Mathematics Review for Business PhD Students

Lecture Notes

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Lecture 1: Introductory Material

I. Sets

1. A *set* is a list or collection of objects. The objects which compose a set are termed the *elements* or *members* of the set.

Remark 1: A set may be written in tabular form, that is, by listing all of its elements. As an example, consider the set A which consists of the positive integers 1,2 and 3. We would write this set as

$A = \{1, 2, 3\}$.

Note that when we define a set, the order of listing is not important. That is, in our above example,

$A = \{1, 2, 3\} = \{3, 1, 2\}$.

Another notation for writing sets is the *set-builder* form. In the above example,

$$A = \{x \mid x \text{ is a positive integer, } 1 \le x \le 3\}.$$

or

$$A = \{x : x \text{ is a positive integer, } 1 \le x \le 3\}$$

The notation ":" or "|" reads "such that". The entire notation reads "A is the set of all numbers such that x is a positive integer and $1 \le x \le 3$."

Remark 2: The symbol "$\in$" reads "is an element of". Thus, in the same example

$1 \in A$ and $1, 2, 3, \in A$.

2. If every element of a set S_1 is also an element of a set S_2, then S_1 is a *subset* of S_2 and we write

$S_1 \subset S_2$ or $S_2 \supset S_1$.

The symbol "$\subset$" reads "is contained in" and "$\supset$" reads "contains". If $S_1 \subset S_2$, then S_2 is said to be a *superset* of S_1.

<u>*Examples*</u>:

#1. If $S_1 = \{1, 2\}$, $S_2 = \{1, 2, 3\}$, then $S_1 \subset S_2$ or $S_2 \supset S_1$.

#2 The set of all positive integers is a subset of the set of real numbers.

Def 1: Two sets S_1 and S_2 are said to be *equal* if and only if $S_1 \subset S_2$ and $S_2 \subset S_1$.

Remark 3: The *largest subset* of a set S is the set S itself. The *smallest subset* of a set S is the set which contains no elements. The set which contains no elements is the *null set*, denoted $\varnothing$.

Remark 4: In Remark 3 we noted that the null set is the smallest subset of any set S. To see that the null set is contained in any set S, consider the following proof by contradiction.

Proof: Assume $\varnothing \not\subset S$, then there exists at least one $x \in \varnothing$ such that $x \notin S$. However, $\varnothing$ has no elements. Thus, we have a contradiction and it must be that $\varnothing \subset S$. ||

Remark 5: The set S which contains zero as its only element, i.e. $S = \{0\}$, is *not* the null set. That is, if $S = \{0\}$, then $S \neq \varnothing$ since $0 \in S$. As an example of the null set, consider the set A where

$A = \{x : x \text{ is a living person 1000 years of age}\}$.

Clearly, $A = \varnothing$. The null set can also be written as $\{\}$.

Def 2: Two sets S_1 and S_2 are *disjoint* if and only if there does not exist an x such that $x \in S_1$ and $x \in S_2$.

Example: If $S_1 = \{0\}$ and $S_2 = \{1, 2, 4\}$, then S_1 and S_2 are disjoint.

Def 3: The operations of *union*, *intersection*, *difference* (relative complement), and *complement* are defined for two sets A and B as follows:

(i) $A \cup B \equiv \{x : x \in A \text{ or } x \in B\}$,

(ii) $A \cap B \equiv \{x : x \in A \text{ and } x \in B\}$,

(iii) $A - B \equiv \{x : x \in A, x \notin B\}$,

(iv) $A' \equiv \{x : x \notin A\}$.

Remark 6: In the application of set theory all sets under investigation are likely to be subsets of a fixed set. We call this set the *universal set* and denote it as U. For example, in human population studies U would be the set of all people in the world.

Remark 7: We have the following laws of the algebra of sets:

1. *Idempotent laws*

1. a. $A \cup A = A$ 1. b. $A \cap A = A$

2. *Associative laws*

2. a. $(A \cup B) \cup C = A \cup (B \cup C)$

2. b. $(A \cap B) \cap C = A \cap (B \cap C)$

3. *Commutative laws*

3. a. $A \cup B = B \cup A$

3. b. $A \cap B = B \cap A$

4. *Distributive laws*

4. a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4. b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

