Lecture 5: Rules of Differentiation

First Order Derivatives

1. Consider functions of a single independent variable, $f: X \rightarrow R$, X an open interval of R.

Rule 1 (Constant function rule). The derivative of the function $y = f(x) = k$ is zero.

Proof Let $y = f(x) = k$. Clearly, for any $x, x' \in X$, $f(x) = f(x') = k$ and therefore

$$\Delta y = f(x) - f(x') = 0.$$

Thus we have that

$$\frac{\Delta y}{\Delta x} \equiv \lim_{\Delta x \to 0} \Delta y / \Delta x = \lim_{\Delta x \to 0} 0 = 0 \quad ||$$

Rule 2 (Power function rule). The derivative with respect to x of the function $y = f(x) = x^N$

is NX^{N-1}, where N is any positive integer.

Proof Let $y = f(x) = x^N$. We have that

$$y + \Delta y = (x + \Delta x)^N.$$

By the binomial theorem, it is true that

$$(x + \Delta x)^N = \begin{cases} x + \Delta x \quad \text{if } N = 1 \\ x^2 + 2x\Delta x + \Delta x^2 \quad \text{if } N = 2 \\ x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 \quad \text{if } N = 3 \\ \text{or } x^3 + 3x^2\Delta x + (3x + \Delta x)\Delta x^2 \quad \text{if } N = 3 \\ \vdots \qquad \vdots \\ x^N + Nx^{N-1}\Delta x + (\text{terms in } x \ \& \ \Delta x) \cdot \Delta x^2, \quad \text{if } N > 3. \end{cases}$$

Since $y + \Delta y = (x + \Delta x)^N$,

$$y + \Delta y = x^N + Nx^{N-1}\Delta x + (\text{terms in } x \ \& \ \Delta x) \cdot \Delta x^2 .$$

Divide by Δx,

$$\frac{\Delta y}{\Delta x} = Nx^{N-1} + (\text{terms in } x \ \& \ \Delta x) \cdot \Delta x$$

$$\lim_{\Delta x \to 0} {\Delta y}/{\Delta x} = Nx^{N-1} = \frac{dy}{dx}. \quad \|$$

<u>Remark 1</u> Rule 2 holds for any real number N.

<u>Rule 3</u> If $z = f(x)$ is a differentiable function of x and k is a constant, then

$$\frac{d(kf(x))}{dx} = kf'(x) = k \cdot \frac{dy}{dx}.$$

<u>Proof</u> Let y be a function of z defined as $y = kz$. Then

$$y + \Delta y = k(z + \Delta z), \text{ where } z + \Delta z = f(x + \Delta x).$$

Subtract $y = kz$ from both sides

$$\Delta y = k(z + \Delta z) - kz$$
$$\Delta y = k\Delta z$$

Divide both sides by Δx

$${\Delta y}/{\Delta x} = k \, {\Delta z}/{\Delta x}$$

$\lim_{\Delta x \to 0} {\Delta y}/{\Delta x} = k \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x}$, since $z = f(x)$ is differentiable. Hence

$$\lim_{\Delta x \to 0} {\Delta y}/{\Delta x} = k \frac{dz}{dx} = kf'(x). \quad \|$$

<u>Remark 2</u> In light of rule 3, it is possible to generalize the power function rule as follows:

<u>Rule 2′</u>: The derivative with respect to x of cx^N is cnx^{N-1}

Examples:

#1 Let y = f(x) = $\sqrt[3]{x}$. Find dy / dx $\sqrt[3]{x}$ = $(x)^{1/3}$. Hence, d / dx $(x)^{1/3}$ = $1/3(x)^{1/3-1}$ = <u>$1/3(x)^{-2/3}$</u>.

#2 Let y = f(x) = 3, where f: $R_+ \to R$. Find dy /dx = 0

#3 Let $y = f(x) = 10x^0$. Find $dy / dx = 0$.

#4 Let $y = f(x) = 20\, x^{4/5}$. Find $dy /dx = 16x^{-1/5}$.

2. Rules of differentiation involving two or more functions of the same independent variable.

Def 1: By the sum (difference) of any two real-valued functions f(x) and g(x), where $f: D \to R$ and $g: E \to R$, we mean the real-valued function $f \pm g: D \cap E \to R$ whose value at any $x \in D \cap E$ is the sum (difference) of two real numbers f(x) and g(x). In symbols we have

$$(f \pm g)(x) = f(x) \pm g(x), \text{ for any } x \in D \cap E.$$

Rule 4: The *derivative of the sum* (difference) of a finite number of differentiable functions of the same independent variable is the sum (difference) of their derivatives.

Proof (By induction only for the sum). Let $y = u_1 + u_2$, where $u_1 = u_1(x)$, $u_2 = u_2(x)$ are both differentiable functions of x. Thus

$$\lim_{\Delta x \to 0} \frac{\Delta u_1}{\Delta x} = du_1/dx, \text{ and } \lim_{\Delta x \to 0} \frac{\Delta u_2}{\Delta x} = du_2/dx.$$

By construction

$$y + \Delta y = (u_1 + \Delta u_2) + (u_2 + \Delta u_2)$$

Subtract $y = u_1 + u_2$ and ÷ by Δx

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u_1}{\Delta x} + \frac{\Delta u_2}{\Delta x}$$

Hence

$$\lim_{\Delta x \to 0} \Delta y/\Delta x = \lim_{\Delta x \to 0} \Delta u_1/\Delta x + \lim_{\Delta x \to 0} \Delta u_2/\Delta x$$

or

$$d(u_1 + u_2) / dx = du_1/dx + du_2/ dx.$$

Thus, Rule 4 is true for the first case. Assume it is true for the Nth case and it remains to be shown that it holds for the (N +1)th case. So assume it is true for the Nth, then for some integer N,

$$\frac{d(u_1 + u_2 + \ldots u_N)}{dx} = \frac{du_1}{dx} + \frac{du_2}{dx} + \ldots + \frac{du_N}{dx}.$$

Now let y = u + v, where

u = u_1 + … + u_N, and

v = u_{N+1}.

We have that

$$\frac{d(u_1 + u_2 + \ldots u_{N+1})}{dx} = \frac{du}{dx} + \frac{dv}{dx},\ \underline{\text{from N=2}}$$

$$= \frac{du}{dx} + \ldots + \frac{du_{N+1}}{dx}.$$

Thus, its is true for any finite sum. ||

Examples:

#1 What is the slope of the curve y= f(x) = x^3 - 3x + 5, when it crosses the y -axis?

First find

$$\frac{dy}{dx} = \frac{d}{dx}x^3 - 3\frac{d}{dx}x + \frac{d}{dx}5$$

$dy / dx = 3x^2 - 3 = f'(x)$.

When the curve crosses the y - axis, we have x = 0. Hence, evaluate f′(x) for x = 0.

$f'(0) = -3$.

