Mathematics Review for Business PhD Students

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Lecture 1: Introductory Material

- Sets
- The Real Number System
- Functions, Ordered Tuples and Product Sets
- Appendix to Lecture 1: Notes on Logical Reasoning

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Sets: Basics

- A *set* is a list or collection of objects. The objects which compose a set are termed the *elements* or *members* of the set.
- Tabular versus set builder notation

  A = {1, 2, 3}

  A = {x | x is a positive integer, $1 \leq x \leq 3$}

  "|" may be interchanged with ":"

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Sets: Basics

- The symbol "$\in$" reads "is an element of" . In our example, $2 \in A$.
- If every element of a set S_1 is also an element of a set S_2, then S_1 is a *subset* of S_2 and we write

  $S_1 \subset S_2$ or $S_2 \supset S_1$.

  The set S_2 is said to be a *superset* of S_1.
- *Def 1:* Two sets S_1 and S_2 are said to be *equal* if and only if $S_1 \subset S_2$ and $S_2 \subset S_1$.

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Sets: Basics

- *Examples*:

#1. If $S_1 = \{1, 2\}$, $S_2 = \{1, 2, 3\}$, then $S_1 \subset S_2$ or $S_2 \supset S_1$.

#2 The set of all positive integers is a subset of the set of real numbers.

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Sets: Basics

- The *largest subset* of a set S is the set S itself.
- The *smallest subset* of a set S is the set which contains no elements. The set containing no elements is called the *null set*, denoted $\varnothing$.
- For all sets S, we have $\varnothing \subset S$.
- If all sets are subsets of a given set, we call that set the *universal set*, denoted U.

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Sets: Basics

- *Def 2*: Two sets S_1 and S_2 are *disjoint* if and only if there does not exist an x such that $x \in S_1$ and $x \in S_2$.

- *Example*: If S = {0} and A = {1, 2, 4}, then S and A are disjoint.

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Operations

- *Def 3*: The operations of *union*, *intersection*, *difference* (relative complement), and *complement* are defined for two sets A and B as follows:

(i) $A \cup B \equiv \{x : x \in A \text{ or } x \in B\}$,

(ii) $A \cap B \equiv \{x : x \in A \text{ and } x \in B\}$,

(iii) $A - B \equiv \{x : x \in A, x \notin B\}$,

(iv) $A' \equiv \{x : x \notin A\}$.

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Operational Laws

1. *Idempotent laws*
   1. a. $A \cup A = A$ 1. b. $A \cap A = A$
2. *Associative laws*
   2. a. $(A \cup B) \cup C = A \cup (B \cup C)$
   2. b. $(A \cap B) \cap C = A \cap (B \cap C)$
3. *Commutative laws*
   3. a. $A \cup B = B \cup A$
   3. b. $A \cap B = B \cap A$

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Operational Laws

4. *Distributive laws*
   4. a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   4. b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. *Identity laws*
   5. a. $A \cup \emptyset = A$
   5. b. $A \cup U = U$
   5. c. $A \cap U = A$
   5. d. $A \cap \emptyset = \emptyset$

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Operational Laws

6. *Complement laws*
   - 6. a. $A \cup A' = U$
   - 6. b. $(A')' = A$
   - 6. c. $A \cap A' = \varnothing$
   - 6. d. $U' = \varnothing$, $\varnothing' = U$

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Operational Laws

7. *De Morgan's laws*
   - 7. a. $(A \cup B)' = A' \cap B'$
   - 7. b. $(A \cap B)' = A' \cup B'$

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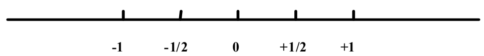
II. The Real Number System

- The real numbers can be geometrically represented by points on a straight line. Numbers to the right of zero are the *positive numbers* and those to the left of zero are the *negative numbers*. Zero is neither positive nor negative.



Real Numbers

- The *integers* are the "whole" real numbers. Let I be the set of integers, so that

  I = {..., -2, -1, 0, 1, 2, ...}.

