Lecture 2: Matrix Algebra

- General Definitions
- Algebraic Operations
- Vector Spaces, Linear Independence and Rank of a Matrix
- Inverse Matrix
- Linear Equation Systems, the Inverse Matrix and Cramer's Rule
- Characteristic Roots and Vectors
- Trace
- Quadratic Forms

General

- A *matrix* is a rectangular array of *objects* or elements. We will take these elements as being real numbers and indicate an element by its row and column position.
- Let $a_{ij} \in R$ denote an *element* of a matrix which occupies the position of the ith row and jth column.
- Denote a matrix by a capital letter and its elements by the corresponding lower case letter. If a matrix A is n x m, we write $\underset{nxm}{A}$

General

Example 1. $\underset{2\times 2}{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Example 2. $\underset{1\times 1}{A} = [a_{11}]$

Example 3. $\underset{n\times n}{A} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$

Example 4. $\underset{2\times 3}{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

General

- A matrix is said to be

(i) *square* if # rows = # columns and a square matrix is said to be

(ii) *symmetric* if $a_{ij} = a_{ji}$ $\forall$i, j, i $\neq$ j.

Example. The matrix $\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$ is square but not symmetric, since $a_{21} = 2 \neq 3 = a_{12}$. The square matrix $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 4 & 3 & 1 \end{bmatrix}$ is symmetric since $a_{12} = a_{21} = 2$, $a_{31} = a_{13} = 4$, and $a_{32} = a_{23} = 3$.

General

- The *principle diagonal elements* of a square matrix A are given by the elements a_{ij}, $i = j$.
- The *principle diagonal* is the ordered n-tuple $(a_{11},\ldots, a_{nn})$.
- The *trace* of a square matrix is defined as the sum of the principal diagonal elements. It is denoted $tr(A) = \Sigma_i a_{ii}$.

Example

- principal diagonal is (1,1,1), Tr(A) = 3

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 4 & 3 & 1 \end{bmatrix}$$

General

- A *diagonal matrix* is a square matrix whose only nonzero elements appear on the principal diagonal.
- A *scalar matrix* is a diagonal matrix with the same value in all of the diagonal elements.
- Examples:

$$\text{Diagonal} \quad : A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \text{Scalar} \quad : A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

General

- The *identity matrix* is a scalar matrix with ones on the diagonal.
- A *triangular matrix* is a square matrix that has only zeros either above or below the principal diagonal. If the zeros are above the diagonal, then the matrix is lower triangular and conversely for upper triangular.

Examples

- upper triangular

$$\begin{bmatrix} 1 & 3 & 6 \\ 0 & 4 & 7 \\ 0 & 0 & 9 \end{bmatrix}$$

- lower triangular

$$\begin{bmatrix} 5 & 0 & 0 \\ 9 & 6 & 0 \\ 5 & 0 & 1 \end{bmatrix}$$

Notation

The following notations for indicating an n $\times$ m matrix A are equivalent

$$\left[a_{ij}\right]_{\substack{i=1,\dots,n \\ j=1,\dots,m}}, \quad \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}, \quad \left[\quad \right], \quad \text{or} \quad \left\| \quad \right\|.$$

General

☆If a matrix A is of dimension $1 \times n$, then it is termed a *row vector*, $\underset{1\times n}{A} = [a_{11} \dots a_{1n}]$. Since there is only one row, the row index is sometimes dropped and A is written $[a_1 \dots a_n] = a'$.

☆A matrix A of dimension $n \times 1$ is termed a *column vector*, $A = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}$. Likewise, since there is only one column, this is sometimes written as $a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$.

Algebraic Operations on Matrices

- *Equality*: A = B if $a_{ij} = b_{ij}$ for all i and j.
- *Addition and Subtraction*: $A \pm B = [a_{ij} \pm b_{ij}]$.