Thus the slope at x = 0 is -3.

#2 Find f′(x) if f(x) = $\frac{x^3}{x^2} - 7x^{-1/2} + 5$.

Clearly we may write f(x) as

$f(x) = x - 7x^{-1/2} + 5$

Hence

$$f'(x) = \frac{d}{dx}x - 7\frac{d}{dx}x^{-1/2} + \frac{d}{dx}5$$

$$f'(x) = 1 + \frac{7}{2}\, x^{-3/2}$$

#3 Find $f'(x)$ if $f(x) = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + 3$

$$f'(x) = \frac{1}{4}\frac{d}{dx}x^4 - \frac{1}{3}\frac{d}{dx}x^3 + \frac{1}{2}\frac{d}{dx}x^2 - \frac{d}{dx}x + \frac{d}{dx}3$$

$$f'(x) = x^3 - x^2 + x - 1$$

3. We consider the product and the quotient of two functions next.

Def By the *product* of any two real valued functions f(x) and g(x), where $f: D \to R$ and $g: E \to R$, we mean the real function

$$fg : D \cap E \to R$$

whose value at any $x \in D \cap E$ is the product of the two real numbers f(x) and g(x).

<u>Rule 5</u>: (Product Rule) The derivative of the product of two differentiable functions, f(x) and g(x), is given by

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)\, g(x) + g'(x)\, f(x).$$

<u>Remark</u>: Rule 5 may be generalized to read.

<u>Rule 5′</u>: The derivative of the product of any finite number of differentiable functions of the same independent variable say $f_1(x), f_2(x), \ldots, f_N(x)$ is given by

$$\frac{d}{dx}[f_1(x) \cdot f_2(x) \ldots f_N(x)] = f'_1(x)[f_2(x) \cdot f_3(x) \ldots f_N(x)] + f'_2(x)[f_1(x) \cdot f_3(x) \ldots f_N(x)]$$

$$+ \ldots + f'_N(x)[f_1(x) \cdot f_2(x) \ldots f_{N-1}(x)].$$

<u>Examples</u>:

#1 Let $g(x) = 2x^{1/2}$ and $f(x) = x^2 + 1$ find

$$\frac{d}{dx}[g(x) \cdot f(x)] = \frac{d}{dx}[2x^{1/2}(x^2 + 1)]$$

Using the product rule, we have

$$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d\left(2x^{1/2}\right)}{dx}(x^2 + 1) + \frac{d\left(x^2 + 1\right)}{dx}(2x^{1/2})$$

$$= x^{-1/2}(x^2 + 1) + 2x\,(2x^{1/2})$$

$$= x^{3/2} + x^{-1/2} + 4x^{3/2}$$

$$\frac{d}{dx}(f \cdot g) = 5x^{3/2} + x^{-1/2}$$

#2 Let $f(x) = \left(\frac{x^5}{5} + x^2\right)(5x + 6)$

Find $\frac{d}{dx}f(x)$. Here we may consider f(x) as the product of two functions, say

$h(x) = \left(\frac{x^5}{5} + x^2\right)$, and $g(x) = (5x + 6)$.

Hence $\frac{d}{dx}f(x) = \frac{d\left(h(x) \cdot g(x)\right)}{dx}$. Using the product rule we obtain

$$f'(x) = (x^4 + 2x)(5x + 6) + x^5 + 5x^2$$

$$= x^4 5x + 6x^4 + 10x^2 + 12x + x^5 + 5x^2$$

$$= x^5 5 + 6x^4 + 15x^2 + 12x + x^5$$

$$f'(x) = 6x^5 + 6x^4 + 15x^2 + 12x.$$

Def. The *quotient* of any two real-valued functions f(x) and g(x), where $f: D \to R$, $g: E \to R$, is the real-valued function $f/g: D \cap E \to R$, defined for $g(x) \neq 0$, whose value at any $x \in D \cap E$ is the quotient of the two real numbers $f(x) / g(x)$, $g(x) \neq 0$.

<u>Rule 6</u>: (Quotient Rule) At a point where $g(x) \neq 0$, the derivative of the quotient of two differentiable functions f(x) and g(x) is given by

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{\left[g(x)\right]^2}.$$

Examples:

#1 Find f′(x), where $f(x) = \frac{x^3}{g(x)}$, g(x) differentiable fn of x.

$$\frac{3x^2 g(x) - g'(x)x^3}{\left[g(x)\right]^2}$$

#2 Find $\frac{d}{dx}\left[\frac{(x^2+1)(x^3+4)}{x^2}\right]$, where $x \neq 0$

Here we must use the product rule and the quotient rule.

$$\frac{\left[2x(x^3+4) + 3x^2(x^2+1)\right]x^2 - 2x\left[(x^2+1)(x^3+4)\right]}{x^4}$$

#3 Find $\frac{d}{dx}\frac{(x^2)}{-3x} = \frac{-2x3x + 3x^2}{(3x)^2} = \frac{-6x^2 + 3x^2}{9x^2} = \frac{-3x^2}{9x^2} = -\frac{1}{3}.$

4. The following rule extends the power function rule.

Rule 7: If f(x) is a differentiable function of x and N is any real number, then

$$\frac{d}{dx}(f(x))^N = N[f(x)]^{N-1}f'(x)$$

Examples:

#1 Let $f(x) = (x^2 + 5)^{20}$, find

$$\frac{d}{dx}(x^2 + 5)^{20} = 20(x^2 + 5)^{19}2x = 40x(x^2 + 5)^{19}$$

#2 Find $\frac{d}{dx}(x^3+x)^{20}(x + 1)^3$

$= 20(x^3 + x)^{19}(3x^2 +1)(x + 1)^3 + 3(x + 1)^2 (x^3 + x)^{20}$

5. Composite functions

a. Suppose that we have a function z = f(y), where f: Y $\rightarrow$Z. Moreover, suppose that y itself is a function of a variable x, or that y = g(x), where g: X $\rightarrow$ Y. Here we have a situation where X is mapped into Y via g and Y is mapped into Z via f. Hence, the domain of f coincides with the range of g. In this case it is possible to define a composite function.

$$h(x) = f(g(x)).$$

b. *Def* The *composite function* of any two functions z = f(y) and y = g(x), where

$$X \xrightarrow{g} Y \xrightarrow{f} Z$$

such that the domain of f coincides with the range of g, is a function h(x), where

$$h : X \rightarrow Z,$$

defined by h(x) = f(g(x)) for every member x $\in$ X.

<u>Notation</u> Sometimes the symbol f o g is used to denote the composite function f(g(x)). Composite functions are also called composition of functions or composed functions.

<u>Examples</u>

#1 Consider the two functions g: R $\rightarrow$ R and f: R $\rightarrow$ R defined by

y = g(x) = 3x -1

z = f(y) = 2y + 3

for any real numbers x, y. From our definition we may form the function f o g or f[g(x)]:

f o g = f[g(x)] = 2(3x-1) + 3 = z,

where f o g is defined for every real x.