5. *Identity laws*

5. a. $A \cup \emptyset = A$

5. b. $A \cup U = U$

5. c. $A \cap U = A$

5. d. $A \cap \emptyset = \emptyset$

6. *Complement laws*

6. a. $A \cup A' = U$

6. b. $(A')' = A$

6. c. $A \cap A' = \emptyset$

6. d. $U' = \emptyset$, $\emptyset' = U$

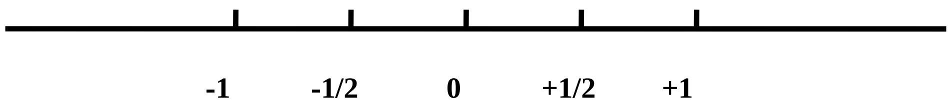
7. *De Morgan's laws*

7. a. $(A \cup B)' = A' \cap B'$

7. b. $(A \cap B)' = A' \cup B'$

II. The Real Number System

1. The real numbers can be geometrically represented by points on a straight line. We would choose a point, called the origin, to represent 0 and another point, to the right, to represent 1. Then it is possible to pair off the points on the line and the real numbers such that each point will represent a unique real number and each real number will be represented by a point. We would call this line the *real line*. Numbers to the right of zero are the *positive numbers* and those to the left of zero are the *negative numbers*. Zero is neither positive nor negative.



We shall denote the set of real numbers and the real line by R.

2. The *integers* are the "whole" real numbers. Let I be the set of integers such that

I = {..., -2, -1, 0, 1, 2, ...}.

The set of integers is *closed* under the operations of *addition*, *subtraction* and *multiplication*. However, I is *not closed* under *division*. By "closed..." we mean that the sum, difference and product of any two integers is also an integer. (This holds for any finite number of integers.)

3. The *rational numbers*, Q, are those real numbers which can be expressed as the *ratio* of two *integers*. Hence,

Def: $Q = \{x \mid x = p/q, p \in I, q \in I, q \neq 0\}$.

Clearly, *each integer is also a rational*, because any $x \in I$ may be expressed as $(x/1) \in Q$. So we have that the set of integers is a subset of the set of rational numbers, i.e.,

$I \subset Q$.

The rational numbers are *closed* under the operations of *addition, subtraction, multiplication and division* (except by zero).

4. The *irrational numbers*, Q', are those real numbers which cannot be expressed as the ratio of two integers. Hence, the irrationals are those real numbers that are not rational. They are precisely the non-repeating infinite decimals. The set of irrationals Q' is just the *complement of the set of rationales* Q in the set of reals (Here we speak of *R as our universal set*). Some examples of irrational numbers are $\sqrt{5}$, $\sqrt{3}$ and $\sqrt{2}$.

5. The *natural numbers* N are the positive integers:

$N = \{x \mid x > 0, x \in I\}$, or

$N = \{1, 2, 3, ...\}$.

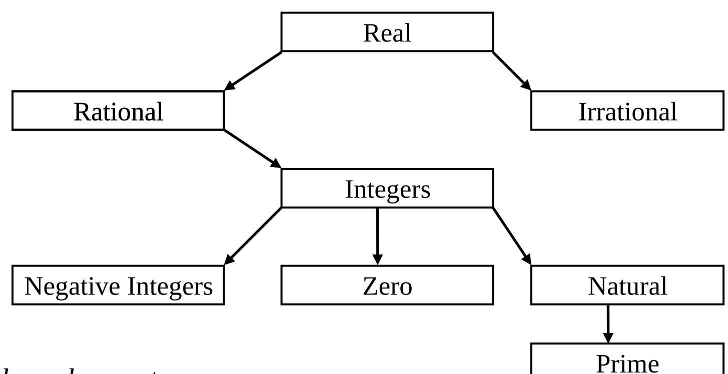
The natural numbers are *closed* under *addition* and *multiplication*.

6. The *prime numbers* are those natural numbers say p, excluding 1, which are divisible (capable of being divided by another number without a remainder) only by 1 and p itself. A few examples are

2, 3, 5, 7, 11, 13, 17, 19, and 23.

The primes are *not closed under addition, subtraction, multiplication or division*.

7. We can illustrate the real number system with the following line diagram.



8. The *extended real number system.*

The set of real numbers R may be extended to include -∞ and +∞. Accordingly we would add to the real line two non-convergent sequences denoted -∞ and +∞. The result would be *the extended real number system* or the *augmented real line*. We denote these by $\hat{R}$. The following operational rules apply.

