

Lecture 2: Matrix Algebra

General

1. A matrix, for our purpose, is a *rectangular array* of *objects* or elements. We will take these elements as being real numbers and indicate an element by its row and column position. A matrix is then an ordered set.

2. Let $a_{ij} \in \mathbb{R}$ denote the *element* of a matrix which occupies the position of the i^{th} row and j^{th} column. The dimension of the matrix is defined or stated by indicating first the number of rows and then the number of columns. We will adopt convention of indicating a *matrix by a capital letter and its elements by the corresponding lower case letter*.

$$\text{Example 1. } \underset{2 \times 2}{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\text{Example 2. } \underset{1 \times 1}{\mathbf{A}} = [a_{11}]$$

$$\text{Example 3. } \mathbf{A}_{n \times n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$\text{Example 4. } \underset{2 \times 3}{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

3. A matrix is said to be (i) *square* if # rows = # columns and a square matrix is said to be (ii) *symmetric* if $a_{ij} = a_{ji} \forall i, j, i \neq j$.

Example. The matrix $\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$ is square but not symmetric, since $a_{21} = 2 \neq 3 = a_{12}$. The square

matrix $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 4 & 3 & 1 \end{bmatrix}$ is symmetric since $a_{12} = a_{21} = 2$, $a_{31} = a_{13} = 4$, and $a_{32} = a_{23} = 3$.

4. The *principle diagonal elements* of a square matrix A are given by the elements a_{ij} , $i = j$. The *principle diagonal* is the ordered n -tuple (a_{11}, \dots, a_{nn}) . The *trace* of a square matrix is defined as the sum of the principal diagonal elements. It is denoted $\text{tr}(A) = \sum_i a_{ii}$.

5. A *diagonal matrix* is a square matrix whose only nonzero elements appear on the principal diagonal.

6. A *scalar matrix* is a diagonal matrix with the same value in all of the diagonal elements.

7. An *identity matrix* is a scalar matrix with ones on the diagonal.

8. A *triangular matrix* is a square matrix that has only zeros either above or below the principal diagonal. If the zeros are above the diagonal, then the matrix is lower triangular and conversely for upper triangular.

Remark: The following notations for indicating an $n \times m$ matrix A are equivalent

$$[a_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,m}} \left(\begin{array}{ccc} \dots & & \\ \vdots & \ddots & \vdots \\ \dots & & \dots \end{array} \right), \left[\begin{array}{ccc} \dots & & \\ \vdots & \ddots & \vdots \\ \dots & & \dots \end{array} \right], \text{or } \parallel \quad \parallel .$$

9. If a matrix A is of dimension $1 \times n$, then it is termed a *row vector*, $A = [a_{11}, \dots, a_{1n}]$. Since there is only one row, the row index is sometimes dropped and A is written $[a_1, \dots, a_n] = \mathbf{a}'$. A matrix A

of dimension $n \times 1$ is termed a *column vector*, $A = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}$. Likewise, since there is only one

column, this is sometimes written as $\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$.

Algebraic Operations on Matrices

1. *Equality.* Two matrices say A and B , $[a_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$, $[b_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$ are said to be equal iff $a_{ij} = b_{ij}$

$\forall i, j$.

2. *Addition and Subtraction.* Take A and B as above with the same dimensions we have

$$A_{n \times m} \pm B_{n \times m} = \begin{bmatrix} a_{11} \pm b_{11} & \cdots & a_{1m} \pm b_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} \pm b_{n1} & \cdots & a_{nm} \pm b_{nm} \end{bmatrix}$$

3. *Scalar Multiplication.* Let $k \in \mathbb{R}$. $k A_{n \times m} = [ka_{ij}]$.

4. *Multiplication.* Two matrices A and B can be multiplied to form AB , only if the column dimension of A = row dimension of B . If this *conformability requirement* is met, then it is possible to define the product AB . In words, the column dimension of the lead matrix must equal the row dimension of the lag matrix, for conformability.

Example If $A_{2 \times 3}$ and $B_{4 \times 2}$, then AB cannot be defined, but $B A_{4 \times 2 \ 2 \times 4}$ can be defined.

In order to precisely present the mechanics of matrix multiplication, let us introduce the idea of an inner (dot) product of two n -tuples of real numbers. Suppose $x, y \in \mathbb{R}^n$. Then the *inner product* of x and y is defined by

$$x \cdot y = \sum_{i=1}^n x_i y_i$$

Note that $x \cdot x = \sum_i x_i^2$ and $x \cdot y = y \cdot x$. That is, the dot product is commutative. Given an $n \times$

m matrix A , let us associate the k^{th} column of A with the ordered n -tuple $a_{0k} = (a_{1k}, \dots, a_{nk})$.

Moreover associate the j^{th} row of A with the ordered m -tuple $a_{j0} = (a_{j1}, \dots, a_{jm})$.

Example. $A_{2 \times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$

$$a_{02} = (2, 4)$$

$$a_{20} = (0, 4, 5)$$

With this notation in hand, consider two matrices $A_{n \times m}$ $B_{m \times k}$ which are conformable for multiplication in the order AB . The *product* AB is then given by

$$A_{n \times m} B_{m \times k} = \begin{bmatrix} a_{10} \cdot b_{01} & \cdots & a_{10} \cdot b_{0k} \\ \vdots & \ddots & \vdots \\ a_{n0} \cdot b_{01} & \cdots & a_{n0} \cdot b_{0k} \end{bmatrix}$$

That is if $AB = C$, then

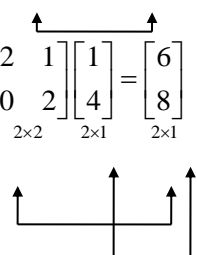
$$c_{jl} = \sum_{i=1}^m a_{ji} b_{il}.$$

Note it must be that $\begin{matrix} AB \\ n \times k \end{matrix}$.

