Lecture 3: Limits

Limits

1. Let y = f(x) be a function mapping a subset of R into R, f : X→R, where X ⊂ R. In studying limits we shall be concerned with a sequence of x's say x^q tending towards some x^0 and the corresponding sequence of images say $f(x^q)$. Loosely speaking, if as $x^q \to x^0$, $f(x^q) \to L$, then we say that L is the limit of f(x) as x approaches x^0 and write $\lim_{x \to x^0} f(x) = L$.

2. Let us now consider the more rigorous definitions of the various types of limits which can occur.

Def 1 The *right-hand limit* of a function f(x) as x approaches a finite number x^0 is a finite number L such that for any ε > 0 there exists a δ > 0 such that if $x' \in (x^0, x^0+\delta)$, then $|L - f(x')| < \varepsilon,$ where x′ is in the domain of f.

Notation: If L is the right-hand limit of f(x) as $x \to x^0$ we write $\lim_{x \to x^{0+}} f(x) = L.$

Def 2: The *left-hand limit* of a function f(x) as x approaches a finite number x^0 is a finite number L′ such that for any ε > 0 there exists δ > 0 such that if $x' \in (x^0-\delta, x^0)$, then $|L' - f(x')| < \varepsilon,$ where x′ is in the domain of f.

Notation: If L′ is a left-hand limit we write $\lim_{x \to x^{0-}} f(x) = L'$.

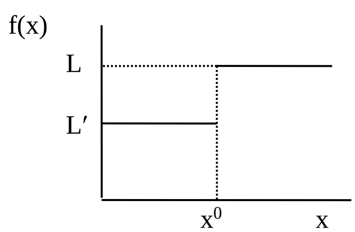
Remark: In general the left and right hand limits of a function f(x) as $x \to x^0$ need not be equal. However, if they are we say that f(x) has a limit as $x \to x^0$ and write $\lim_{x \to x^0} f(x) = L,$ where

$$L = \lim_{x \to x^{0+}} f(x) = \lim_{x \to x^{0-}} f(x).$$

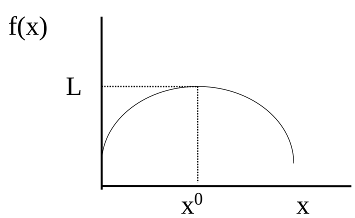
Remark: One point made implicit in definitions 1 and 2 is that the sequence of x's, x^q, must be contained in the domain of f and, hence every x in such a sequence must have an image. However, it is not necessary that the point x^0 be contained in the domain of f nor that the limit L be contained in the range of f. Hence, we can have $\lim_{x \to x^0} f(x) = L$ and at the same time have the ordered pair (x^o, L) $\notin$ Gr(f).

Def 3 A function f(x) has the finite *limit* L as x approaches a finite number x^0 if, for every positive number ε, there is a number δ > 0 such that if $0 < |x^1 - x^0| < \delta$, then $|f(x^1) - L| < \varepsilon$, where x^1 is in the domain of f.

<u>Remark:</u> We may illustrate the definitions 1,2 and 3 graphically



$$\lim_{x \to x^{0-}} f(x) = L'$$
$$\lim_{x \to x^{0+}} f(x) = L$$



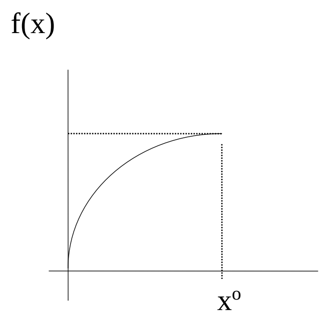
$$\lim_{x \to x^{0-}} f(x) = \lim_{x \to x^{0+}} f(x) = \lim_{x \to x^0} f(x) = L$$

<u>Remark :</u> An equivalent definition of a limit may be formulated using the concept of a neighborhood.

Def 4 A neighborhood of a number L is an open interval containing L. We denote a neighborhood of L by N(L) where $N(L) \equiv (L - \varepsilon, L + \varepsilon)$, and where ε is a positive number. Hence $N(L) = \{x \mid |x - L| < \varepsilon\}$.

Def 5 The $\lim_{x \to x^0} f(x) = L$ if for every N(L) there exists a N(x^o), in the domain of f, such that for every $x' \in N(x^0)$, $f(x') \in N(L)$.

<u>Remark:</u> We note that Def 5 is not applicable in the following illustration.