  Note that for these operations, A and B must be of the same dimension.

$$\underset{n\times m}{A} + \underset{n\times m}{B} = \begin{bmatrix} (a_{11}+b_{11}) & & \dots & (a_{1m}+b_{1m}) \\ \vdots & & & \\ (a_{n1}+b_{n1}) & \dots & \dots & (a_{nm}+b_{nm}) \end{bmatrix} A-B = \begin{bmatrix} a_{11}-b_{11} & \dots & a_{1m}-b_{1m} \\ \vdots & & \vdots \\ a_{n1}-b_{n1} & \dots & a_{nm}-b_{nm} \end{bmatrix}.$$

Algebraic Operations on Matrices

- *Scalar multiplication*: . Let $k \in R$. $kA = [ka_{ij}]$.

$$3\begin{bmatrix} 1 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ 15 & 18 \end{bmatrix}$$

Algebraic Operations on Matrices: Multiplication

- *Conformability*: Two matrices A and B can be multiplied to form AB, only if the column dimension of A = row dimension of B. (col. dim. lead = row dim. of lag)

<u>Example</u> If $\underset{2\times3}{A}$ and $\underset{4\times2}{B}$, then AB cannot be defined, but $\underset{4\times2}{B}\,\underset{2\times4}{A}$ can be defined.

Multiplication

- Inner product of two n-tuples: Suppose x, y $\in R^n$. Then the *inner product* (also called the dot product) of x and y is defined by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

Multiplication

- Associate with the kth col of A (n x m) the n-tuple

  $a_{ok} = (a_{1k},\ldots,a_{nk}) \in R^n$
- Associate with the jth row of A the m-tuple

  $a_{jo} = (a_{j1},\ldots,a_{jm}) \in R^m$

Example. $\underset{2\times 3}{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$ $a_{02} = (2, 4)$

$a_{20} = (0, 4, 5)$

Multiplication

The *product AB* is then given by

$$\underset{n\times m}{A}\ \underset{m\times k}{B} = \begin{bmatrix} a_{10}\cdot b_{01} & \cdots & a_{10}\cdot b_{0k} \\ \vdots & & \vdots \\ a_{n0}\cdot b_{01} & \cdots & a_{n0}\cdot b_{0k} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} a_{1i} b_{i1} & \cdots & \sum_{i=1}^{m} a_{1i} b_{ik} \\ \vdots & & \vdots \\ \sum_{i=1}^{m} a_{ni} b_{i1} & \cdots & \sum_{i=1}^{m} a_{ni} b_{ik} \end{bmatrix}$$

n x k

Multiplication

- Note that the product matrix is n x k. It takes on the row dimension of the lead and the column dimension of the lag.
- Example:

$$A = \underset{2\times 2}{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}} \quad B = \underset{2\times 1}{\begin{bmatrix} 1 \\ 4 \end{bmatrix}}$$

$$\underset{2x1}{AB} = \underset{2\times 2}{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 6 \\ 8 \end{bmatrix}}$$

Multiplication: Example

$$\underset{2\times 2}{A}\ \underset{2\times 2}{B} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{2} a_{1i}b_{i1} & \sum_{i=1}^{2} a_{1i}b_{i2} \\ \sum_{i=1}^{2} a_{2i}b_{i1} & \sum_{i=1}^{2} a_{2i}b_{i2} \end{bmatrix} = \begin{bmatrix} a_{10}\cdot b_{01} & a_{10}\cdot b_{02} \\ a_{20}\cdot b_{01} & a_{20}\cdot b_{02} \end{bmatrix}$$

Multiplication

- *Scalar product*:

Suppose that A is a $1 \times n$ row vector $A = a' = (a_{11}\ a_{12}\ \dots\ a_{1n})$ and B an $n \times 1$ col vector

$B = b = \begin{pmatrix} b_{11} \\ \vdots \\ b_{n1} \end{pmatrix}$. Hence we have

$$\underset{1\times n}{a'}\ \underset{n\times 1}{b} = \begin{bmatrix} a_{11} & \dots & a_{1n} \end{bmatrix}\begin{bmatrix} b_{11} \\ \vdots \\ b_{n1} \end{bmatrix} = \underset{1\times 1}{\begin{bmatrix} \sum_i a_{1i}b_{i1} \end{bmatrix}}$$