Example 1

$$\begin{aligned} \begin{matrix} A & B \\ 2 \times 2 & 2 \times 2 \end{matrix} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^2 a_{1i} b_{i1} & \sum_{i=1}^2 a_{1i} b_{i2} \\ \sum_{i=1}^2 a_{2i} b_{i1} & \sum_{i=1}^2 a_{2i} b_{i2} \end{bmatrix} = \begin{bmatrix} a_{10} \cdot b_{01} & a_{10} \cdot b_{02} \\ a_{20} \cdot b_{01} & a_{20} \cdot b_{02} \end{bmatrix} \end{aligned}$$

Example 2

$$\begin{aligned} A &= \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}_{2 \times 2} & B &= \begin{bmatrix} 1 \\ 4 \end{bmatrix}_{2 \times 1} \\ AB &= \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}_{2 \times 2} \begin{bmatrix} 1 \\ 4 \end{bmatrix}_{2 \times 1} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}_{2 \times 1} \end{aligned}$$


Example 3. Suppose that A is a $1 \times n$ row vector $A = a' = (a_{11} \ a_{12} \ \dots \ a_{1n})$ and B an $n \times 1$ col vector

$$B = b = \begin{bmatrix} b_{11} \\ \vdots \\ b_{n1} \end{bmatrix}. \text{ Hence, we have}$$

$$a'b = \sum_{i=1}^n a_{1i} b_{i1}.$$

This is a scalar and the operation is termed a *scalar product*. Note that $a'b = a' \bullet b'$. (The scalar product is same as the inner product of 2 row vectors.)

Moreover suppose that $\mathbf{a}' = (a_{11}, \dots, a_{1n})$ and $\mathbf{b} = \begin{bmatrix} b_{11} \\ \vdots \\ b_{m1} \end{bmatrix}$. Then

$$\mathbf{ba}' = \begin{bmatrix} b_{11}a_{11} & \cdots & b_{11}a_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1}a_{11} & \cdots & b_{m1}a_{1n} \end{bmatrix}.$$

5. The operation of addition is both commutative and associative. We have

$$\text{(Com. Law)} \quad \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

$$\text{(Associative)} \quad (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$

The operation of multiplication is not commutative but it does satisfy the associative and distributive laws.

$$\text{(Associative)} \quad (\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

$$\text{(Distributive)} \quad \mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$$

$$(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$$

To see that $\mathbf{AB} \neq \mathbf{BA}$ consider the example $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

$$\mathbf{AB} = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

6. Generally, when we take the product of a matrix and a vector, we can write the result as

$$\mathbf{c} = \mathbf{Ab}.$$

In this example, the matrix \mathbf{A} is n by n and the column vectors \mathbf{c} and \mathbf{b} are n by 1 . This product can be interpreted in two different ways. Taking the case of a 2×2 matrix \mathbf{A} , we have

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}.$$

First, this can be a compact way of writing the two equations

$$1 = a + 3b$$

$$4 = 3a + 2b.$$

Alternatively, we can write the relationship as a linear combination of the columns of A

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + b \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

In the general case where A is $n \times n$, we have

$$c = Ab$$

$$= b_1 \mathbf{a}_1 + \dots + b_n \mathbf{a}_n,$$

where \mathbf{a}_i is the i th column of A. Further, in the product $C = AB$, each column of the matrix C is a linear combination of the columns of A where the coefficients are the elements in the corresponding columns of B. That is,

$$C = AB \text{ if and only if } \mathbf{c}_i = A\mathbf{b}_i.$$

7. Transpose of a Matrix. The *transpose* of a matrix A, denoted A' , is the matrix formed by interchanging the rows and columns of the original matrix A.

Example 1. Let $A = (1 \ 2)$ then $A' = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Example 2. Let $A_{3 \times 2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, then $A'_{2 \times 3} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

Properties :

(i) $(A')' = A$ (obvious)

(ii) $(A + B)' = A' + B'$

Proof: Let $A + B = C$, then $c_{ij} = a_{ij} + b_{ij}$. Let c'_{ij} denote an element of C' .

Clearly, $c'_{ij} = c_{ji} = a_{ji} + b_{ji}$. Let a'_{ij}, b'_{ij} be elements of A' and B' respectively such that

$$a'_{ij} = a_{ji} \text{ and } b'_{ij} = b_{ji}$$

$$c'_{ij} = a_{ji} + b_{ji} = a'_{ij} + b'_{ij}.$$

Thus, the elements of C' and $A' + B'$ are identical.

(iii) $(AB)' = B'A'$

Proof: Let A , B then $A_{n \times m} B_{m \times k} = \begin{bmatrix} a_{10} \cdot b_{01} & \cdots & a_{10} \cdot b_{0k} \\ \vdots & \ddots & \vdots \\ a_{n0} \cdot b_{01} & \cdots & a_{n0} \cdot b_{0k} \end{bmatrix}$. Then

$$(AB)' = \begin{bmatrix} a_{10} \cdot b_{01} & \cdots & a_{n0} \cdot b_{01} \\ \vdots & \ddots & \vdots \\ a_{10} \cdot b_{0k} & \cdots & a_{n0} \cdot b_{0k} \end{bmatrix}. \text{ We have that } B'_{k \times m} = \begin{bmatrix} b_{01}' \\ \vdots \\ b_{0k}' \end{bmatrix} \text{ and } A'_{m \times n} = [a_{10}' \quad \cdots \quad a_{n0}'].$$

$$\text{Thus, } B'A' = \begin{bmatrix} b_{01}' \cdot a_{10}' & \cdots & b_{01}' \cdot a_{n0}' \\ \vdots & \ddots & \vdots \\ b_{0k}' \cdot a_{10}' & \cdots & b_{0k}' \cdot a_{n0}' \end{bmatrix}. \parallel$$

7. The Identity and Null Matrices.

- a. An *identity matrix* is a square matrix with ones in its principle diagonal and zeros elsewhere. An $n \times n$ identity matrix is denoted I_n .