(i) If "a" is a real number, then $-\infty < a < +\infty$

(ii) $a + \infty = \infty + a = \infty$, if $a \neq -\infty$

(iii) $a + (-\infty) = (-\infty) + a = -\infty$, if $a \neq +\infty$

(iv) If $0 < a \leq +\infty$, then

$a \bullet \infty = \infty \bullet a = \infty$

$a \bullet (-\infty) = (-\infty) \bullet a = -\infty$

(v) If $-\infty \leq a < 0$, then

$a \bullet \infty = \infty \bullet a = -\infty$

$a \bullet (-\infty) = (-\infty) \bullet a = +\infty$

(vi) If "a" is a real number, then $\frac{a}{\infty} = \frac{a}{-\infty} = 0$

9. Absolute Value

Def 1: The *absolute value* of any real number x, denoted |x|, is defined as follows:

$$|x| \equiv \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Remark 1: If x is a real number, its absolute value |x| geometrically represents the distance between the point x and the point 0 on the real line. If a, b are real numbers, then $|a-b| = |b-a|$ would represent the distance between a and b on the real line.

Remark 2: The following properties characterize absolute values:

(i) $|a| \geq 0$,

(ii) $|a| + |b| \geq |a + b|$,

(iii) $|a| \cdot |b| = |a \cdot b|$,

(iv) $\frac{|a|}{|b|} = \left|\frac{a}{b}\right|$, $b \neq 0$,

where $a, b \in R$.

10. *Intervals* on the real line.

Let a, b $\in$ R where a < b, then we have the following terminology:

(i) The set A = {x | a $\leq$ x $\leq$ b}, denoted A = [a, b], is termed a *closed interval* on R. (note that a, b $\in$ A)

(ii) The set B = {x | a < x $\leq$ b}, denoted B = (a, b], is termed an *open-closed interval* on R. (note a $\notin$ B, b $\in$ B)

(iii) The set C = {x | a $\leq$ x < b}, denoted C = [a, b), is termed a *closed-open interval* on R. (note a $\in$ C, b $\notin$ C)

(iv) The set D = {x | a < x < b}, denoted D = (a, b), is termed an *open-interval* on the real line. (note a, b $\notin$ D)

III. Functions, Ordered Tuples and Product Sets

Ordered Pairs and Ordered Tuples

1. If we are given a set {a, b}, we know that {a, b} = {b, a}. As we noted above, order does not matter. We could call {a, b} an *unordered pair*.

2. However, if we were to designate the element “a” as the first listing of the set and the element “b” as the second listing of the set, then we would have an *ordered pair*. We denote this ordering by

(a, b).

Similarly it would be possible to define an *ordered triple* say

(x_1, x_2, x_3),

where the ordered triple is a set with three elements such that x_1, is the first, x_2 the second, and x_3 the third element of the set.

Likewise, an *ordered N-tuple*

$(x_1, x_2, ..., x_n)$

could be defined analogously. An ordered N-tuple of numbers may be given various interpretations. For example, it could represent a vector whose components are the n numbers, or it could represent a point in n-space, the n numbers being the coordinates of the point. (Note (a, b) = (b, a) iff b = a, (a, b) = (c, d) iff a = c, d = b)

The product set

1. Let X and Y be two sets. The *product set* of X and Y or the *Cartesian product* of X and Y consists of all of the possible ordered pairs (x, y), where $x \in X$ and $y \in Y$. That is, we construct every possible ordered pair (x, y) such that the first element comes from X and the second from Y. The product set of X and Y is denoted

$$X \times Y,$$

which reads "X cross Y." More formally we have

Def 1: The *product set* of two sets X and Y is defined as follows:

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

Examples:

#1. If A = {a, b}, B = {c, d}, then A × B = {(a, c), (a, d), (b, c), (b, d)}

#2. If A = {a, b}, B = {c, d, e}, then A × B = {(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)}.

#3 The Cartesian plane or Euclidean two-space, R^2, is formed by

$$R \times R = R^2.$$

Each point x in the plane is an ordered pair $x = (x_1, x_2)$, where x_1 represents the coordinate on the axis of abscissas and x_2 represents the coordinate on the axis of ordinates.

The n-fold Cartesian product of R is Euclidean n-space.

Functions

Def 2: A *function* from a set X into a set Y is a rule f which assigns to every member x of the set X a single member y = f(x) of the set Y. The set X is said to be the *domain* of the function f and the set Y will be referred to as the *codomain* of the function f.

Remark 1: If f is a function from X into Y, we write

$$f : X \rightarrow Y,$$

or

$$X \xrightarrow{f} Y.$$

This notation reads "f is a function from X into Y" or "f maps from X into Y".

Def. 2. a: The element in Y assigned by f to an $x \in X$ is the *value of f at x* or the *image of x under f*.

Remark 2: We denote the value of f at x or the image of x under f by f(x). Hence, the function may be written

$$y = f(x),$$

which reads "y is a function of x", where $y \in Y$, $x \in X$.