Multiplication: Scalar Product Continued

- Note that $a'b = a \bullet b$ where $a,b \in R^n$ (The scalar product is same as the inner product of two equivalent ordered n-tuples.)
- Let i be a column vector of ones and x an n x 1 column vector, then

$$i'x = \Sigma_i x_i.$$

Multiplication: Special Case

- The product of a conformable column vector (m x 1) and a row vector (1 x n) is an m x n matrix:

$$ba' = \underset{m\times 1}{\begin{pmatrix} b_{11} \\ \vdots \\ b_{m1} \end{pmatrix}} \underset{1\times n}{\begin{pmatrix} a_{11} & \dots & a_{1n} \end{pmatrix}} = \underset{m\times n}{\begin{bmatrix} b_{11}a_{11} & \dots & b_{11}a_{1n} \\ \vdots & & \vdots \\ b_{m1}a_{11} & \dots & b_{m1}a_{1n} \end{bmatrix}}.$$

Addition and Multiplication: Properties

- The operation of addition is both commutative and associative. We have

  (Com. Law) $A + B = B + A$

  (Associative) $(A + B) + C = A + (B + C)$

- The operation of multiplication is not commutative but it does satisfy the associative and distributive laws.

  (Associative) $(AB)C = A(BC)$

  (Distributive) $A(B + C) = AB + AC$

  $(B + C) A = BA + CA$

Multiplication is not commutative

- To see that $AB \neq BA$ consider the example

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

- We have that

$$AB = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \qquad BA = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Equation Systems

- Generally, when we take the product of a matrix and a vector, we can write the result as

  c = Ab.
- In this example, the matrix A is n by n and the column vectors c and b are n by 1.

Equation Systems

- Taking an example of a 2×2 matrix A, we have

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

- This can be a short-hand way to write two equations in the unknowns a and b

  1 = a + 3b

  4 = 3a + 2b.

Equation Systems

- This same system can be written as a linear combination of the columns of A

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Transpose

- The *transpose* of a matrix A, denoted A′, is the matrix formed by interchanging the rows and columns of the original matrix A.

$$\textit{Example}\ 1.\ \text{Let } A = (1\ \ 2) \text{ then } A' = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\textit{Example}\ 2.\ \text{Let } \underset{3\times 2}{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \text{ then } \underset{2\times 3}{A'} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}.$$

Key Properties of Transpose

1. $(A')' = A$
2. $(A + B)' = A' + B'$
3. $(AB)' = B'A'$

The Identity Matrix

- An *identity matrix* is a square matrix with ones in its principle diagonal and zeros elsewhere. An n $\times$ n identity matrix is denoted I_n. For example

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of I_n

1. Let A be $n \times p$. Then we have $I_nA = AI_p = A$.
2. Let A be $n \times p$ and B be $p \times m$. Then we have

$$\underset{n\times p}{A}\;\underset{p\times p}{I_p}\;\underset{p\times m}{B} = \left(AI_p\right)B = AB.$$

3. $\underbrace{I_n I_n I_n \cdots I_n}_{p\ \text{terms}} = I_n.$

- In general, a matrix is termed *idempotent*, when it satisfies the property AA = A.

The Null Matrix

- The *null matrix*, denoted [0] is a matrix whose elements are all zero.

The Null Matrix: Properties

1. A + [0] = [0] + A = A
2. [0]A = A[0] = [0].

3. Remark: If AB = [0], it need not be true that A = [0] or B [0]. Example where AB = 0:

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$$

Determinants and Related Concepts.