  The positive integers are called the *natural numbers*.
- The *rational numbers*, Q, are those real numbers which can be expressed as the *ratio* of two *integers*. Hence,

  Def: $Q = \{x \mid x = p/q, p \in I, q \in I, q \neq 0\}$.
- Note that $I \subset Q$.

Real Numbers

- The *irrational numbers*, Q', are those real numbers which cannot be expressed as the ratio of two integers. They are the non-repeating infinite decimals. The set of irrationals is just the *complement of the set of rationales* Q in the set of reals

  Examples: $\sqrt{5}$, $\sqrt{3}$ and $\sqrt{2}$.

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Real Numbers

- The *prime numbers* are those natural numbers say p, excluding 1, which are divisible only by 1 and p itself. A few examples are 2, 3, 5, 7, 11, 13, 17, 19, and 23.

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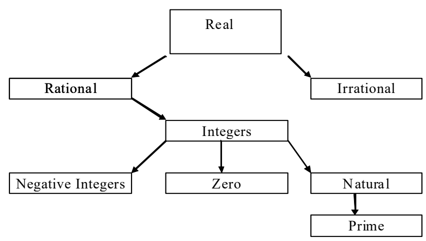
Illustration



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The extended real number system.

- The set of real numbers R may be extended to include -∞ and +∞. These notions mean to become negatively infinite or positively infinite, respectively.
- The result would be *the extended real number system* or the *augmented real line*, $\hat{R}$

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Rules: Extended Real Numbers

The following operational rules apply.

(i) If "a" is a real number, then $-\infty < a < +\infty$

(ii) $a + \infty = \infty + a = \infty$, if $a \neq -\infty$

(iii) $a + (-\infty) = (-\infty) + a = -\infty$, if $a \neq +\infty$

(iv) If $0 < a \leq +\infty$, then

$a \bullet \infty = \infty \bullet a = \infty$

$a \bullet (-\infty) = (-\infty) \bullet a = -\infty$

(v) If $-\infty \leq a < 0$, then

$a \bullet \infty = \infty \bullet a = -\infty$

$a \bullet (-\infty) = (-\infty) \bullet a = +\infty$

(vi) If "a" is a real number, then $a/-\infty = a/+\infty = 0$

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Absolute Value of a Real Number

- *Def 1:* The *absolute value* of any real number x, denoted |x|, is defined as follows:

$$|x| \equiv \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

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Properties of |x|

- If x is a real number, its *absolute value* |x| geometrically represents the distance between the point x and the point 0 on the real line. If a, b are real numbers, then |a-b| = |b-a| would represent the distance between a and b on the real line.

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Properties of |x|

- We have, for a, b $\in$ R
  (i) $|a| \geq 0$
  (ii) $|a| + |b| \geq |a+b|$
  (iii) $|a| \times |b| = |ab|$
  (iv) $|a| / |b| = |a/b|$

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Intervals on R

- Let a, b $\in$ R where a < b, then we have the following terminology:

(i) The set A = {x | a $\leq$ x $\leq$ b}, denoted A = [a, b], is termed a *closed interval* on R. (note that a, b $\in$ A)

(ii) The set B = {x | a < x $\leq$ b}, denoted B = (a, b], is termed an *open-closed interval* on R. (note a $\notin$ B, b $\in$ B)

(iii) The set C = {x | a $\leq$ x < b}, denoted C = [a, b), is termed a *closed-open interval* on R. (note a $\in$ C, b $\notin$ C)

(iv) The set D = {x | a < x < b}, denoted D = (a, b), is termed an *open-interval* on the real line. (note a, b $\notin$ D)

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III. Functions, Ordered Tuples and Product Sets

- *Ordered Pairs and Ordered Tuples:*

An *ordered pair* is a set consisting of two elements with a designated first element and a designated second element. If a,b are the two elements, we write

$$(a, b)$$

An *ordered n-tuple* is the generalization of this idea to n elements

$$(x_1,\ldots,x_n)$$

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Product Set

- Let X and Y be two sets. The *product set* of X and Y or the *Cartesian product* of X and Y consists of all of the possible ordered pairs (x, y), where $x \in X$ and $y \in Y$.