Def. 2. b: The *graph* Gr(f) of the function $f : X \rightarrow Y$ is defined as follows:

$$Gr(f) \equiv \{(x, f(x)) : x \in X\},$$

where $Gr(f) \subset X \times Y$.

Def 2. c: The *range* f [X] of the function $f : X \rightarrow Y$ is the set of images of $x \in X$ under f or

$$f[X] \equiv \{ f(x): x \in X\}.$$

Remark 3: Note that the range of a function f is a subset of the codomain of f, that is

$$f[X] \subset Y.$$

Def 3: A function $f: X \rightarrow Y$ is said to be *surjective* ("onto") if and only if

$$f[X] = Y.$$

Def 4: A function f: X → Y is said to be *injective* ("one-to-one") if and only if images of distinct members of the domain of f are always distinct; in other words, if and only if, for any two members x, x′ ∈ X, f(x) = f(x′) implies x = x′.

Def 5: A function f: X → Y is said to be *bijective* ("one-to-one" and "onto") if and only if it is both surjective and injective.

Examples:

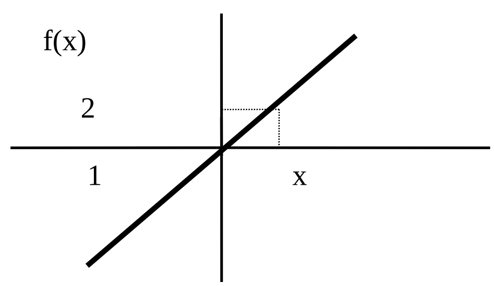
#1 Let the function f assign to every real number its double. Hence, for every real number x,

$$f(x) = 2x,$$

or

$$y = 2x.$$

Both the domain and the codomain of f are the set of real numbers, R. Hence, f: R →R. The image of the real number 2 is f(2) = 4. The range of f is given by f[R] = {2x : x ∈ R}. The Gr(f) is given by G r (f) = {(x, 2x): x ∈ R}. We illustrate a portion of G r (f) ⊂ R × R = R^2.

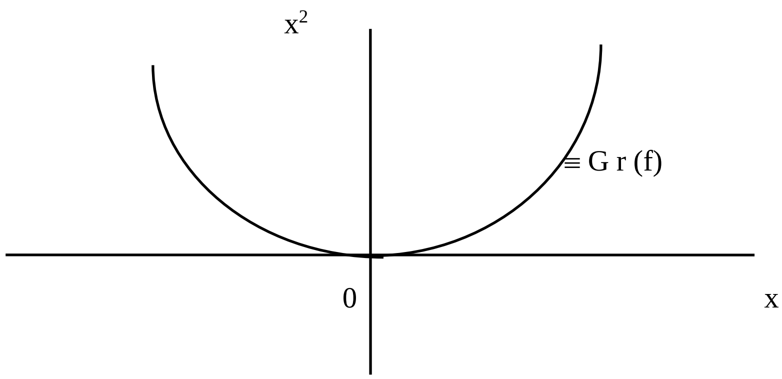


Clearly, the function f(x) = 2x is *injective* ("one-to-one").

Moreover, f(x) = 2x is *surjective* ("onto"), that is f[R] = R.

Hence, by Def 5. f is *bijective*.

#2 Let $f(x) = x^2$, where $f: R \to R$.



(i) The range of f is $f[X] = [0, +\infty)$, while the domain and codomain are each the set of real numbers. It follows that f is not onto or surjective.

(ii) $f(2) = 4$ and $f(-2) = 4$. It follows that f is not injective or one-to-one.

Appendix to Lecture 1: Notes on Logical Reasoning

1. Notation for logical reasoning:

a. $\forall$ means "for all"

b. ~ means "not"

c. $\exists$ means "there exists"

2. A Conditional

Let A and B be two statements.

$A \Rightarrow B$ means all of the following:

- If A, then B
- A implies B
- A is sufficient for B
- B is necessary for A

3. Proving a Conditional: Methods of Proof

a. *Direct*: Show that B follows from A.

b. *Indirect*: Find a statement C where $C \Rightarrow B$. Show that $A \Rightarrow C$.

c. *Contrapositive*: Show that $(\sim B) \Rightarrow (\sim A)$.

d. *Contradiction*: Show that (~ B and A) $\Rightarrow$ (false statement).

3. A Biconditional

Let A and B be two statements.

$A \Leftrightarrow B$ means all of the following:

- A if and only if B (A iff B)
- A is necessary and sufficient for B
- A and B are equivalent
- A implies B and B implies A

4. Proving a Biconditional

Use any of the above methods for proving a conditional and show that $A \Rightarrow B$ and that $B \Rightarrow A$.