- A determinant is defined only for square matrices. When taking the determinant of a matrix we attach a sign + or - to each element:

$$\text{sign attached to } a_{ij} = \text{sign } (-1)^{i+j}.$$

Determinants

- The determinant of a scalar x, is the matrix itself.
- The determinant of a 2 $\times$ 2 matrix A, denoted |A| or det A, is defined as follows:

$$|A| = \left\| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right\| = +1(a_{11})(a_{22}) - (1)(a_{21}) \cdot a_{12} \,.$$

Determinants

Example $A = \begin{bmatrix} 3 & 6 \\ -5 & 6 \end{bmatrix}$. $|A| = 3 \cdot 6 - (-5)(6) = 18 + 30 = 48.$

Determinants n x n: *Laplace Expansion* process

- *Definition.* The *minor* of the element a_{ij}, denoted $|M_{ij}|$ is the determinant of the submatrix formed by deleting the ith row and jth column.

- Example: If $A = [a_{ij}]$ is 3 x 3, then $|M_{13}| = a_{21}a_{32} - a_{31}a_{22}$. $|M_{12}| = a_{21}a_{33} - a_{31}a_{23}$.

Determinants n x n: *Laplace Expansion* process

- *Definition.* The *cofactor* of the element a_{ij} denoted $|C_{ij}|$ given by $(-1)^{i+j}|M_{ij}|$.

- Example: In the above 3 x 3 example

  $|C_{13}| = a_{21}a_{32} - a_{31}a_{2}$

  $|C_{12}| = -a_{21}a_{33} + a_{31}a_{23}$

Determinants n x n: *Laplace Expansion* process

- *Laplace Expansion*: Let A be n × n, n ≥ 2. Then

$$|A| = \sum_{i=1}^{n} a_{ij}|C_{ij}| \text{ (expansion by } j^{th} \text{ col)} \qquad |A| = \sum_{j=1}^{n} a_{ij}|C_{ij}| \text{ (expansion by } i^{th} \text{ row)}$$

Examples

- A is

$$\begin{bmatrix} 0 & 4 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

- What is |A|? (answer: -3)

Properties of Determinants

1. |A| =|A'|

2. The interchange of any two rows (or two col.) will change the sign of the determinant, but will not change its absolute value.

Examples of Properties 1 and 2

#1 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$|A| = -2$ $|A'| = -2$

#2 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

$|A| = -2$ $|B| = +2$

Properties of Determinants

3. The multiplication of any p rows (or col) of a matrix A by a scalar k will change the value of the determinant to $k^p |A|$.
4. The addition (subtraction) of any multiple of any row to (from) another row will leave the value of the determinant unaltered, if the linear combination is placed in the initial (the transformed) row slot. The same holds true if we replace the word "row" by column.

Examples of Properties 3 and 4

- Take A, 2 x 2, and multiply by 2.

  $|2A| = 2a_{11} 2a_{22} - 2a_{21} 2a_{12} = 4|A|$

- Take A, 2 x 2, and add 2 times the second row to the first row.

$$\tilde{A} = \begin{bmatrix} a_{11} + 2a_{21} & a_{12} + 2a_{22} \\ a_{21} & a_{22} \end{bmatrix}, |\tilde{A}| = a_{11}a_{22} - a_{12}a_{21}$$

Properties of Determinants

5. If one row (col) is a multiple of another row (col), the value of the determinant will be zero.

6. If A and B are square, then |AB| = |A||B|.

Examples of Properties 5 and 6

- Let

$$A = \begin{bmatrix} 3a & 3b \\ a & b \end{bmatrix}, |A| = 3ab - 3ab = 0$$

- Let

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 1 \end{bmatrix}, |A| = -3, B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, |B| = -2, |A||B| = 6.$$

$$AB = \begin{bmatrix} 12 & 18 \\ 5 & 8 \end{bmatrix}, |AB| = 96 - 90 = 6.$$

Vector Spaces, Linear Independence and Rank

- Define

Def. An *n-component vector* a is an ordered n tuple of real numbers written as a row