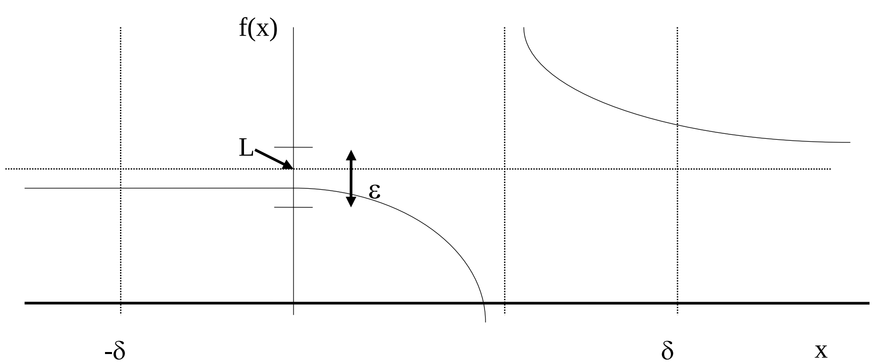


However, Def 3 is applicable. Seeing that only the left-hand limit is relevant in this case, we can conclude that $\lim_{x \to x^{0-}} f(x) = L$ by Def 2.

Limits and Nonconvergent Sequences

1. *Def 6* The function f(x) has a finite *limit* L as $|x|$ becomes infinite if, for every positive number ε, ∃ a number δ > 0 such that if $|x^1| > \delta$ then $|f(x^1) - L| < \varepsilon,$ where $x^1 \in$ domain of f.

Remark: We may illustrate Def 6 with the following graph.



$\lim_{x \to +\infty} f(x) = L$

$\lim_{x \to -\infty} f(x) = L$

2. Next, we wish to consider the "infinite" limit which in some cases is said to exist.

Def 7 The function f(x) has an infinite limit as $x \to x^0$, x^0 finite, if for any positive number ε there exists a number δ > 0 such that if $0 < |x^1 - x^0| < \delta,$ then $|f(x^1)| > \varepsilon,$ where $x^1 \in$ domain of f.

Remark: In Def 7 we would write $\lim_{x \to x^0} f(x) = \pm\infty$.

Remark: Definition 6 may be augmented to define such limits as

$$\lim_{|x| \to +\infty} f(x) = \pm\infty .$$

We consider this next

Def 8 The function f(x) has an infinite limit as $|x|$ becomes infinite if, for every positive number ε there exists a number δ > 0 such that if $|x^1| > \delta$, then $|f(x^1)| > \varepsilon$, where x^1 ε domain of f.

3. With these definitions in hand, let us consider the properties of limits.

Proposition 1. If y = f(x) is such that

(i) f(x) = ax + b, then $\lim_{x \to x^0} f(x) = ax^0 + b$,

(ii) f(x) = b, then $\lim_{x \to x^0} f(x) = b$,

(iii) f(x) = x, then $\lim_{x \to x^0} f(x) = x^0$,

(iv) f(x) = x^k, then $\lim_{x \to x^0} f(x) = (x^0)^k$.

Suppose now that we have two functions of the same variable x say $y_1 = f_1(x)$ and $y_2 = f_2(x)$.

Proposition 2. If $\lim_{x \to x^0} f_1(x) = L_1$ and $\lim_{x \to x^0} f_2(x) = L_2$ both exist and are finite, then

(i) $\lim_{x \to x^0} (f_1(x) \pm f_2(x)) = L_1 \pm L_2$,

(ii) $\lim_{x \to x^0} f_1(x) \cdot f_2(x) = L_1 L_2$,

(iii) $\lim_{x \to x^0} \frac{f_1(x)}{f_2(x)} = \frac{L_1}{L_2}$, if $L_2 \neq 0$.

4. Algebraic example.

Find $\lim_{x \to 3} f(x)$ where

$$f(x) = \frac{x^2 - 9}{x - 3}$$

Here, if we substitute x = 3 in f(x) we obtain

$$\frac{9-9}{3-3} = \frac{0}{0},$$

which is meaningless. However, from the definition of a limit we know that f(3) need not be defined. All that counts is what happens to f(x) when x approaches 3. Hence, manipulate f(x), for all $x \neq 3$.

$$f(x) = \frac{(x-3)(x+3)}{x-3} = (x+3), \ \forall x \neq 3$$

We conclude that as $x \to 3$, $f(x) \to 6$ and write

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3}(x+3) = 6.$$

We next wish to consult definition 3 and see if L = 6 qualifies as a limit:

For every $\varepsilon > 0$, $\exists\ \delta > 0$ such that $0 < |x^1 - 3| < \delta \Rightarrow |f(x^1) - 6| < \varepsilon$. Substitute, $f(x^1) = x^1 + 3$ in the second expression $|x^1 + 3 - 6| < \varepsilon$, or $|x^1 - 3| < \varepsilon$. Then $0 < |x^1 - 3| < \delta \Rightarrow |x^1 - 3| < \varepsilon$ will hold if we choose $\delta = \varepsilon$, for example. We conclude that L = 6 is the limit.