Properties:

- (i) Let A be $n \times p$. Then we have $I_n A = A I_p = A$.

Proof: Exercise

- (ii) Let A be $n \times p$ and B be $p \times m$. Then we have

$$A I_p B = (A I_p) B = AB.$$

- (iii) $I_n \cdot I_n \cdot I_n \cdots I_n = I_n$. The product of any number of identity

matrices is the identity matrix. In general, a matrix is termed

idempotent, when it satisfies the property $AA = A$.

- b. The *null matrix*, denoted $[0]$ is a matrix whose elements are all zero. Subject to dimensional conformability we have

Properties:

$$(i) \quad A + [0] = [0] + A = A$$

$$(ii) \quad [0]A = A[0] = [0].$$

Proofs: Exercise

Remark : If $AB = [0]$, it need not be true that $A = [0]$ or $B = [0]$.

Example.

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$$

It is easy to show that $AB = [0]$.

8. Sums of Values.

a. Let \mathbf{i} represent a column vector of ones. Then the sum of the elements of any vector x is given by

$$\sum_{i=1}^n x_i = \mathbf{i}'x.$$

b. If all of the elements of x are the same and equal to k , then $x = k\mathbf{i}$. Further,

$$\sum_{i=1}^n x_i = \mathbf{i}'k\mathbf{i} = k\mathbf{i}'\mathbf{i} = kn.$$

c. Obviously, if a is a constant, then

$$a \sum_{i=1}^n x_i = \sum_{i=1}^n ax_i = a\mathbf{i}'x.$$

Setting $a = 1/n$, we obtain the simple mean

$$\bar{x} = \frac{1}{n} \mathbf{i}'x.$$

9. Determinants and Related Concepts.

a. A determinant is defined only for square matrices. When taking the determinant of a matrix we attach a sign + or - to each element:

$$\text{sign attached to } a_{ij} = \text{sign } (-1)^{i+j}.$$

Thus, for example, sign $a_{12} = -$, sign of $a_{43} = -$, and sign $a_{13} = +$.

b. The determinant of a scalar x , $|x|$, is the matrix itself. The determinant of a 2×2 matrix A , denoted $|A|$ or $\det A$, is defined as follows:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = +1(a_{11})(a_{22}) - (1)(a_{21}) \cdot a_{12}.$$

Example $A = \begin{bmatrix} 3 & 6 \\ -5 & 6 \end{bmatrix}$. $|A| = 3 \cdot 6 - (-5)(6) = 18 + 30 = 48.$

c. The determinant of an arbitrary $n \times n$ ($n \geq 2$) matrix A can be found via the *Laplace Expansion* process. In order to introduce this process, let us consider some preliminary definitions. Let $A_{n \times n}$.

Definition. The *minor* of the element a_{ij} , denoted $|M_{ij}|$, is the determinant of the submatrix formed by deleting the i^{th} row and j^{th} column.

Example 1. Let A be 2×2 , $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. $|M_{11}| = |a_{22}| = a_{22}$. Moreover

$$|M_{12}| = a_{21}, |M_{21}| = a_{12} \text{ and } |M_{22}| = a_{11}.$$

Example 2. Let A be 3×3 .

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. |M_{13}| = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}.$$

Definition. The *cofactor* of the element a_{ij} denoted $|C_{ij}|$ is given by

$$(-1)^{i+j} |M_{ij}|.$$

Example Let A be 3×3 . Then $|C_{21}| = -1 \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = -1(a_{12}a_{33} - a_{32}a_{13})$.

Definition. The *principle minor* of the principle diagonal element a_{ii} , denoted $|PM_i|$ is the determinant of the submatrix formed by retaining only the first i rows and first i columns. The order of $|PM_i|$ is its row = col. dimension.

Example

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|PM_1| = a_{11} \quad |PM_2| = a_{11}a_{22} - a_{21}a_{12} \quad |PM_3| = |A|.$$

10. *Laplace Expansion:* Let A be $n \times n$. Then

$$|A| = \sum_{i=1}^n a_{ij} |C_{ij}| \quad (\text{expansion by } j^{\text{th}} \text{ col}) \quad |A| = \sum_{j=1}^n a_{ij} |C_{ij}| \quad (\text{expansion by } i^{\text{th}} \text{ row})$$

Note that eventually cofactors degenerate to the 2×2 case.

Example 1. 3×3 . Expansion by 2nd col.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$$|A| = \sum_{i=1}^3 a_{i2} C_{i2} = -a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{22}(a_{11}a_{33} - a_{31}a_{13}) - a_{32}(a_{11}a_{23} - a_{21}a_{13}).$$

Next consider expansion via the 3rd row.

$$|A| = \sum_{j=1}^3 a_{3j} c_{3j} = a_{31}(a_{12}a_{23} - a_{22}a_{13}) - a_{32}(a_{11}a_{23} - a_{21}a_{13}) + a_{33}(a_{11}a_{22} - a_{21}a_{12}).$$

Let's check the two terms to see if they are equal. The middle term of the second expression is the same as the last term of the first expression. Checking the remaining two terms, we have the following. In the first case $-a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{22}a_{11}a_{33} - a_{22}a_{31}a_{13}$. In the second case $a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} + a_{33}a_{11}a_{22} - a_{33}a_{21}a_{12}$. Thus, they are the same.