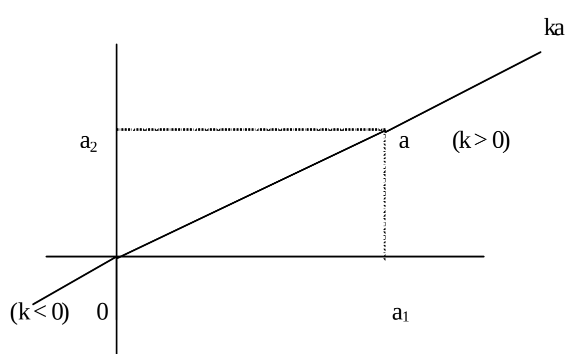
$$\left(a_1 \ldots a_n\right) = a' \text{ or as a col } a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$ The a_i, $i = 1, \ldots, n$, are termed the components of the vector.

- The elements of such a vector can be viewed as the coordinates of a point in R^n or as the definition of the line segment connecting the origin and this point. We will take these as ordered n-tuples:

$$(a_1,\ldots,a_n) \in R^n$$

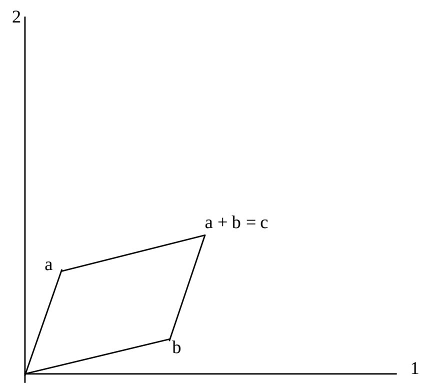
Two basic operations

- *Scalar multiplication*: $ka = (ka_1,\ldots,ka_n)$



Two basic operations

- *Addition*: $a + b = (a_1 + b_1, \ldots, a_n + b_n)$



Vector Space

- *Def.* A *vector space* is a collection of vectors that is closed under the operations of addition and scalar multiplication.
- Remark: R^n is a vector space.
- *Def.* A set of vectors span a vector space if any vector in that space can be written as a linear combination of the vectors in that set.

Basis

- A set of vectors spanning a vector space which contains the smallest number of vectors is called a basis. This set must be must be linearly independent. If the set were dependent, then some one could be expressed as a linear combination of the others and it could be eliminated. In this case we would not have the smallest set.

Linear Independence

- Def. A set of vectors $a^1,\ldots,a^m$ is said to be *linearly dependent* if $\exists\ \lambda^i \in R$ not all zero such that

  $\lambda^1 a^1 + \cdots + \lambda^m a^m = (0,\ldots,0) \in R^n$.

  If the only set of λ^i for which this holds is $\lambda^i = 0$, for all i, then the set $a^1,\ldots,a^m$ is said to be *linearly independent*.

Results

- *Proposition 1.* The vectors $a^1,\ldots,a^n$ from R^n are linearly dependent iff some one of the vectors is a linear combination of the others.

  Proof: (i) Let a^1 be a lin combo of the others and show dependence.

  (ii) Assume dependence and show that a^k can be written as a lin combo of the others.

Special Case

- Remark: If the set of vectors has but one member $a \in R^n$, then a is linearly dependent if $a = 0$ and a is linearly independent, if $a \neq 0$.

Results

- *Proposition 2*. No set of linearly independent vectors can contain the zero vector.

  Proof: Let $a^1 = 0^n$. Set $\lambda^1 = 1$ and all others $= 0$.

Results

- *Proposition 3*. Any subset of a set of linearly independent vectors is linearly independent.
- Proof: Assume to the contrary that 1,…,k of a^i are linearly dependent and show that they all are.

Results

- *Proposition 4.* Any superset of a set of linearly dependent vectors is linearly dependent.

  Proof: Use direct proof.

Def of Basis

- *Def*. A *basis* for a vector space of n dimensions is any set of n linearly independent vectors in that space.
- In R^n, exactly n vectors can form a basis. That is, it takes n (independent) vectors to create any other vector in R^n through a linear combination.