#2 Consider two functions g(x) and f(x), where g: R $\rightarrow$ R, f: R $\rightarrow$ R. Define these by

f(x) = 5x + 2

g(x) = 3x - 1

Here we may construct f o g and g o f, since

$$R \xrightarrow{g} R \xrightarrow{f} R \quad \text{and} \quad R \xrightarrow{f} R \xrightarrow{g} R.$$

f[g(x)] = f o g = 5(3x - 1) + 2

g[f(x)] = g o f = 3(5x + 2) - 1

Thus

f o g = 15x - 3

g o f = 15x + 5

note that f o g ≠ g o f.

Rule 8: (Chain Rule) If z = f(y) is a differentiable function of y and y = g(x) is a differentiable function of x, then the composite function f o g or f[g(x)] is a differentiable function of x and

$$\frac{d}{dx} f[g(x)] = \frac{dz}{dy}\frac{dy}{dx} = f'[g(x)] \cdot g'(x).$$

Remark: The chain rule may be extended to any finite number of functions. For example, if z = f(y), y = g(x) and x = h(v) then

$$\frac{d}{dx} z = \frac{d}{dx} f\{g[h(v)]\} = \frac{dz}{dy}\frac{dy}{dx}\frac{dx}{dv}, \text{ or}$$

$$= f'\{g[h(v)]\}\ g'[h(v)]\ h'(v).$$

Examples:

#1 Let $z = 6y + 1/2y^2$ and y =3x. Then

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

$$=(6 + y)3$$

but y = 3x

$$\frac{dz}{dx} = (6 + 3x)\ 3 = 9x + 18$$

#2 Let z = $\frac{y+1}{y^2}$, y = 5x

$$\frac{dz}{dx} = \frac{1(y^2) - 2y(y+1)}{(y^2)^2}(5)$$

$$= \frac{5(y^2 - 2y^2 - 2y)}{y^4} = \frac{5(-y^2 - 2y)}{y^4} = \frac{-5y^2 - 10y}{y^4} = \frac{y(-5y - 10)}{y^4}$$

$$\frac{dz}{dx} = \frac{-(5y+10)}{y^3}$$

but y = 5x

$$\frac{dz}{dx} = \frac{-5(5x) - 10}{(5x)^3}$$

#3 Using Rule 7, let z = $(x^3 + 5x)^{90}$

define y = $x^3 + 5x$

and then z = y^{90}, where y = g(x)

$$\frac{dz}{dx} = 90y^{89}(3x^2 + 5)$$

$$\frac{dz}{dx} = 90(x^3 + 5x)^{89}(3x^2 + 5).$$

Remark: The chain rule, then involves a chain of steps:

1. Differentiable the outside fn. with respect to y
2. Differentiate the inside fn with respect to x
3. Take the product of the two derivatives and substitute y for x according to y = g(x).

#4 Let z = 3y, y = x^2 and x = 5v. Find dz / dv

$$\frac{dz}{dv} = \frac{dz}{dy}\frac{dy}{dx}\frac{dx}{dv}$$

= (3) (2x) (5) = 30x

but x = 5v $\therefore$

$$\frac{dz}{dv} = 150v.$$

6. Inverse functions.

Def. The *inverse function* of a function y =f(x), where f: X $\rightarrow$ Y, is a function x =f $^{-1}$(y), where f $^{-1}$: f[X] $\rightarrow$ X. We have that y = f(x) if and only if x = f^{-1}(y) for all (x,y) $\in$ (X,f[X]).

Proposition. The function y = f(x) is one-to-one if and only if the inverse function x = f $^{-1}$(y) exists.

<u>Remark</u>: Clearly if f is bijective, then an inverse function f $^{-1}$ exists and f $^{-1}$: Y $\rightarrow$ X, since f[X] = Y.

<u>Remark</u>: When our variables x and y refer to real numbers, an injective (one-to-one) function is called *monotonic.* Monotonic functions will, then, always have inverse functions from the proposition above. Monotonic functions are classified as *monotonically increasing* or *monotonically decreasing.*

Def. A monotonic function f(x), f: R $\rightarrow$ R is *monotonically increasing* iff for any x′, x″ $\in$ R, if x′ > x″ then f(x′) > f(x″).

Def. A monotonic function f(x), f: R $\rightarrow$ R is a *monotonically decreasing* function iff for any x′, x″ $\in$ R, x′ > x″implies f(x′) < f(x″).

<u>Remark</u>: A practical method of determining whether a particular differentiable function is monotonic is to see if its derivative never changes sign. (> 0 if $\uparrow$, < 0 if $\downarrow$)

Proposition. (Inverse function Rule) If the differentiable function y = f(x) is injective, the inverse function x = f $^{-1}$(y) is differentiable and

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(x)}, \text{ or}$$

$$\frac{dx}{dy} = \frac{1}{dy/dx}.$$

Proposition. If a function $y = f(x)$, $f : X \to Y$, is injective, then

(i) the inverse function $x = f^{-1}(y)$ exists,

(ii) $f^{-1}(y)$ is injective,

(iii) $(f^{-1})^{-1} = f$.

<u>Examples</u>

#1 Let $y = f(x) = 5x + 4$, $f: R \to R$.

f is bijective $\therefore \exists\, f^{-1}(y)$.

$$f^{-1}(y) = \frac{y-4}{5} = x$$

Now find $\frac{d}{dx} f(x)$ and $\frac{d}{dy} f^{-1}(y)$ and compare.

$\frac{d}{dx} f(x) = 5$, then by inverse fn rule $\frac{d}{dy} f^{-1}(y)$ should $= 1/5$.

$$\frac{d}{dy} f^{-1}(y) = \frac{(1)5-0}{25} = \frac{5}{25} = \frac{1}{5}$$

From above we see that

$$y = 5x + 4$$

$$x = \tfrac{1}{5}y - 4/5$$

are both bijective.

#2 If $y = f(x) = (20x^2 + x^{10})^{90}$, where $f : R_+ \to R_+$ and $f^{-1}(y)$ exists, what is

$\frac{d}{dy} f^{-1}(y)$.

Here it would be extremely difficult to solve for $f^{-1}(y)$ so we use the inverse function rule.

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{dy/dx}$$

$$\frac{dy}{dx} = 90(20x^2 + x^{10})^{89}(40x + 10x^9)$$

Hence,

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{90(20x^2 + x^{10})^{89}(40x + 10x^9)}.$$

Higher Order Derivatives

1. If a function is differentiable, then its derivative function is itself a function which may possess a derivative. If this is the case, then the derivative function may be differentiated. This derivative is called the second derivative. If a derivative of the second derivative function exists, then the resultant derivative is called the third derivative. Generally, if successive derivatives exist, a function may have any number of higher order derivatives.