Example 2. A is 3×3 and given by

$$\begin{bmatrix} 0 & 4 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \text{ In this case it is easiest to expand via the first col.}$$

$$|A| = -2 \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -2(4-1) + 1(4-1) = -2(3) + 3 = \underline{\underline{-3}}.$$

11. Properties of Determinants.

(i) $|A| = |A'|$

(ii) The interchange of any two rows (or two col.) will change the sign of the determinant, but will not change its absolute value.

Examples of (i) and (i)

#1 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$|A| = -2$ $|A'| = -2$

#2 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

$|A| = -2$ $|B| = +2$

(iii) The multiplication of any p rows (or col) of a matrix by a scalar k will change the value of the determinant to $k^p|A|$.

(iv) The addition (subtraction) of any multiple of any row to (from) another row will leave the value of the determinant unaltered, if the linear combination is placed in the initial (the transformed) row slot. The same holds true if we replace the word “row” by column.

(v) If one row (col) is a multiple of another row (col), the value of the determinant will be zero.

(vi) If A and B are square, then $|AB| = |A||B|$.

Vector Spaces, Linear Independence and Rank of a Matrix

1. As pointed out above, an $n \times 1$ matrix is termed a col vector and a $1 \times n$ matrix is called a row vector. In general we have

Def. An n -component vector a is an ordered n tuple of real numbers written as a row vector a' or a column vector a . The a_i for, $i = 1, \dots, n$, are termed the components of the vector.

The elements of such a vector can be viewed as the coordinates of a point in R^n or as the definition of the line segment connecting the origin and this point.

2. R^n can be defined as the collection of all vectors $a' = (a_1, \dots, a_n)$. It is the n -fold product of R .

3. The two basic operations defined for vectors are scalar multiplication and addition. Recall that

for a vector a , $ka = \begin{bmatrix} ka_1 \\ \cdot \\ \cdot \\ \cdot \\ ka_n \end{bmatrix}$, and that $a + b = \begin{bmatrix} a_1 + b_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n + b_n \end{bmatrix}$. The set of all possible scale multiples of

a vector a is the line through the $n \times n$ zero vector and a . Any given scale multiple of a is a segment of this line. We can illustrate these concepts geometrically.

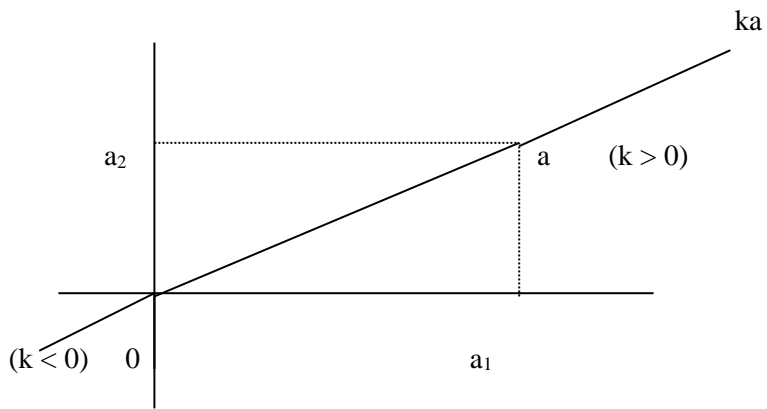


Figure 1

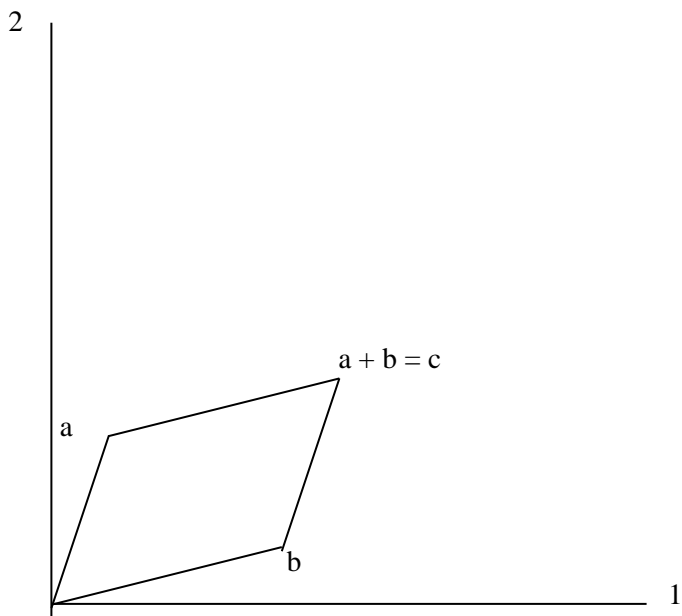


Figure 2

To find $c = a + b$ geometrically, we move "a" parallel to the tip of "b" or conversely.

A *vector space* is a collection of vectors that is closed under the operations of addition and scalar multiplication. Clearly \mathbb{R}^n is a vector space.

4. Linear Combinations of Vectors and Basis Vectors

a. *Def.* A set of vectors span a vector space if any vector in that space can be written as a linear combination of the vectors in that set.

b. A set of vectors spanning a vector space which contains the smallest number of vectors is called a basis. This set must be linearly independent. If the set were dependent, then some one could be expressed as a linear combination of the others and it could be eliminated. In this case we would not have the smallest set.

Def. A set of vectors $a^1, \dots, a^m \in \mathbb{R}^n$ is said to be *linearly dependent* if there exist scalars $\lambda^i \in \mathbb{R}$, not all zero such that

$$\lambda^1 a^1 + \lambda^2 a^2 + \dots + \lambda^m a^m = 0^n.$$

If the only set of λ^i for which the above holds is $\lambda^i = 0$, for all i , then the vectors $a^1, \dots, a^m \in \mathbb{R}^n$ are said to be *linearly independent*.

5. From this definition we can derive the following result:

Proposition 1. The vectors a^1, \dots, a^m from \mathbb{R}^n are linearly dependent iff some one of the vectors is a linear combination of the others.