Example

- Let a,b be linearly independent in R^2. Let c be any third vector. We can show that c can be created from a linear combination of a and b.
- Select λ^1 and λ^2 such that $\lambda^1 a + \lambda^2 b = c$. That is $\lambda^1 a_i + \lambda^2 b_i = c_i$, i = 1,2. We have

$\lambda^1 = (b_2c_1 - b_1c_2)/(b_2a_1 - b_1a_2)$ and $\lambda^2 = (a_1c_2 - c_1a_2)/(a_1b_2 - b_1a_2)$.

Example

- To solve for λ^i, it must be true that $(a_1b_2 - b_1a_2) \neq 0$. This is true if $a_1/a_2 \neq b_1/b_2$. That is, a can not be a scale multiple of b.

Rank

- *Def*. The *rank* of an $n \times m$ matrix A, r(A), is defined as the largest # of linearly independent columns or rows.
- *Proposition 1*. Given an $n \times m$ matrix A, we have

(i) $r(A) \leq \min \{m, n\}$

(ii) largest # lin indep. col. = largest # lin indep. rows.

An Operational Test for Rank

- *Proposition 2.* The rank, r(A), of an $m \times n$ matrix A is equal to the order of the largest submatrix of A whose determinant is nonzero. (By submatrix we mean a matrix selected from A by eliminating rows and columns of A.)
- An n x n matrix with a nonvanishing det has n linearly independent rows or columns.

Example

- Determine that the rank is 2

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 6 & 10 \\ 5 & 1 & 0 & 0 \end{bmatrix}$$

Inverse Matrix

- *Def. G*iven an $n \times n$ square matrix A, the *inverse matrix* of A, denoted A^{-1}, is that matrix which satisfies

  $A^{-1} A = A A^{-1} = I_n$.

  When such a matrix exists, A is said to be *nonsingular*. If A^{-1} exists it is unique.

Result

- *Proposition.* An n x n matrix A is nonsingular iff $r(A) = n$.

Computation of Inverse

- Assume that A is n x n and has $|A| \neq 0$.
- *Cofactor matrix* of A is $C = [|C_{ij}|]$.
- The *adjoint matrix* is adj $A = C'$.
- $A^{-1} = (\text{adj } A) / |A|$.

Example

- Compute the inverse of

$$A = \begin{bmatrix} 1 & 3 \\ 9 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-25}\begin{bmatrix} 2 & -3 \\ -9 & 1 \end{bmatrix} = \begin{bmatrix} -2/25 & 3/25 \\ 9/25 & -1/25 \end{bmatrix}.$$

Key Properties

- $(AB)^{-1} = B^{-1}A^{-1}$

Proof: $B^{-1}A^{-1}AB = I$ and $ABB^{-1}A^{-1} = I$.

- $(A^{-1})^{-1} = A$

Proof: $AA^{-1} = I$ and $A^{-1}A = I$.

- $I^{-1} = I$

Proof: $II = I$

Remarks

- Note that AB = 0 does not imply that A = 0 or that B = 0. If either A or B is nonsingular and AB = 0, then the other matrix is the null matrix.

$AB = 0$ and $|A| \neq 0 \Rightarrow B = 0$

Proof : Let $|A| \neq 0$ and $AB = 0$. Then $A^{-1}AB = B = 0$.

Remarks

- If A and B are square, then $|AB| = 0$ iff $|A| = 0$, $|B| = 0$ or both.

Proof: Note that $|AB| = |A||B|$.

Linear Equation Systems, the Inverse Matrix and Cramer's Rule.

- Let A be n x n, let x be a column vector of unknown variables and let d be a column vector of constants.
- $Ax = d$ is a linear equation system of n equations in n unknowns, x_i.
- A unique solution is possible if $|A| \neq 0$ in which case A^{-1} exists. That is the rows and columns of A are linearly independent.