2. The rules of differentiation for higher order derivatives are the same as those above for the first derivative.

3. The second derivative is denoted $f''(x)$ or d^2f/dx^2 and the n^{th} order derivative is given by d^nf/dx^n.

4. Example: Let $f(x) = 3x^5 + 10x$. We have that $f' = 15x^4 + 10$, $f'' = 60x^3$, and $d^3f/dx^3 = 180x^2$.

Partial Differentiation

1. Here we consider a function of the form

$$y = f(x_1, \ldots, x_N),\ f: R^N \rightarrow R^1.$$

The function f is real valued and is a function of N independent variables. For efficiency of notation, we sometimes write

$$y = f(x), \text{ where } x \equiv (x_1, \ldots, x_N).$$

Def. The *partial derivative* of the function $f(x_1, x_2, \ldots, x_n)$, $f: R^N \rightarrow R^1$, at a point $(x_1^0, \ldots, x_N^0)$ *with respect to* x_i is given by

$$\lim_{\Delta x_i \to 0} {}^{\Delta y}\!\!/\!_{\Delta x_i} = \frac{f(x_1^o,\ldots,x_i^0+\Delta x_i,\ldots,x_N^0) - f(x_1^0,\ldots,x_N^0)}{\Delta x_i}.$$

Notation: We denote the partial derivative, $\lim_{\Delta x_i \to 0} {}^{\Delta y}\!\!/\!_{\Delta x_i}$, in each of the following ways.

$f_i(x)$, $\partial f(x)/\partial x_i$ or $\frac{\partial}{\partial x_i} f(x)$.

Remark: The partial derivative will be illustrated in class. Just as with the simple derivative, a function is said to be partially differentiable in x_i if it is partially differentiable in x_i at each point in its domain.

2. The mechanics of differentiation are very simple:

- When differentiating with respect to x_i, regard all other independent variables as constants
- Use the simple rules of differentiation for x_i.

Examples:

#1 Let $y = f(x_1,x_2) = x_1^2 + x_2 + 3$

then $\frac{\partial f}{\partial x_1} = 2x_1$ and $\frac{\partial f}{\partial x_2} = 1$

#2 Let $y = f(x_1, x_2, x_3) = (x_1 + 3)(x_2^2 + 4)(x_3)$

$$\frac{\partial f}{\partial x_1} = 1\,(x_2^2 + 4)(x_3)$$

#3 Let $y = f(x_1,x_2) = x_1^2 x_2^4 + 10x_1$

$$\frac{\partial f}{\partial x_1} = 2x_1 x_2^4 + 10$$

$$\frac{\partial f}{\partial x_2} = 4x_1^2 x_2^3$$

#4 $f(x_1,x_2) = (x_1^3 + 4)^2 (x_2 + 5)$

$$\frac{\partial f}{\partial x_1} = 2(x_1^3 + 4)\, 3x_1^2 (x_2 + 5)$$

$$\frac{\partial f}{\partial x_2} = 1(x_1^3 + 4)^2$$

3. Some extensions of the chain rule

a. Let $y = f(x_1,...,x_n)$, where $x_i = x_i(x_1)$, for $i = 2,...,n$. We have that

$$dy/dx_1 = \partial f/\partial x_1 + \sum_{i=2}^{n} \frac{\partial f}{\partial x_i} \frac{dx_i}{dx_1}.$$

b. Let $y = f(x_1,...,x_n)$, where $\forall\, i\ \ x_i = x_i(u)$. Then we have that

$$dy/du = \sum_{i=1}^{n} f_i \frac{dx_i}{du}.$$

c. Let $y = f(x_1,...,x_n)$, where $x_i = x_i(v_1,...,v_m)$, for all i. Then we have that

$$\partial y/\partial v_j = \sum_{i=1}^{n} f_i \frac{\partial x_i}{\partial v_j}.$$

<u>Examples</u>

#1 Let $y = f(x_1, x_2)$ be defined by $y = (x_1^3 + 5x_1x_2)$

where

$$x_2 = g(x_1) = 3x_1.$$

Find dy / dx_1.

$$\frac{dy}{dx_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dx_1}$$

$$\frac{dy}{dx_1} = 3x_1^2 + 5x_2 + 5x_1 \frac{dx_2}{dx_1}$$

$$= 3x_1^2 + 5x_2 + 5x_1 (3)$$

$$= 3x_1^2 + 15x_1 + 5x_2$$

Now we must substitute for

$x_2 = 3x_1$

$$\frac{dy}{dx_1} = 3x_1^2 + 15x_1 + 5(3x_1)$$

$$\frac{dy}{dx_1} = 3x_1^2 + 30x_1.$$

Now lets check our rule. Derive the composite fn $f(x_1, g(x_1))$,

$$f(x_1, g(x_1)) = x_1^3 + 5x_1(3x_1)$$

$$= x_1^3 + 15x_1^2$$

Now find dy / dx_1.

$dy / dx_1 = 3x_1^2 + 30x_1$. It checks.

#2 Let $y = f(x_1, x_2)$ be defined by $f(x_1, x_2) = x_1x_2 + 5x_1$.

where $x_1 = 3w^2$

$x_2 = w + 6$.

We have that

$$df/dw = (\partial f/\partial x_1)dx_1/dw + (\partial f/\partial x_2)dx_2/dw$$

$$= (x_2 + 5)\ 6w + x_1(1)$$

$$= (x_2 + 5)\ 6w + x_1.$$

Now substitute for the ultimate variable w,

$$\frac{dy}{dw} = 6wx_2 + 30w + x_1$$

$$= 6w\,(w+6) + 30w + 3w^2$$

$$= 6w^2 + 36w + 30w + 3w^2$$

$$\frac{dy}{dw} = 9w^2 + 66w.$$

Let us again check our result. Form the composite function

$f[g(w), h(w)]$

$f[g(w), h(w)] = 3w^2 (w + 6) + 5 (3w^2)$

$= 3w^3 + 18w^2 + 15w^2$

$= 3w^3 + 33w^2$

$$\frac{d}{dw} = f[g(w), h(w)] = 9w^2 + 66w.$$

#3 Let $y = f(x_1, x_2)$ be defined by $f(x_1, x_2) = 3x_1 + x_2^2$

where

$x_1 = v^2 + u.$

$x_2 = u + 5v.$

Find $\partial y/\partial u = \partial f/\partial x_1 \; \partial x_1/\partial u + \partial f/\partial x_2 \; \partial x_2/\partial u$

$\partial y/\partial u = 3(1) + 2x_2(1) = 3 + 2x_2$

now substitute for the ultimate variables u,v

$\partial y/\partial u = 3 + 2(u + 5v)$

$\partial y/\partial u = 3 + 2u + 10v$

Find $\partial y/\partial v = \partial f/\partial x_1 \; \partial x_1/\partial v + \partial f/\partial x_2 \; \partial x^2/\partial v$.