Proof Suppose one vector, say a^1 , is a linear combination of the others. Then $a^1 = \lambda^2 a^2 + \dots + \lambda^m a^m$. Thus, $-a^1 + \lambda^2 a^2 + \dots + \lambda^m a^m = 0$, with $\lambda^1 = -1 \neq 0$. Hence, the set is linearly dependent.

Suppose the set is linearly dependent. Then the above condition is satisfied and $\exists \lambda^k \neq 0$. Thus,

$$a^k = \frac{\lambda^1}{-\lambda^k} a^1 + \dots + \frac{\lambda^{k-1}}{-\lambda^k} a^{k-1} + \frac{\lambda^{k+1}}{-\lambda^k} a^{k+1} + \dots + \frac{\lambda^m}{-\lambda^k} a^m.$$

and a^k has been expressed as a linear combination of the other vectors. ||

Remark 1. If the set of vectors under consideration has but one member $a \in \mathbb{R}^n$, then a is linearly dependent if $a = 0$ and a is linearly independent, if $a \neq 0$. Here, linear dependence means $\exists \lambda \neq 0$ such that $\lambda a = 0 \Rightarrow a = 0$. Now, if a is not linearly dependent we have that the only λ for which $\lambda a = 0$ is $\lambda = 0$. Thus, $\forall \lambda \neq 0 \lambda a \neq 0$ and $a \neq 0$.

Proposition 2. No set of linearly independent vectors can contain the zero vector.

Proof: Suppose that $a^1 = 0^n$. Set $\lambda^2 = \dots = \lambda^m = 0$ and $\lambda^1 = 1$. Then

$$\lambda^1 a^1 + \lambda^2 a^2 + \dots + \lambda^m a^m = 0^n, \text{ and the set is linearly dependent. } \parallel$$

Proposition 3. Any subset of a set of linearly independent vectors is linearly independent.

Proof: Suppose a subset of a linearly independent set, a^1, \dots, a^m , is linearly dependent. Let this subset be a^1, \dots, a^k . Then $\exists \lambda^1, \dots, \lambda^k$ not all zero such that

$$\lambda^1 a^1 + \lambda^2 a^2 + \dots + \lambda^k a^k = 0^n.$$

Set $\lambda^{k+1}, \dots, \lambda^m = 0$. Then $\lambda^1 a^1 + \lambda^2 a^2 + \dots + \lambda^m a^m = 0^n, \lambda^i$ not all zero and we contradict linear independence. \parallel

Proposition 4. Any superset of a set of linearly dependent vectors is linearly dependent.

Proof: Suppose that a subset of a^1, \dots, a^m is linearly dependent. Let this subset be given by a^1, \dots, a^k . Then $\exists \lambda^1, \dots, \lambda^k$ not all zero such that $\lambda^1 a^1 + \lambda^2 a^2 + \dots + \lambda^k a^k = 0^n$. Set $\lambda^{k+1}, \dots, \lambda^m = 0$ and the result follows. \parallel

We then have the following definition for the basis of a vector space:

Def. A basis for a vector space of n dimensions is any set of n linearly independent vectors in that space.

Remark: Can you see why this definition is equivalent to the one given above? In \mathbb{R}^n exactly n independent vectors can form a basis for that space. That is, it takes n independent vectors to create any other vector in \mathbb{R}^n through a linear combination.

Example. Let a, b be linearly independent in \mathbb{R}^2 . Let c be any third vector. We can show that c can be created from a linear combination of a and b . We only need to select λ^1 and λ^2 such that

$\lambda^1 a + \lambda^2 b = c$. That is we wish to find $\lambda^1 a_i + \lambda^2 b_i = c_i$, for $i = 1, 2$. Solving these equations for the λ^i we have

$$\lambda^1 = (b_2 c_1 - b_1 c_2) / (b_2 a_1 - b_1 a_2) \text{ and } \lambda^2 = (a_1 c_2 - c_1 a_2) / (a_1 b_2 - b_1 a_2).$$

It is possible to solve for the λ^i if $(a_1 b_2 - b_1 a_2) \neq 0$. This is true if $a_1/a_2 \neq b_1/b_2$. That is, a can not be a scale multiple of b .

6. *Def.* The *rank* of an $n \times m$ matrix A , $r(A)$, is defined as the largest # of linearly independent columns or rows.

Proposition 1. Given an $n \times m$ matrix A , we have

(i) $r(A) \leq \min\{n, m\}$

(ii) largest # lin indep. col. = largest # lin indep. rows.

Proposition 2. The rank, $r(A)$ of an $m \times n$ matrix A is equal to the order of the largest submatrix of A whose determinant is nonzero. (By submatrix we mean a matrix selected from A by taking out rows and columns of A .)

Remark 1. If A is $n \times n$ and $r(A) = n$ then $|A| \neq 0$ and the n rows or n col each form a set of linearly independent vectors. Moreover if $|A| \neq 0$, then, from Proposition 1, there are n linearly independent rows (col) in A . We have

$$|A|_{n \times n} \neq 0 \Leftrightarrow \text{rows (col) of } A \text{ are lin indep.} \Leftrightarrow r(A) = n.$$

Example. Find the rank of $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 6 & 10 \\ 5 & 1 & 0 & 0 \end{bmatrix}$. You should obtain a rank of 2.

Subspaces

1. The set of all linear combinations of a set of vectors is called the vector space *spanned* by those vectors.

2. As an example, the space spanned by a basis for \mathbb{R}^n is \mathbb{R}^n . Moreover, if a, b , and c are a basis for \mathbb{R}^3 and d is a fourth vector in \mathbb{R}^3 , then the space spanned by a, b, c, d is \mathbb{R}^3 . Obviously, d is superfluous.