Solution

- The solution can be written

  $A^{-1}Ax = A^{-1}d$

  $x = A^{-1}d$
- Alternatively, define $|A_j|$ =

$$\begin{array}{c} j^{th}\text{ col} \\ \left| \begin{bmatrix} a_{11} & \cdots & \begin{bmatrix} d_1 \\ \\ d_n \end{bmatrix} & a_{1n} \\ & \vdots & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} \right| \end{array}$$

Solution

- We have

$x_j = |A_j| / |A|$. (Cramer's Rule)

Example

- Solve

$$3x_1 + 4x_2 = 10$$
$$6x_1 + x_2 = 20$$

Answer: $x_1 = 70/21$ and $x_2 = 0$

Characteristic Roots

- Let D be an $n \times n$ matrix. Does there exist a scalar r and an $n \times 1$ vector $x \neq 0$ such that

$$Dx = rx?$$

If so, then r is said to be a *characteristic root of D*.

Characteristic Roots

- Rewrite

Characteristic Matrix of D

(*) $$[D - rI]x = 0.$$

- For (*) to be true it is necessary that $|D - rI| = 0$, given that $x \neq 0$.

Proof: To see that this is true, let $A = [D - rI]$ and suppose to the contrary that $Ax = 0$, $x \neq 0$, and $|A| \neq 0$. Then $A^{-1}Ax = 0$ and $x = 0$, so that we have a contradiction.

Characteristic Roots

- The condition

  $$(**) \quad |D - rI| = 0$$

  is called the *characteristic equation* of D and x is called a *characteristic vector* of D.
- By definition, an x is not unique. If $[D - rI]x = 0$, then $[D - rI]kx = 0$, for any k.
- To remove the indeterminacy, x is normalized so that $x'x = 1$.

Characteristic Roots

- $(**) \quad |D - rI| = 0$ represents an n^{th} degree polynomial in r which has n roots.

- If D is symmetric, then these roots are real numbers.

An Example

$$D = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}.$$

$$[D - rI] = \begin{bmatrix} 2-r & 2 \\ 2 & -1-r \end{bmatrix}.$$

Solution

$$r^2 - r - 6 = 0.$$

$$r_1,\ r_2 = \frac{1 \pm (1 - 4 \cdot 1 \cdot -6)^{1/2}}{2} = 1/2 \pm 5/2 = 3, -2.$$

Recall that the solution to $ax^2 + bx + c = 0$ is

$x_1, x_2 = (-b \pm (b^2 - 4ac)^{1/2})/2a$

Solution

$$[D - rI]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 2x_2 = 0$$

$$2x_1 - 4x_2 = 0.$$

Solution

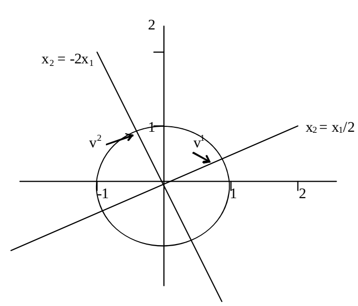
- Equation 1 is just a multiple of equation 2. They are not independent as expected. All that we can conclude from these equations is that
  (1) $x_1 = 2x_2$.
- If we impose the normalization constraint
  (2) $x_1^2 + x_2^2 = 1$,
  then (1) and (2) give us two equations in two unknowns. Solving (1) and substituting
  $(2x_2)^2 + x_2^2 = 1$
  $x_1 = 2/(5)^{1/2}$ and $x_2 = 1/(5)^{1/2}$.

Solution

- The characteristic vector is
  $v^1 = (v_1^1, v_2^1) = (2/(5)^{1/2}, 1/(5)^{1/2})$.

- Using the same technique for $r_2 = -2$, we can show that ($x_2 = -2x_1$)
  $v^2 = (v_1^2, v_2^2) = (-1/(5)^{1/2}, 2/(5)^{1/2})$.

Illustration



General Results for Characteristic Roots and Vectors

- For a symmetric matrix, characteristic vectors corresponding to distinct characteristic roots are pairwise orthogonal.