$\partial y/\partial u = 3(2v) + 2x_2(5)$

$= 6v + 10x_2$

again subs. for u, v,

$= 6v + 10(u + 5v)$

$= 6v + 10u + 50v$

$\partial y/\partial v = 56v + 10u$

Again, let us check our results. Form the composite fn

$f[h(u, v), g(u, v)] = 3 (v^2 + u) + (u + 5v)^2$

$= 3v^2 + 3u + u^2 + 10uv + 25v^2$

$= 28v^2 + u^2 + 10uv + 3u.$

Now find $\partial y / \partial v$,

$\partial y / \partial v = 56v + 10u$

and $\partial y / \partial u$

$\partial y / \partial u = 2u + 10v + 3$

$\therefore$ our results check.

4. Higher order partial derivatives

a. A first partial derivative function is itself a function of the n independent variables of the original function. Thus, if this function has partial derivatives, then we can define higher order partial derivative functions.

b. The most interesting partial derivative functions for applications would be the second order partial derivatives of a function $y = f(x_1,...,x_n)$. The mechanics are the same as with higher order derivatives of a function of a single independent variable. We just apply the rules of differentiation to the first partial derivative to obtain the set of second order partial derivatives.

c. A function $y = f(x_1,...,x_n)$ has n partial derivatives at each point given by f_i, $i = 1,...,n$. If each of these functions is differentiable in each of the n variables, we would have n^2 second order partial derivatives at each point. These are given by

$$f_{ij} = \frac{\partial}{\partial x_j} f_i \text{ for } j = 1,...,n.$$

Another notation for f_{ij} is $\partial^2 f/\partial x_i \partial xj$. If i and j are equal, then f_{ii} is called a direct second order partial derivative and if $i \neq j$ f_{ij} is called a cross second order partial derivative.

d. We can associate a matrix of partial derivatives to each point $(x_1,...,x_n)$ in the domain of f. The matrix is defined by $[f_{ij}(x^o)]_{i,j=1,...,n}$ and it is called the Hessian of f at the arbitrary point x^o.

e. Examples:

#1. Let $f = (x_1 + 3)(x_2^2 + 4)x_3$. We have that

$f_1 = (x_2^2 + 4)x_3$, $f_{11} = 0$, $f_{12} = 2x_2x_3$, $f_{13} = (x_2^2 + 4)$

$f_2 = (x_1 + 3)(x_3)(2x_2)$, $f_{22} = 2(x_1 + 3)x_3$, $f_{21} = 2x_2x_3$, $f_{23} = (x_1 + 3)(2x_2)$.

$f_3 = (x_1 + 3)(x_2^2 + 4)$, $f_{32} = (x_1 + 3)(2x_2)$, $f_{31} = (x_2^2 + 4)$, $f_{33} = 0$.

#2. Let $f = x_1x_2$.

$f_1 = x_2$, $f_{12} = 1$, $f_{11} = 0$,

$f_2 = x_1$, $f_{21} = 1$, $f_{22} = 0$.

The Hessian at any point (x_1,x_2) is given by

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}.$$

f. *Young's Theorem.* If the a function $f(x_1,...,x_n)$ possesses continuous cross partial derivatives a point, then $f_{ij} = f_{ji}$ for all $i \neq j$, at that point.

Remark: Check Young's theorem for the above examples

Differentials

1. Given that $y = f(x)$, a Δx will generate a Δy as discussed above. When the Δx is infinitesimal we write dx, which, thus, generates an infinitesimal change in y, dy. However, because

$$\frac{dy}{dx} \equiv f'(x)$$

we have that

$$dy = f'(x)\,dx = \frac{dy}{dx}\,dx.$$

2. Once we know $f'(x)$, it is a simple matter to calculate the differential, dy.

Example.

Let $f(x) = 3x + x^2$, find dy:

$$dy = f'(x)\,dx$$

$$dy = (3 + 2x)\, dx.$$

3. When we have a function of many independent variables, we must distinguish between partial and total differentials. Let $y = f(x_1, x_2, \ldots, x_n)$, then the partial differential with respect to x_i is given by

$$\frac{\partial f}{\partial x_i}\, dx_i$$

The total differential is given by

$$dy = \frac{\partial f}{\partial x_1}\, dx_1 + \ldots + \frac{\partial f}{\partial x_n}\, dx_n.$$

4. Rules of differentials

Proposition 1. If $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$ are differentiable functions of the variables x_i ($i = 1, \ldots, n$), then

(i) $d(f^n) = nf^{n-1}df$

(ii) $d(f \pm g) = df \pm dg$

(iii) $d(f \cdot g) = gdf + fdg$

(iv) $d\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}$, $g \neq 0$.

<u>Remark</u>: Part (ii) holds for any finite sum or difference of differentiable functions of x_i($i = 1, \ldots, n$) and part (iii) holds for any finite number of differentiable functions taken as a product

<u>Examples</u>:

#1 Let $f(x_1, x_2) = 2x_1 + x_2$, find df. Consider each part as a separate function and use part (ii).

$$df = d(2x_1) + d(x_2)$$

$$df = 2(1)\, dx_1 + dx_2 = 2dx_1 + dx_2.$$

Let us check this by using the form $df = f_1\, dx_1 + f_2 dx_2$

$$df = 2dx_1 + dx_2.$$

#2 Let $f(x_1, x_2) = \dfrac{x_1^2 + 3x_2}{x_1x_2}$

$$df = \frac{d(x_1^2 + 3x_2) \cdot x_1x_2 - d(x_1x_2) \cdot (x_1^2 + 3x)}{x_1^2x_2^2}$$

$$= \{x_1x_2\,(2x_1dx_1 + 3dx_2) - (x_2dx_1 + x_1dx_2)\,(x_1^2 + 3x_2)\}/(x_1x_2)^2$$

$$= \{2x_1^2x_2dx_1 + 3x_1x_2dx_2 - (x_1^2x_2dx_1 + 3x_2^2dx_1 + x_1^3dx_2 + 3x_1x_2dx_2)\}/(x_1x_2)^2$$

$$= \{2x_1^2x_2dx_1 + 3x_1x_2dx_2 - x_1^2x_2dx_1 - 3x_2^2dx_1 - x_1^3dx_2 - 3x_1x_2dx_2\}/(x_1x_2)^2$$

$$= \frac{(2x_1^2x_2 - x_1^2x_2 - 3x_2^2)dx_1 + (-x_1^3)dx_2}{(x_1x_2)^2}$$

$$df = \frac{(x_1^2x_2 - 3x_2^2)}{(x_1x_2)^2}dx_1 - \frac{(x_1^3)}{(x_1x_2)^2}dx_2$$

Let us check this with the other formula:

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2$$

$$= \frac{2x_1(x_1x_2) - x_2(x_1^2 + 3x_2)}{(x_1x_2)^2}dx_1 + \frac{3x_1x_2 - x_1(x_1^2 + 3x_2)}{(x_1x_2)^2}dx_2$$

$$= \frac{(2x_1^2x_2 - x2x_2^1 - 3x_2^2)}{(x_1x_2)^2}dx_1 + \frac{3x_1x_2 - x_1^3 - 3x_2x_1}{(x_1x_2)^2}dx_2$$

$$df = \frac{x_1^2x_2 - 3x_2^2}{(x_1x_2)^2}dx_1 - \frac{x_1^3}{(x_1x_2)^2}dx_2$$. It checks.

