# Lecture 4: Continuity and Differentiability

- Continuity
- The derivative

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# Continuity

- Let $N(x^o)$ denote neighborhood of $x^o$, meaning an open interval covering $x^o$:

$$N(x^o) = (x^o - \varepsilon, x^o + \varepsilon)$$

- We have

*Def 1* A function $f(x)$, $f: X \to R$, *is continuous at a point $x^0$*, in the domain of f, iff for any $N\left(f\left(x^0\right)\right)$ there is a neighborhood $N\left(x^0\right)$ such that if (for every) $x \in N\left(x^0\right)$, we have $f(x) \in N\left(f\left(x^0\right)\right)$, where $x \in X$.

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# Continuity: A Result

*Proposition 1.* If the $\lim_{x \to x^0} f(x) = L$ exists and is finite and the ordered pair $(x^o, L) \in Gr(f)$, then $f(x)$ is continuous at $x^0$.

Proof: If $\lim_{x \to x^0} f(x) = L$ then for every $N(L)$ $\exists$ $N(x^o)$ such that if $x \in N(x^o)$ then $f(x) \in N(L)$. However, we have that $(x^o, L) \in Gr(f)$ which implies that $L = f(x^o)$. Hence, all the conditions of Def 1 are satisfied. ||

*Proposition 2.* If $f(x)$ is continuous at $x^0$, then $\exists \lim_{x \to x^0} f(x) = L$ and $(x^o, L) \in Gr(f)$.

Proof: See Def 1.

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# Continuity

- *Def 2* A function , f: X→R, is *continuous* if it is continuous at every point x in the domain of f.

- *Def 3* A function , f: X→R, is *continuous on the interval* (x', x''), x' < x'', if it is continuous at all points in this interval.

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# The Derivative

- Consider a function f : X→ R, where X is an open interval of R
- Consider a change in the independent variable x and the corresponding change in the image of x. Let the change in x be given by Δx and let the initial value of x be $x^o$

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# The Derivative

- Form the difference quotient

$$\frac{\Delta y}{\Delta x} \equiv \frac{f\left(x^{o} + \Delta x\right) - f\left(x^{0}\right)}{\Delta x}$$

- If the following limit exists, it is said to be the *derivative of f at $x^o$*

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x^{o} + \Delta x) - f(x^{o})}{\Delta x}$$

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# The Derivative

- Notation: $f'(x^o)$, $df(x^o)/dx$ or $dy/dx|_{x^o}$

- f is said to be differentiable if it is has a derivative at each point in its domain.

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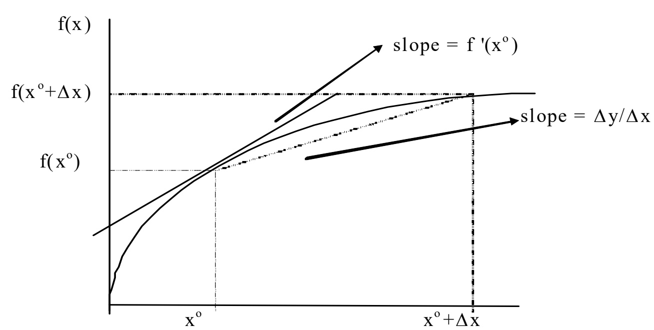
# Illustration

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# The relationship between differentiability and continuity

*Proposition 1.* The function $y = f(x)$ is differentiable at a point $x^0$ in its domain only if $f(x)$ is continuous at $x^0$.

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# Proof

<u>Proof</u> To show necessity of continuity we need to show that differentiability implies continuity. If $f(x)$ is differentiable at $x^0$,

$$f'(x^0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x \to x^0} \frac{f(x) - f(x^0)}{x - x^0}.$$

Since, $x \to x^0$ implies $x \neq x^0$, then

$$f(x) - f(x^0) \equiv \frac{f(x) - f(x^0)}{(x - x^0)}(x - x^0)$$

Take limits as $x \to x^0$ of both sides

$$\lim_{x \to x^0} f(x) - f(x^0) = f'(x^0)(0) = 0$$

Hence,

$$\lim_{x \to x^0} f(x) = f(x^0)$$

Since $\exists \lim_{x \to x^0} f(x) = f(x^0)$ and since $x^0$ is contained in the domain of f, $f(x)$ is continuous at $x^0$.‖

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# The converse is not ture

- Although differentiability implies continuity, the converse is not true. To show this, consider the function

$$f(x) = \begin{cases} ax & \text{if } x < \bar{x} \\ a\bar{x} & \text{if } x \geq \bar{x} \end{cases}$$

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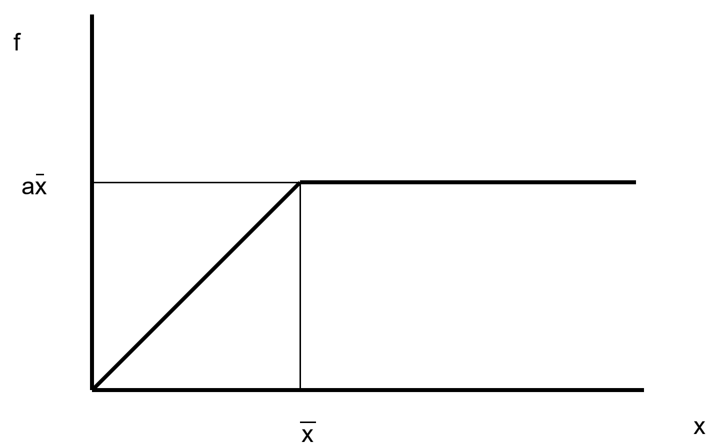
# Illustration

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