Lecture 3: Limits

Let $y = f(x)$ be a function mapping a subset of R into R, $f : X \to R$, where $X \subset R$.

Def 1 The *right-hand limit* of a function f(x) as x approaches a finite number x^0 is a finite number L such that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x' \in (x^0, x^0+\delta)$, then $|L - f(x')| < \varepsilon$, where x' is in the domain of f.

Notation: If L is the right-hand limit of f(x) as $x \to x^0$ we write $\lim_{x \to x^{0+}} f(x) = L$.

Limits

Def 2: The *left-hand limit* of a function f(x) as x approaches a finite number x^0 is a finite number L' such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x' \in (x^0-\delta, x^0)$, then $|L' - f(x')| < \varepsilon$, where x' is in the domain of f.

Notation: If L' is a left-hand limit we write $\lim_{x \to x^{0-}} f(x) = L'$.

Limits

Remark: In general the left and right hand limits of a function f(x) as $x \to x^0$ need not be equal. However, if they are we say that f(x) has a limit as $x \to x^0$ and write $\lim_{x \to x^0} f(x) = L$, where

$$L = \lim_{x \to x^{0+}} f(x) = \lim_{x \to x^{0-}} f(x).$$

Remark: One point made implicit in definitions 1 and 2 is that the sequence of x's, x^q, must be contained in the domain of f and, hence every x in such a sequence must have an image. However, it is not necessary that the point x^0 be contained in the domain of f nor that the limit L be contained in the range of f. Hence, we can have $\lim_{x \to x^0} f(x) = L$ and at the same time have the ordered pair $(x^o, L) \notin Gr(f)$.

Limits: An Alternative Definition

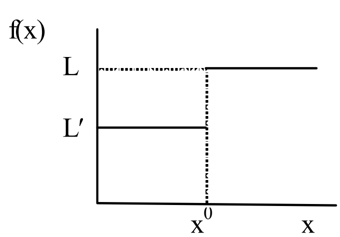
Def 3 A function f(x) has the finite *limit* L as x approaches a finite number x^0 if, for every positive number ε, there is a number δ > 0 such that if $0 < |x^1 - x^0| < \delta$, then $|f(x^1) - L| < \varepsilon$, where x^1 is in the domain of f.

Limits: An Alternative Definition 2

Def 4 A neighborhood of a number L is an open interval containing L. We denote a neighborhood of L by N(L) where $N(L) \equiv (L - \varepsilon, L + \varepsilon)$, and where ε is a positive number. Hence $N(L) = \{x : | x - L | < \varepsilon\}$.

Def 5 The $\lim_{x \to x^0} f(x) = L$ if for every N(L) there exists a $N(x^o)$, in the domain of f, such that for every $x' \in N(x^0)$, $f(x') \in N(L)$.

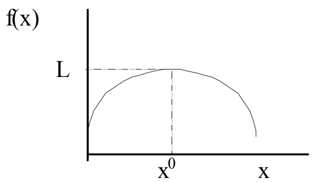
Illustrations



$$\lim_{x \to x^{0-}} f(x) = L'$$

$$\lim_{x \to x^{0+}} f(x) = L$$

Illustrations



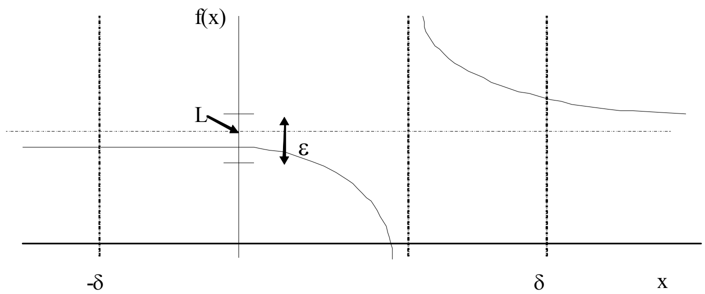
$$\lim_{x \to x^{0-}} f(x) = \lim_{x \to x^{0+}} f(x) = \lim_{x \to x^0} f(x) = L$$

Limits and Nonconvergent Sequences

Def 6 The function f(x) has a finite *limit* L as $|x|$ becomes infinite if, for every positive number ε, $\exists$ a number $\delta > 0$ such that if $|x^1| > \delta$ then $|f(x^1) - L| < \varepsilon$, where $x^1 \in$ domain of f.

Illustration

Remark: We may illustrate Def 6 with the following graph.



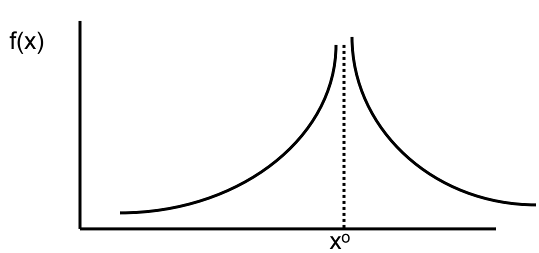
$$\lim_{x \to +\infty} f(x) = L$$

$$\lim_{x \to -\infty} f(x) = L$$

Limits and Nonconvergent Sequences

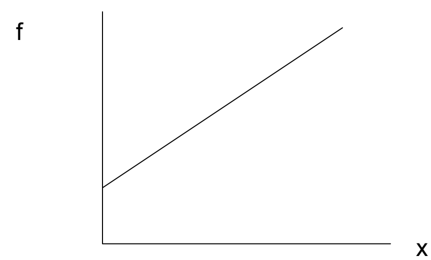
Def 7 The function f(x) has an infinite limit as $x \to x^0$, x^0 finite, if for any positive number ε there exists a number $\delta > 0$ such that if $0 < |x^1 - x^0| < \delta$, then $|f(x^1)| > \varepsilon$, where $x^1 \in$ domain of f.

Remark: In Def 7 we would write $\lim_{x \to x^0} f(x) = \pm\infty$.



Limits and Nonconvergent Sequences

Def 8 The function f(x) has an infinite limit as $|x|$ becomes infinite if, for every positive number ε there exists a number $\delta > 0$ such that if $|x^1| > \delta$, then $|f(x^1)| > \varepsilon$, where x^1 ε domain of f.



Properties

Proposition 1. If y = f(x) is such that

(i) $f(x) = ax + b$, then $\lim_{x \to x^0} f(x) = ax^0 + b$,

(ii) $f(x) = b$, then $\lim_{x \to x^0} f(x) = b$,

(iii) $f(x) = x$, then $\lim_{x \to x^0} f(x) = x^0$,

(iv) $f(x) = x^k$, then $\lim_{x \to x^0} f(x) = (x^0)^k$.

Properties

Proposition 2. If $\lim_{x \to x^0} f_1(x) = L_1$ and $\lim_{x \to x^0} f_2(x) = L_2$ both exist and are finite, then

(i) $\lim_{x \to x^0} \left(f_1(x) \pm f_2(x) \right) = L_1 \pm L_2$,

(ii) $\lim_{x \to x^0} f_1(x) \cdot f_2(x) = L_1 L_2$,

(iii) $\lim_{x \to x^0} \dfrac{f_1(x)}{f_2(x)} = \dfrac{L_1}{L_2}$, if $L_2 \neq 0$.

Example

Find $\lim_{x \to 3} f(x)$ where

$$f(x) = \frac{x^2 - 9}{x - 3}$$