It is also possible to take a differential of a first order differential so as to define a second order differential. Let

$$df = \sum_{i=1}^{n} f_i dx_i \,.$$

Then

$$d^2f = \sum_{i=1}^{n} \sum_{i=1}^{n} f_{ij}dx_idx_j.$$

If we let dx be an n×1 column vector, we can write this expression as

$$dx'Hdx,$$

where H is the Hessian Matrix of the function f. The second order differential is then expressed as a quadratic form with discriminant H.

Derivatives of Implicit Functions

1. So far we have dealt with functions of the form $y = f(x)$, which expresses y explicitly in terms of x. If for example

$$y = (x^2 + 10x)^{-1}, x \neq 0,$$

the function may be rewritten in the form

$$y(x^2 + 10x) - 1 = 0$$

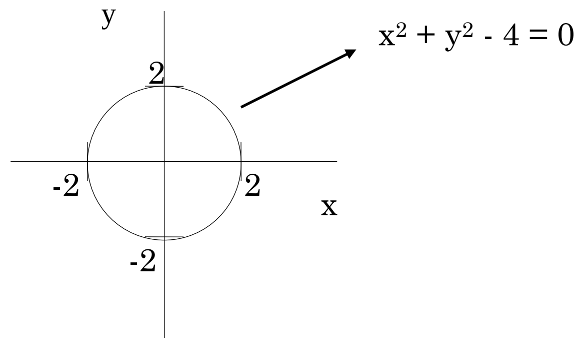
In this case, the function $y = f(x) = (x^2 + 10x)^{-1}$ is said to be defined implicitly by the above equation. That is, the equation $F(y, x) = -1 + y(x^2 +10x) = 0$ implies a function $y = f(x)$ in the sense that when x changes y must change in a unique manner in order to make $-1 + y(x^2 +10x) = 0$.

2. In general we would call the expression $F(x, y) = 0$ an implicit equation where y is implicitly defined as a function of x, f(x). Actually, the equation $F(x, y) = 0$ will constitute, in general, a relation between the variables x and y. When a definite number say x^1, from some domain, is substituted for x in $F(x, y) = 0$, the resulting equation will determine <u>one</u> <u>or</u> <u>more</u> values of y to be associated with the given x^1 such that $F(x^1, y) = 0$. We, therefore say that the equation F(x, y) will determine y as one or more implicit functions of x.

3. As an example, consider the equation defined by

$$F(x, y) = x^2 + y^2 - 4 = 0$$

which plots a circle of radius 2. ($x^2 + y^2 - r^2 = 0$ is the equation for a circle with center = 0)



Here every x $\in$ (-2, 2) is associated with two values of y. Clearly, $F(x, y) = x^2 + y^2 - 4 = 0$ represents a relation. However, in this case, it is possible to define two implicit functions of y in terms of x. Define the functions

$y = f(x) = \sqrt{4 - x^2}$, where

f: [-2, +2] $\rightarrow$ [0, 2], and

$y = g(x) = -\sqrt{4 - x^2}$, where

g: [-2, +2] $\rightarrow$ [-2, 0].

4. Given a relation of the form F(x, y) = 0, the derivatives of differentiable implicit functions may be found by finding the implicit functions themselves and differentiating. However, it may be that solving F(x, y) for an implicit function y = f(x) entails extreme difficulties. As an example, consider $F(x, y) = x^5 + 4xy^3 - 3y^5 - 2 = 0$. In cases of this type it is still possible to find dy / dx by using the method of implicit differentiation.

Proposition 1. The relation F(x, y) = 0 defines one or more differentiable implicit functions of the form y = f(x), at some point x, if $\partial F / \partial y \neq 0$. We have that

$$\frac{dy}{dx} = \frac{-\partial F / \partial x}{\partial F / \partial y}.$$

Proof: Take total differentials of both sides of the equation

F(x, y) = 0

dF= d(0) = 0

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0$$

solving

$$\frac{dy}{dx} = \frac{-\partial F / \partial x}{\partial F / \partial y} . \|$$

Example:

#1 Let us apply this rule to the equation of the circle given above.

$$F(x, y) = x^2 + y^2 - 4 = 0$$

We defined two implicit functions from this equation:

$$y = f(x) = \sqrt{4 - x^2} \text{ ; f: } [-2, +2] \rightarrow [0, 2]$$

$$y = g(x) = -\sqrt{4 - x^2} \text{ ; g: } [-2, +2] \rightarrow [-2, 0]$$

Differentiating we find

$$f(x) = (4 - x^2)^{1/2}$$

$$g(x) = - (4 - x^2)^{1/2}$$

$$f'(x) = 1/2\ (4 - x^2)^{-1/2}\ (-2x)$$

$$f'(x) = -x(4 - x^2)^{-1/2} = -x/y$$

and

$$g'(x) = -1/2\ (4 - x^2)^{-1/2}\ (-2x)$$

$$g'(x) = +x\ (4 - x^2)^{-1/2} = -x/y$$

Let us check the implicit function theorem:

$$\frac{dy}{dx} = \frac{-\partial F / \partial x}{\partial F / \partial y} = \frac{-2x}{2y} = \frac{-x}{y},$$

and the theorem holds.

#2 Let $F(x, y) = x^5 + 4xy^3 - 3y^5 - 2 = 0$ find dy / dx

$$\frac{dy}{dx} = \frac{-\partial F / \partial x}{\partial F / \partial y} = \frac{-(5x^4 + 4y^3)}{(12xy^2 - 15y^4)}, \text{ for all points where}$$

$12xy^2 - 15y^4 \neq 0.$

5. We may generalize Proposition 1 as follows:

Proposition 2. The relation $F(x_1, x_2, x_3, \ldots, x_N) = 0$ defines one or more differentiable implicit functions of the form $x_i = f_i(x_j)$, at x_j, if $\partial F / \partial x_i \neq 0$, i, j=1, …, N, $i \neq j$. We have that

$$\frac{\partial x_i}{\partial x_j} = \frac{-\partial F / \partial x_j}{\partial F / \partial x_i} .$$

Proof: Take the total differential of both sides of the equation

$$F(x_1, \ldots, x_N) = 0$$

$$\frac{\partial F}{\partial x_i} dx_1 + \ldots + \frac{\partial F}{\partial x_N} dx_N = 0.$$