3. Consider two vectors $a, b \in \mathbb{R}^3$, where $a_3 = b_3 = 0$. It is clear that a and b can not span \mathbb{R}^3 , because all linear combinations of a and b will have a third coordinate equal to zero. While a and b do not span \mathbb{R}^3 , they do span that subspace of \mathbb{R}^3 , namely the set of all vectors in \mathbb{R}^3 which have a zero third coordinate. This subspace is a plane in \mathbb{R}^3 and it is called a two dimensional subspace of \mathbb{R}^3 . Generally, the space spanned by a set of vectors in \mathbb{R}^n has at most n dimensions. If this space has less than n dimensions, it is called a subspace of \mathbb{R}^n or a hyperplane in \mathbb{R}^n .

Inverse Matrix

1. *Def.* Given an $n \times n$ square matrix A , the *inverse matrix* of A , denoted A^{-1} , is that matrix which satisfies

$$A^{-1}A = AA^{-1} = I_n.$$

When such a matrix exists, A is said to be *nonsingular*. If A^{-1} exists it is unique.

Theorem. An $n \times n$ matrix A is nonsingular iff $r(A) = n$.

Remark. Now we have the following equivalence. Let A be $n \times n$

$$\left| A \right|_{n \times n} \neq 0 \Leftrightarrow \text{rows (col) of } A \text{ lin. indep.} \Leftrightarrow r(A) = n \Leftrightarrow A^{-1} \text{ exists.}$$

2. Computing the Inverse.

a. Let us begin by assuming that the matrix we wish to invert is an $n \times n$ matrix A with $|A| \neq 0$.

b. *Def.* The *cofactor matrix* of A is given by

$$C = \left[[C_{ij}] \right].$$

Def. The *adjoint matrix* of A is given by $\text{adj } A = C'$.

c. *Computation of Inverse:* $A^{-1} = \frac{\text{adj } A}{|A|}$.

Example: Let $A = \begin{bmatrix} 1 & 3 \\ 9 & 2 \end{bmatrix}$ $|A| = 2 - 27 = -25$, $C = \left[[C_{ij}] \right] = \begin{bmatrix} 2 & -9 \\ -3 & 1 \end{bmatrix}$,

$$\text{adj } A = \begin{bmatrix} 2 & -3 \\ -9 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-25} \begin{bmatrix} 2 & -3 \\ -9 & 1 \end{bmatrix} = \begin{bmatrix} -2/25 & 3/25 \\ 9/25 & -1/25 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 3 \\ 9 & 2 \end{bmatrix} \begin{bmatrix} -2/25 & 3/25 \\ 9/25 & -1/25 \end{bmatrix} = \begin{bmatrix} -\frac{2}{25} + \frac{27}{25} & \frac{3}{25} - \frac{3}{25} \\ -\frac{18}{25} + \frac{18}{25} & \frac{27}{25} - \frac{2}{25} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

3. Key Properties of the Inverse Operation

a. $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: $B^{-1}A^{-1}AB = B^{-1}IB = I$ and $ABB^{-1}A^{-1} = I$. It follows that $B^{-1}A^{-1}$ is the inverse of AB . ||

b. $(A^{-1})^{-1} = A$.

Proof: $AA^{-1} = I$ and $A^{-1}A = I$. Thus, the result holds. ||

c. $(A')^{-1} = (A^{-1})'$.

Proof: $AA^{-1} = A^{-1}A = I$. Transposing and noting that $I' = I$, we have $(A^{-1})'A' = I = A'(A^{-1})'$. ||

d. $I^{-1} = I$.

Proof: $II = I$.

4. a. Note that $AB = 0$ does not imply that $A = 0$ or that $B = 0$. If either A or B is nonsingular and $AB = 0$, then the other matrix is the null matrix. That is, the product of two non-singular matrices cannot be null.

Proof: Let $|A| \neq 0$ and $AB = 0$. Then $A^{-1}AB = B = 0$. ||

b. For square matrices, it can be shown that $|AB| = |A||B|$, so that, in this case, $|AB| = 0$ if and only if $|A| = 0$, $|B| = 0$, or both.

Linear Equation Systems, the Inverse Matrix and Cramer's Rule.

1. Consider an equation system with n unknowns x_i , $i = 1, \dots, n$. In matrix notation this system can be written as

$$Ax = d,$$

where $A = [a_{ij}]_{\substack{i=1, \dots, n \\ j=1, \dots, n}}$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $d = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$. If $|A| \neq 0$, A^{-1} exists and we can write

$$A^{-1}Ax = A^{-1}d$$

$$I_n x = A^{-1}d$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A^{-1}d.$$

Thus, if there is no linear dependence in the rows or columns of the coefficient matrix we can obtain a solution to the equation system. Since A^{-1} is unique if it exists, this solution is unique. Hence, an easy way to test for existence and uniqueness of a solution to a set of linear equations is to determine whether the coefficient matrix has a nonvanishing determinant.

2. This solution gives us values of the solution variables, in terms of A^{-1} , in vector form. A formula known as *Cramer's Rule* gives explicit solutions for each x_i . If $|A| \neq 0$, we have

$$x = A^{-1}d = \frac{1}{|A|} [\text{adj}A]d = \frac{1}{|A|} \begin{bmatrix} |C_{11}| & \cdots & |C_{n1}| \\ \vdots & \ddots & \vdots \\ |C_{1n}| & \cdots & |C_{nn}| \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix},$$

$$x = \frac{1}{|A|} \begin{bmatrix} d_1|C_{11}| & \cdots & d_n|C_{n1}| \\ \vdots & \ddots & \vdots \\ d_1|C_{1n}| & \cdots & d_n|C_{nn}| \end{bmatrix},$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} \sum_{i=1}^n |C_{i1}|d_i \\ \vdots \\ \sum_{i=1}^n |C_{in}|d_i \end{bmatrix}$$

Thus,

$$x_j = \frac{1}{|A|} \sum_{i=1}^n |C_{ij}|d_i.$$

Consider the term $\sum_{i=1}^n |C_{ij}| d_i$. Recall $|A| = \sum_{i=1}^n a_{ij} |C_{ij}|$ (expansion by j th col.). Thus

$$\sum_{i=1}^n d_i |C_{ij}| = \begin{vmatrix} a_{11} & \dots & d_1 & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & d_n & \dots & a_{nn} \end{vmatrix} \equiv |A_j|.$$

Using this notation, we obtain for $j = 1, \dots, n$, $x_j = \frac{|A_j|}{|A|}$. (Cramer's Rule)

Remark. This method does not involve computation of A^{-1} .