Def 1: The *product set* of two sets X and Y is defined as follows:

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

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Generalization

- The product set of the n sets X_i, i = 1,…,n, is given by

$X_1 \times \bullet\bullet\bullet \times X_n = \{(x_1,\dots,x_n) : x_i \in X_i, i = 1,\dots,n\}$ (n-terms)

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Product Set: Examples

- If A = {a, b}, B = {c, d, e}, then

$A \times B$ = {(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)}.

- The Cartesian plane or Euclidean two-space, R^2, is formed by

$$R \times R = R^2 .$$

- The n-fold Cartesian product of R is Euclidean n-space

$$R \times R \times \ldots \times R = R^n.$$

(n-terms)

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Functions

- *Def 2*: A *function* from a set X into a set Y is a rule f which assigns to every member x of the set X a single member y = f(x) of the set Y. The set X is said to be the *domain* of the function f and the set Y will be referred to as the *codomain* of the function f.

- If f is a function from X into Y, we write

$$f : X \to Y.$$

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Functions

- *Def. 2. a*: The element in Y assigned by f to an $x \in X$ is the *value of f at x* or the *image of x under f.* We write $y = f(x)$.

- Def. 2. b: The *graph* Gr(f) of the function $f : X \rightarrow Y$ is defined as follows:

$$Gr(f) \equiv \{(x, f(x)) : x \in X\},$$

where $Gr(f) \subset X \times Y$.

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Functions

- *Def 2. c*: The *range* f [X] of the function $f : X \rightarrow Y$ is the set of images of $x \in X$ under f or

$$f[X] \equiv \{ f(x): x \in X\}.$$

- Note that the range of a function f is a subset of the codomain of f, that is

$$f[X] \subset Y.$$

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Functions

- *Def 3*: A function $f: X \to Y$ is said to be *surjective* ("onto") if and only if

  $$f[X] = Y.$$

- *Def 4:* A function $f: X \to Y$ is said to be *injective* ("one-to-one") if and only if images of distinct members of the domain of f are always distinct; in other words, if and only if, for any two members $x, x' \in X$, $f(x) = f(x')$ implies $x = x'$.

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Functions

- *Def 5:* A function $f: X \to Y$ is said to be *bijective* ("one-to-one" and "onto") if and only if it is both surjective and injective.

- Examples: $y = 2x$, $y = x^2$

  In the first case, the function is one-to-one and onto and in the second case the function is neither.

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Notes on Logical Reasoning

1. Notation for logical reasoning:
a. $\forall$ means "for all"
b. ~ means "not"
c. $\exists$ means "there exists"

2. A Conditional
Let A and B be two statements. A $\Rightarrow$ B means all of the following:
- If A, then B
- A implies B
- A is sufficient for B
- B is necessary for A

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Notes on Logical Reasoning

3. Proving a Conditional: Methods of Proof
a. *Direct*: Show that B follows from A.
b. *Indirect*: Find a statement C where C $\Rightarrow$ B. Show that A $\Rightarrow$ C.
c. *Contrapositive*: Show that (~B) $\Rightarrow$ (~ A).
d. *Contradiction*: Show that (~ B and A) $\Rightarrow$ (false statement).

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Notes on Logical Reasoning

3. A Biconditional

Let A and B be two statements. A $\Leftrightarrow$ B means all of the following:

- A if and only if B (A iff B)
- A is necessary and sufficient for B
- A and B are equivalent
- A implies B and B implies A

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Notes on Logical Reasoning

4. Proving a Biconditional

Use any of the above methods for proving a conditional and show that A $\Rightarrow$ B and that B $\Rightarrow$ A.

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