Since we search for the expression $\partial x_i / \partial x_j$ we hold x_k constant for all $k \neq i$ and $k \neq j$. Hence, we have that $dx_K = 0$ for all $k \neq i$ or j, so that

$$\frac{\partial F}{\partial x_i} dx_i + \frac{\partial F}{\partial x_j} dx_j = 0$$

or that $\left(\frac{dx_i}{dx_j}\right) = \frac{-\partial F / \partial x_j}{\partial F / \partial x_i}$ all other variables constant.

or that $\left(\frac{\partial x_i}{\partial x_j}\right) = \frac{-\partial F / \partial x_j}{\partial F / \partial x_i}$, for $\partial F / \partial x_i \neq 0$ ||

Example

#1 Let $F(x_1, x_2, x_3)$ be defined by

$$F(x_1, x_2, x_3) = x_1^3 x_2 + x_3 x_2 = 0$$

Find $\partial x_2 / \partial x_3$

$$\frac{\partial x_2}{\partial x_3} = \frac{-\partial F / \partial x_3}{\partial F / \partial x_2} = \frac{-x_2}{(x_1^3 + x_3)}, \text{ for } (x_1^3 + x_3) \neq 0.$$

#2 Let $F(y, x_1, x_2)$ implicitly define a production function:

$$y - F(x_1, x_2) = 0$$

Find $\frac{\partial x_1}{\partial x_2} = \frac{-\partial F / \partial x_2}{\partial F / \partial x_1}$ = M.R.T.S.

this measures the slope of an isoquant.

A Generalized Implicit Function Theorem

1. Consider the set of n equations

$$0 = F^i(y_1,...y_n,x_1,...,x_m),\ i = 1,...,n.$$

The variables $y_1,...y_n$ represent n independent variables and the $x_1,...,x_m$ represent m parameters.

2. First let us define the Jacobian matrix of this system with respect to $y \in R^n$ as the matrix of first order partial derivatives of F^i with respect to y_i. This Jacobian is denoted J_y.

$$J_y = \begin{bmatrix} \partial F^1 / \partial y_1 & \bullet & \bullet & \bullet & \partial F^1 / \partial y_n \\ \bullet & & & & \bullet \\ \bullet & & & & \bullet \\ \bullet & & & & \bullet \\ \partial F^n / \partial y_1 & \bullet & \bullet & \bullet & \partial F^n / \partial y_n \end{bmatrix}$$

3. *Proposition 3.* Let $F^i(y,x) = 0$ possess continuous partial derivatives in y and x. If at a point (y^o,x^o), $|J_y| \neq 0$, then $\exists$ n functions $y_i = f^i(x_1,...,x_m)$ which are defined in a neighborhood of x^o. We have

(i) $F^i(f^1,...f^n,x^o) = 0,\ \forall i,$

(ii) $y_i = f^i(\cdot)$ are continuously differentiable in x locally

(iii) $$J_y \begin{bmatrix} \partial y_1 / \partial x_k \\ \bullet \\ \bullet \\ \bullet \\ \partial y_n / \partial x_k \end{bmatrix} = \begin{bmatrix} -\partial F^1 / \partial x_k \\ \bullet \\ \bullet \\ \bullet \\ -\partial F^n / \partial x_k \end{bmatrix} \text{ and } \begin{bmatrix} \partial y_1 / \partial x_k \\ \bullet \\ \bullet \\ \bullet \\ \partial y_n / \partial x_k \end{bmatrix} = J_y^{-1} \begin{bmatrix} -\partial F^1 / \partial x_k \\ \bullet \\ \bullet \\ \bullet \\ -\partial F^n / \partial x_k \end{bmatrix}.$$

Remark: Using Cramer's Rule, we have that

$$\partial y_i/\partial x_k = |J_{yi}|/|J_y|.$$

Example: Solve the following system for system $\partial y_1/\partial x$ and $\partial y_2/\partial x$.

$$2y_1 + 3y_2 - 6x = 0$$

$$y_1 + 2y_2 = 0.$$

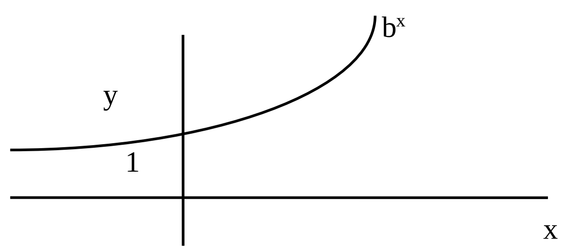
Exponential and Logarithmic Functions

1. An exponential function is a function in which the independent variable appears as an exponent:

$$y = b^x, \text{ where } b > 1.$$[1]

A logarithmic function is the inverse function of b^x. That is,

$$x = \log_b y.$$



2. A preferred base is the number $e \cong 2.72$. More formally, $e = \lim_{n\to\infty} [1 + 1/n]^n$. e is called the natural logarithmic base. It has the desirable property that $\frac{d}{dx} e^x = e^x$. The corresponding log function is written $x = \ln y$, meaning $\log_e y$.

3. Rules of logarithms.

[1] We take $b > 0$ to avoid complex numbers. $b > 1$ is not restrictive because we can take $(b^{-1})^x$ =

R. 1: If x, y > 0, then log (yx) = log y + log x.

Proof: $e^{\ln x} = x$ and $e^{\ln y} = y$ so that $xy = e^{\ln x} e^{\ln y} = e^{\ln x + \ln y}$. Moreover $xy = e^{\ln xy}$. Thus, $e^{\ln x+\ln y} = e^{\ln xy}$. ||

R. 2: If x, y > 0, then log (y/x) = log y - log x.

Proof: $e^{\ln x} = x$ and $e^{\ln(1/y)} = y^{-1}$ so that $x/y = e^{\ln x} e^{\ln(1/y)}$. Moreover $x/y = e^{\ln x/y}$. Thus, $e^{\ln x} e^{\ln(1/y)} = e^{\ln x/y}$ and $\ln x + \ln(y^{-1}) = \ln x/y$. Using rule 3 (to be shown) the result holds.

R. 3: If x > 0, then log x^a = a log x.

Proof: $e^{\ln x} = x$ and $e^{\ln x^a} = x^a$. However, $x^a = (e^{\ln x})^a = e^{a\ln x}$. Thus, $e^{\ln x^a} = e^{a\ln x}$. ||

4. Conversion and inversion of bases.

a. conversion

$\log_b u = (\log_b c)(\log_c u)$ ($\log_c u$ is known)

Proof: Let $u = c^p$. Then $\log_c u = p$. We know that $\log_b u = \log_b c^p = p\log_b c$. By definition, $p = \log_c u$, so that $\log_b u = (\log_c u)(\log_b c)$. ||

b. inversion

$\log_b c = 1/(\log_c b)$.