Example

$$3x_1 + 4x_2 = 10$$

$$6x_1 + 1x_2 = 20$$

$$\begin{bmatrix} 3 & 4 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{10 - 80}{3 - 24} = \frac{-70}{-21} = \frac{70}{21}$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{60 - 60}{-21} = 0.$$

Let's check this by computing A^{-1}

$$C = \begin{bmatrix} 1 & -6 \\ -4 & 3 \end{bmatrix} \Rightarrow C' = \text{adj}A = \begin{bmatrix} 1 & -4 \\ -6 & 3 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{21} \begin{bmatrix} 1 & -4 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} -1/21 & 4/21 \\ 6/21 & -3/21 \end{bmatrix}$$

Check

$$A^{-1}A = \begin{bmatrix} -\frac{1}{21} & \frac{4}{21} \\ \frac{6}{21} & -\frac{3}{21} \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{21} + \frac{24}{21} & -\frac{4}{21} + \frac{4}{21} \\ \frac{18}{21} - \frac{18}{21} & \frac{24}{21} - \frac{3}{21} \end{bmatrix} = I$$

$$x = \begin{bmatrix} -\frac{1}{21} & \frac{4}{21} \\ \frac{6}{21} & -\frac{3}{21} \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{10}{21} + \frac{80}{21} \\ \frac{60}{21} - \frac{60}{21} \end{bmatrix} = \begin{bmatrix} \frac{70}{21} \\ 0 \end{bmatrix}.$$

Characteristic Roots and Vectors

1. Let D be an $n \times n$ matrix. Does there exist a scalar r and an $n \times 1$ vector $x \neq 0$ such that

$$Dx = rx?$$

If so, then r is said to be a *characteristic root of D* . Rewriting,

$$(Dx - rx) = 0, \text{ or}$$

$$(*) \quad [D - rI]x = 0.$$

x is called a *characteristic vector of D* . Clearly, if x is a solution vector, then so is kx for any value of k . To remove the indeterminacy, x is normalized so that $x'x = 1$. The solution then consists of r and the n unknown elements of x .

2. Equation (*) defines the matrix $[D - rI]$. This is called the *characteristic matrix of D* . For (*) to be true, it is necessary that $|D - rI| = 0$, given that $x \neq 0$. (To see that this is true, let $A = [D - rI]$ and suppose to the contrary that $Ax = 0$, $x \neq 0$, and $|A| \neq 0$. Then $A^{-1}Ax = 0$ and $x = 0$, so that we have a contradiction.) This condition is called the *characteristic equation of D* :

$$(**) \quad |D - rI| = 0.$$

3. The characteristic equation is an n^{th} degree polynomial in r which has n roots. If D is symmetric, then these roots are real numbers.

4. An Example.

$$\text{Let } D = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}. \quad [D - rI] = \begin{bmatrix} 2-r & 2 \\ 2 & -1-r \end{bmatrix}. \quad \text{Taking the determinant of the latter and}$$

setting it equal to zero, we have

$$(2 - r)(-1 - r) - 4 = 0.$$

Whence,

$$r^2 - r - 6 = 0.$$

This is a quadratic in r (2^{nd} degree polynomial). It has the solution¹

¹ Note that $r_1, r_2 = [-b/2a] \pm [(b^2 - 4ac)^{1/2}/2a]$, for $ar^2 + br + c = 0$.

$$r_1, r_2 = \frac{1 \pm (1 - 4 \cdot 1 \cdot -6)^{1/2}}{2} = 1/2 \pm 5/2 = 3, -2.$$

Given that $|D - rI| = 0$, it is clear that there are an infinity of x satisfying (*), for each characteristic root. We can obtain a unique characteristic vector by normalizing as $\sum_{i=1}^n x_i^2 = 1$, for each root. Going back to the example, we have, for the first root $r_1 = 3$,

$$[D - rI] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 2x_2 = 0$$

$$2x_1 - 4x_2 = 0.$$

Note that equation 1 is just a multiple of equation 2. They are not independent as expected. All that we can conclude from these is that

$$(1) \quad x_1 = 2x_2.$$

If we impose the normalization constraint

$$(2) \quad x_1^2 + x_2^2 = 1,$$

then (1) and (2) give us two equations in two unknowns. Solving²

$$(2x_2)^2 + x_2^2 = 1$$

$$x_1 = 2/(5)^{1/2} \text{ and } x_2 = 1/(5)^{1/2}.$$

The characteristic vector under this normalization is written as $v^1 = (v_1^1, v_2^1) = ((2/(5)^{1/2}), 1/(5)^{1/2})$.

Using the same method for $r_2 = -2$ (In this case, $x_2 = -2x_1$ from the characteristic equation), we can show that $v^2 = (-1/(5)^{1/2}, 2/(5)^{1/2})$. Figure 1 illustrates this procedure. The v^i are taken on the unit circle.

² We take the positive root for each x_i or the negative root for each x_i .