  $v^i \cdot v^i = 1$ and $v^j \cdot v^i = 0$.

- If the characteristic roots of a symmetric matrix $n \times n$ are distinct, then they form a basis (orthonormal basis) for R^n.

General Results for Characteristic Roots and Vectors

- The matrix of characteristic vectors of a matrix A is (v^i is n x 1 here)

  $Q = [v^1 \dots v^n]$.

- By definition,

  $Q'Q = I$ so that $Q' = Q^{-1}$.

When this condition is met Q is said to be *orthogonal*.

General Results for Characteristic Roots and Vectors

- From the characteristic equation,

$$AQ = QR, \text{ where } R \equiv \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 \\ 0 & . & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & r_n \end{bmatrix}.$$

To see this, note that $AQ = [Av^1 \ldots \ Av^n] = [r_1 v^1 \ldots \ r_n v^n] = QR$

General Results for Characteristic Roots and Vectors

- We conclude that

(*) Q'AQ = Q'QR = R.

- (*) is called the *diagonalization of A*. We have found a matrix Q such that the transformation Q'AQ produces a diagonal matrix with A's characteristic roots along the diagonal.

General Results for Characteristic Roots and Vectors

- For a square matrix A, we have

i. The product of the characteristic roots is equal to the determinate of the matrix.

ii. The rank of A is equal to the number of nonzero characteristic roots.

iii. The characteristic roots of A^2 are the squares of the characteristic roots A, but the characteristic vectors of both matrices are the same.

iv. The characteristic roots of A^{-1} are the reciprocal of the characteristic roots of A, but the characteristic vectors of both matrices are the same.

General Results on the Trace of a Matrix

- $tr(cA) = c(tr(A))$.
- $tr(A') = tr(A)$.
- $tr(A+B) = tr(A) + tr(B)$.
- $tr(I_k) = k$.
- $tr(AB) = tr(BA)$.

(Note this can be extended to any permutation: $tr(ABCD) = tr(BCDA) = tr(CDAB) = tr(DABC)$.

Quadratic Forms

- A *quadratic form* is a homogeneous polynomial of the second degree.
- It takes on the form x'Ax, where A is symmetric and n by n, and x is n by 1.
- We have $x'Ax = \Sigma\Sigma\, a_{ij}x_ix_j$.
- x'Ax is termed *negative definite* if it is negative for all $x \neq 0^n$. The form and the matrix are termed *negative definite* in this case.

Quadratic Forms

- The definitions for *positive definite* are analogous with the inequality sign reversed.
- Next we consider a new concept defined as the *principal minor* of an n x n matrix A:
- $|PM_i|$ is the determinant of the submatrix of A formed by retaining only the first i rows and columns of A.

Example of PM_i

Example

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|PM_1| = a_{11} \qquad |PM_2| = a_{11}a_{22} - a_{21}a_{12} \qquad |PM_3| = |A|.$$

Operational Tests for Definite Quadratic Forms

- *Proposition 1*. A and its quadratic form are *negative definite* if and only if principal minors of order i are of sign $(-1)^i$.

- Is the following definite?

$$\begin{bmatrix} -1 & 1/2 \\ 1/2 & -1 \end{bmatrix}.$$

Operational Tests for Definite Quadratic Forms

- *Proposition 2*. A and its quadratic form are *positive definite* if and only if principal minors of order i are of positive sign.

Example: Is the following definite?

$$\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

Example

- If A is 2 x 2 and negative definite, is it true that $a_{22} < 0$?
- If A is 2 x 2 and positive definite, is it true that $a_{22} > 0$?

Other Operational Tests for Definite Quadratic Forms and Matrices

Another equivalent condition is given in

- *Proposition 3.* A matrix A is *negative (positive) definite* if and only if all of its characteristic roots are negative (positive).

- Remark: Semidefinite matrices are defined as above with $\geq$ replacing >.