Proof: Using the conversion rule, $1 = \log_b b = (\log_b c)(\log_c b)$. Thus, $\log_b c = 1/(\log_c b)$. ||

5. Derivatives of Exponential and Logarithmic functions.

R. 1: The derivative of the log function y = ln f(x) is given by

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}.$$

R. 2: The derivative of the exponential function $y = e^{f(x)}$ is given by

$$\frac{dy}{dx} = f'(x)e^{f(x)}.$$

R. 3: The derivative of the exponential function $y = b^{f(x)}$ is given by

---

b^{-x} for this case.

$$\frac{dy}{dx} = f'(x)b^{f(x)}\ln b.$$

A special case is where $y = f(x)^{g(x)}$, $f(x) > 0$. We can take $\ln y = g(x)\ln f(x)$. For this case,

$$d\ln y/dx = (dy/dx)(1/y) = [g'(x)\ln f + g(x)f'(x)/f].$$

Thus,

$$dy/dx = f(x)^{g(x)}[g'(x)\ln f + g(x)f'(x)/f].$$

R. 4: The derivative of the logarithmic function $y = \log_b f(x)$ is given by

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}\frac{1}{\ln b}$$

Examples:

#1 Let $y = e^{2x^2+4}$, then

$$\frac{dy}{dx} = 4x\,e^{2x^2+4}.$$

#2 Let $y = [\ln(16x^2)]x^2$

$$\frac{dy}{dx} = \frac{32x}{16x^2}x^2 + 2x\ln(16x^2)$$

$$= 2x + 2x\ln 16x^2 = 2x + 2x(\ln 16 + 2\ln x)$$

$$= 2x(1 + \ln 16) + 4x\ln x.$$

#3 Let $y = 6^{x^2+17x}$, then $f' = (2x + 17)\,6^{x^2+17x}\ln 6$.

The Taylor Series Approximation

1. Given a function, it is sometimes of interest to approximate or estimate that function at a point with the use of a polynomial function. This approximation uses the derivatives of the function in the coefficients of the various terms in the estimated polynomial function. Linear and quadratic approximations are typically used.

2. First consider a single variable function $y = f(x)$. A *Taylor's Series Approximation* of f at x^o is given by

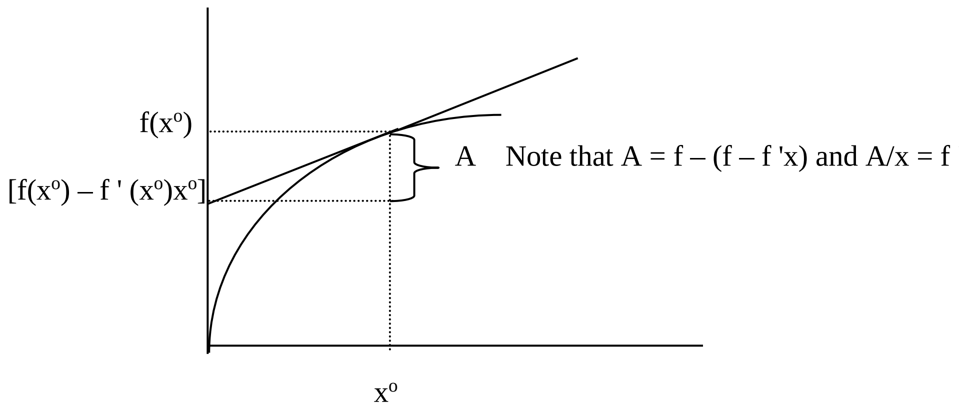
$$f(x) \approx f(x^o) + \sum_{i=1}^{k} \frac{1}{i!} \frac{d^i f(x^o)}{dx^i} (x - x^o)^i.$$

A *linear approximation* takes $k = 1$. This is given by

$$f(x) \approx f(x^o) + [df(x^o)/dx](x-x^o) = [f(x^o) - f\,'(x^o)x^o] + f\,'(x^o)x.$$

Defining, $a \equiv [f(x^o) - f\,'(x^o)x^o]$ and $b \equiv f\,'(x^o)$, we have

$$f(x) \approx a + bx.$$



A quadratic approximation takes $k = 2$. We have that

$$f(x) \approx f(x^o) + f\,'(x^o)(x-x^o) + \frac{f\,''(x^o)}{2}(x - x^o)^2.$$ This expression can be written as

$$f(x) \approx [f(x^o) - f\,'(x^o)x^o + f\,''(x^o)(x^o)^2/2] + [f\,'(x^o) - x^o f\,''(x^o)]x + [f\,''(x^o)/2]x^2,$$

which is of the form

$$f(x) \approx a + bx + cx^2.$$

3. In the general case of a function of many variables, the Taylor's Series Approximation can be generalized for the cases above. A general linear approximation for a function $y = f(x_1,\ldots,x_n)$ is written as

$$f(x) \approx f(x^o) + \sum_{i=1}^{n} f_i(x^o)(x_i - x_i^o).$$

Define $\nabla f(x)' \equiv (f_1(x)\ldots f_n(x))$ and term this the gradient of f. Then we have

$$f(x) \approx f(x^o) + \nabla f(x^o)'(x - x^o),$$

in matrix notation. A quadratic approximation is given by

$$f(x) \approx f(x^o) + \nabla f(x^o)'(x - x^o) + \frac{1}{2}\sum_i \sum_j f_{ij}(x^o)(x_i - x_i^o)(x_j - x_j^o).$$

In matrix notation,

$$f(x) \approx f(x^o) + \nabla f(x^o)'(x - x^o) + \frac{1}{2}(x - x^o)'H(x^o)(x - x^o).$$

4. *Examples*

a. Consider the function $y = 2x^2$. Let us construct a linear approximation at $x = 1$.

$$f \approx 2 + 4(x - 1) = -2 + 4x.$$

A quadratic approximation at $x = 1$ is

$$f \approx 2 + 4(x - 1) + (4/2)(x-1)^2 = 2 + 4x - 4 + 2(x^2 - 2x + 1) = 2 + 4x - 4 + 2x^2 - 4x + 2)$$

$$f = 2x^2.$$

Given that the original function is quadratic, the approximation is exact.

b. Let $y = x_1^3 x_2$. Construct a linear and a quadratic approximation of f at (1,1). The linear approximation is given by

$$f \approx 1 + 3(x_1 - 1) + 1(x_2 - 1).$$

Rewriting,

$$f \approx -3 + 3x_1 + x_2.$$

A quadratic approximation is constructed as follows:

$$f \approx 1 + 3(x_1 - 1) + 1(x_2 - 1) + \frac{1}{2}\,[x_1 - 1 \;\; x_2 - 1]\begin{bmatrix} 6 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}.$$

Expanding terms we obtain the following expression.

$$f \approx 3 - 6x_1 - 2x_2 + 3x_1x_2 + 3x_1^2.$$