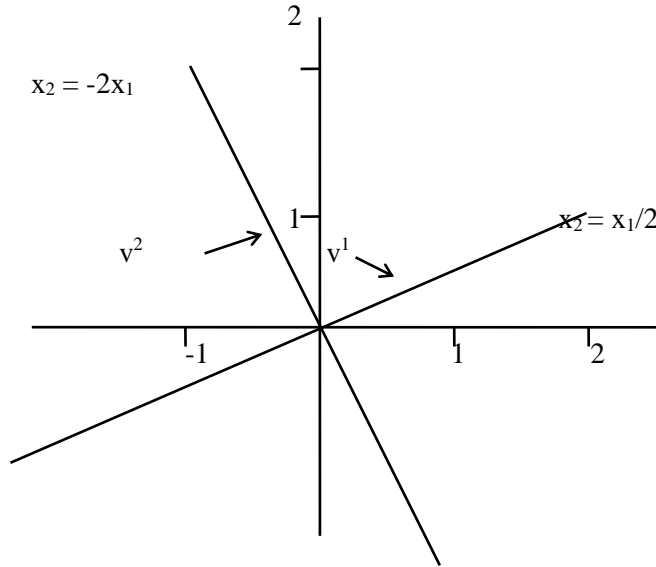


Figure 1

5. General Results for Characteristic Roots and Vectors

- Characteristic roots of a symmetric matrix are real, but need not be distinct.
- For a symmetric matrix, characteristic vectors corresponding to distinct characteristic roots are pairwise orthogonal.³ If the characteristic roots of a symmetric matrix $n \times n$ are distinct, then they form a basis (orthonormal basis) for \mathbb{R}^n . We have

$$v^i \cdot v^i = 1 \text{ and } v^j \cdot v^i = 0.$$

It is conventional to form the following matrix of characteristic vectors corresponding to A

$$Q = [v^1 \dots v^n].$$

It is clear that $Q'Q = I$, so that $Q' = Q^{-1}$. Generally, we have

Def. The square matrix A is *orthogonal* if $A^{-1} = A'$.

Thus, the matrix of characteristic vectors is orthogonal. From the characteristic equation, we have that

³ If the roots are repeating, then it is still possible to find orthogonal characteristic vectors. We ignore this detail here and assume distinct roots.

$$AQ = QR, \text{ where } R \equiv \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 \\ 0 & . & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & r_n \end{bmatrix}.$$

To see this, note that $AQ = [Av^1 \dots Av^n] = [r_1v^1 \dots r_nv^n] = QR$. It follows that

$$(*) \quad Q'AQ = Q'QR = R.$$

Condition (*) is called the *diagonalization of A*. That is, we have found a matrix Q such that the transformation $Q'AQ$ produces a diagonal matrix. In this case, the matrix diagonalizing A is the corresponding matrix of characteristic vectors and the diagonal matrix is the matrix with A's characteristic roots along the diagonal. It is always possible to diagonalize a symmetric matrix in this way. As an exercise, it is useful to work through this process for the numerical example provided above.

c. For a square matrix A, we have

- i. The product of the characteristic roots is equal to the determinate of the matrix.
- ii. The rank of A is equal to the number of nonzero characteristic roots.
- iii. The characteristic roots of A^2 are the squares of the characteristic roots A, but the characteristic vectors of both matrices are the same.
- iv. The characteristic roots of A^{-1} are the reciprocal of the characteristic roots of A, but the characteristic vectors of both matrices are the same.

General Results on the Trace of a Matrix

1. We defined the *trace* of a square matrix as the sum of the diagonal elements.
2. The following results are easily shown.
 - a. $\text{tr}(cA) = c(\text{tr}(A))$.
 - b. $\text{tr}(A') = \text{tr}(A)$.
 - c. $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$.
 - d. $\text{tr}(I_k) = k$.

e. $\text{tr}(AB) = \text{tr}(BA)$.

Remark: The product rule can be extended to any cyclic permutation in a product.

$$\text{tr}(ABCD) = \text{tr}(BCDA) = \text{tr}(CDAB) = \text{tr}(DABC).$$

3. It can also be shown that the trace of a matrix equals the sum of its characteristic roots.

Quadratic Forms

A *quadratic form* is a homogeneous polynomial of the second degree. It takes the form $x'Ax$, where A is symmetric and n by n , and x is n by 1 . We have $x'Ax = \sum \sum a_{ij}x_i x_j$. Now $x'Ax$ is termed *negative definite* if it is negative for all $x \neq 0^n$. The form and the matrix are termed *negative definite* in this case. The matrix A defining the quadratic form is called the *discriminate* of the quadratic form. The definitions for positive definite is analogous with the inequality sign reversing.

The condition given above is not computationally convenient for determining that a form is definite. However, there are more convenient equivalent conditions. The first involves the principal minors of A .

Proposition 1. A and its quadratic form are *negative definite* if and only if principal minors of order i are of sign $(-1)^i$.

Proposition 1 states that

$$|a_{11}| < 0, |[a_{ij}]_{i,j=1,2}| > 0, |[a_{ij}]_{i,j=1,2,3}| < 0 \dots$$

As an example, consider the matrix

$$\begin{bmatrix} -1 & 1/2 \\ 1/2 & -1 \end{bmatrix}.$$

Is this definite?

Proposition 2. A and its quadratic form are *positive definite* if and only if principal minors of order i are of positive sign.

Proposition 2 says that

$$|a_{11}| > 0, |[a_{ij}]_{i,j=1,2}| > 0, |[a_{ij}]_{i,j=1,2,3}| > 0, \dots, |[a_{ij}]_{i,j=1,\dots,n}| > 0.$$

Another equivalent condition is given in

Proposition 3. A matrix A is *negative (positive) definite* if and only if all of its characteristic roots are negative (positive).

Remark: Semidefinite matrices are defined as above with \geq replacing $>$.