \mathbb{I}\{y \geq v_0\} - q$.

Part a): Fix some k and consider the corresponding semi-infinite constraint in Eq. (11):

$$\max_{v \in [p_{k-1}, p_k)} \theta + \lambda_1 v + \lambda_2 \mathbb{I}\{v \geq v_0\} - \lambda_2 q \ \leq \ \sum_{j=0}^{k-1} p_j Q_j,$$

If $v_0 \notin [p_{k-1}, p_k)$, then the objective function on the left is a linear function, and we can replace this constraint with the two linear constraints corresponding to the end points:

$$\theta + \lambda_1 p_{k-1} + \lambda_2 \mathbb{I}\{p_{k-1} \geq v_0\} - \lambda_2 q \ \leq \ \sum_{j=0}^{k-1} p_j Q_j,$$
$$\theta + \lambda_1 p_k + \lambda_2 \mathbb{I}\{p_k \geq v_0\} - \lambda_2 q \ \leq \ \sum_{j=0}^{k-1} p_j Q_j.$$

On the other hand, if $v_0 \in [p_{k-1}, p_k)$, then the objective function on the left is a piecewise linear function with one breakpoint. In general, there may be a discontinuity at this breakpoint. Hence we can replace the semi-infinite constraint by four constraints: the two constraints above corresponding to the endpoints and two additional constraints corresponding to the values $v = v_0$ and $v \uparrow v_0$:

$$\theta + \lambda_1 v_0 + \lambda_2(1-q) \leq \sum_{j=0}^{k-1} p_j Q_j, \qquad \theta + \lambda_1 v_0 - \lambda_2 q \leq \sum_{j=0}^{k-1} p_j Q_j.$$

Making these replacements for each k yields an explicit linear optimization problem.

Part b): Again, an optimizer of Eq. (EC.30) occurs either at endpoint p_k, p_{k+1}, or else at a critical point. When $a_3 = 0$, the unique critical point is at $-\frac{a_2}{2a_1}$, if this value occurs in $[p_k, p_{k+1})$. When $a_3 \neq 0$, a critical point occurs when $\frac{a_1}{a_3} + \frac{2a_2}{a_3} t = \mathbb{I}\{t \geq v_0\} - q$. We have two cases depending on the value of the indicator:

If $t \geq v_0$, then a critical point occurs when $\frac{a_1}{a_3} + \frac{2a_2}{a_3} t = 1 - q$, i.e., at $t_1 \equiv \frac{a_3}{2a_2}\left(1 - q - \frac{a_1}{a_3}\right)$, provided $a_2 \neq 0$, $t_1 \in [p_k, p_{k+1})$ and $t_1 \geq v_0$. Otherwise, there is no such critical point.

If $t < v_0$, then a critical point occurs when $\frac{a_1}{a_3} + \frac{2a_2}{a_3} t = -q$, i.e., at $t_0 \equiv -\frac{a_3}{2a_2}\left(q + \frac{a_1}{a_3}\right)$, provided $a_2 \neq 0$, $t_0 \in [p_k, p_{k+1})$ and $t_0 < v_0$. Otherwise, there is no such critical point.

Checking these at most 4 values yields an optimizer.

Part c): Consider the first of the two optimization problems in Eq. (EC.31). By definition,

$$H(t, m_c) \ = \ \mathbb{E}[\overline{h}(Y_t)] \ = \ \mathbb{P}(Y_t \geq v_0) - q \ = \ G(v_0, m_c, t) - q,$$

where we now $Y_t \sim \text{Unif}[m_c, t]$. Moreover, this last function is non-decreasing in t. Hence, we see that an optimum of the first problem in Eq. (EC.31) either occurs at an endpoint or, if $a_2 a_1 < 0$, in the interior. We add the two endpoints ℓ, u to the set of potential optimizers and next search for potential optimizers on the interior. To this end, we assume $a_2 a_1 < 0$.

Notice that $\overline{h}(t)$ is continuous whenever $t \neq v_0$, Hence, by the fundamental theorem of calculus, we can differentiate Eq. (12) when $t \neq v_0$, yielding

$$\partial_t H(t, m_c) \;=\; -\frac{1}{m_c - t} h(t) + \frac{1}{(m_c - t)^2} \int_t^{m_c} h \;=\; \frac{H(t) - h(t)}{m_c - t} \;=\; \frac{G(v_0, m_c, t) - \mathbb{I}\{t \geq v_0\}}{m_c - t}.$$

This implies, for $t \neq v_0$,

$$\partial_t \left(a_1 H(t, m_c) + a_2 t\right) \;=\; \frac{a_1}{m_c - t}\left(G(v_0, m_c, t) - \mathbb{I}\{t \geq v_0\}\right) + a_2,$$

which, by inspection, is not well-defined when $t = m_c$. Thus, we conclude that, excluding the points v_0 and m_c, any potential optimizer in the interior must satisfy $\frac{a_1}{m_c - t}\left(G(v_0, m, t) - \mathbb{I}\{t \geq v_0\}\right) + a_2 = 0$. We add both v_0 and m_c to the set of potential optimizers and restrict attention in the remainder to solutions of this equation.

Multiplying through by $(m_c - t)$ shows such critical points must satisfy

$$G(v_0, m_c, t) = \mathbb{I}(t \geq v_0\} + \frac{a_2}{a_1}(t - m_c). \tag{EC.33}$$

Notice this equation is piecewise continuous in t. We solve it by considering 6 cases corresponding to all combinations of the two branches of the indicator and the 3 branches which define $G(\cdot)$ where $m_c \neq t$.

Case 1: $v_0 \leq t$.

Subcase i) $\max(m_c, t) < v_0$**.** This subcase is impossible since we assume $v_0 \leq t$.

Subcase ii) $v_0 < \min(m_c, t)$**.** Here Eq. (EC.33) reduces to $1 = \frac{a_2}{a_1}(t - m_c) + 1$, whose only solution is $t = m_c$. Since we already added m_c as a potential maximizer, we ignore this case.

Subcase iii) $\min(m_c, t) \leq v_0 \leq \max(m_c, t)$ **and** $m_c \neq t$**.** Since $v_0 \leq t$, it follows that $m_c \leq v_0 \leq t$ and $m_c < t$. Then Eq. (EC.33) reduces to $\frac{t - v_0}{t - m_c} = \frac{a_2}{a_1}(t - m_c) + 1$. Since $t \neq m_c$ by assumption, we can multiply through and solve for t, yielding

$$t_{1,2} = m_c \pm \sqrt{\frac{a_1}{a_2}(m_c - v_0)}.$$

We disregard t_2 since $m_c < t$ by assumption. Thus, we add t_1 to the set of potential optimizers.

Case 2: $v_0 > t$.

Subcase i) $\max(m_c, t) < v_0$**.** Equation (EC.33) reduces to $0 = \frac{a_2}{a_1}(t - m_c)$, whose only solution is $t = m_c$ Since we already added m_c as a potential maximizer, we ignore this case.

Subcase ii) $v_0 < \min(m_c, t)$**.** This case is impossible since we assume $v_0 > t$.

Subcase iii) $\min(m_c, t) \leq v_0 \leq \max(m_c, t)$ **and** $m_c \neq t$**.** Since $v_0 > t$, it follows that $t < v_0 \leq m_c$. Simplifying Eq. (EC.33) gives

$$\frac{m_c - v_0}{m_c - t} = \frac{a_2}{a_1}(t - m_c).$$

Again, since $t \neq m_c$, we can multiply through and solve for t yielding two roots

$$t_{3,4} = m \pm \sqrt{\frac{a_1}{a_2}(v_0 - m_c)}.$$

We disregard t_3 since $t < m_c$ and add t_4 to the set of potential optimizers.

In summary, we have shown that an optimizer to the first problem in Eq. (EC.31) occurs at one of the following points: $\left\{l, u, m_c, v_0, m_c + \sqrt{\frac{a_1}{a_2}(m_c - v_0)}, m_c - \sqrt{\frac{a_1}{a_2}(v_0 - m_c)}\right\}$. Checking these at most 6 points thus yields an optimizer.

A procedure for solving the second problem in Eq. (EC.31) was given in Part b).

□