Lecture 5: Rules of Differentiation

- First order derivatives
- Higher order derivatives
- Partial differentiation
- Higher order partials
- Differentials
- Derivatives of implicit functions
- Generalized implicit function theorem
- Exponential and logarithmic functions
- Taylor series approximation

First Order Derivatives

- Consider functions of a single independent variable, $f : X \to R$, X an open interval of R.

R1(constant function) $f(x) = k \Rightarrow f'(x) = 0$.

R2 (power function) $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

R3 (multiplicative constant) $f(x) = kg(x) \Rightarrow f'(x) = kg'(x)$.

Examples R1-R3

#1 Let $y = f(x) = \sqrt[3]{x}$. Find dy / dx $\sqrt[3]{x} = (x)^{1/3}$. Hence, d / dx $(x)^{1/3} = 1/3(x)^{1/3-1} = \underline{1/3(x)^{-2/3}}$.

#2 Let y = f(x) = 3, where f: $R_+ \rightarrow R$. Find dy /dx = 0

#3 Let $y = f(x) = 10x^0$. Find dy / dx = 0

#4 Let $y = f(x) = 20\ x^{4/5}$. Find dy /dx = $16x^{-1/5}$.

Two or more functions of the same variable: sum-difference

- Def 1. By the sum (difference) of any two real-valued functions f(x) and g(x), where f: $D \rightarrow R$ and g: $E \rightarrow R$, we mean the real-valued function $f \pm g: D \cap E \rightarrow R$ whose value at any $x \in D \cap E$ is the sum (difference) of two real numbers f(x) and g(x). In symbols we have

  $(f \pm g)(x) = f(x) \pm g(x)$, for any $x \in D \cap E$.

Rules

R4 (sum-difference) $g(x) = \Sigma_i f_i(x) \Rightarrow g'(x) = \Sigma_i f_i'(x)$.

Remark: To account for differences, simply multiply any of the f_i by -1 and use the multiplicative constant rule.

Examples

#1 What is the slope of the curve $y= f(x) = x^3 - 3x + 5$, when it crosses the y -axis?

First find

$$\frac{dy}{dx} = \frac{d}{dx}x^3 - 3\frac{d}{dx}x + \frac{d}{dx}5$$

$dy / dx = 3x^2 - 3 = f'(x)$.

When the curve crosses the y - axis, we have x = 0. Hence, evaluate $f'(x)$ for x = 0.

$f'(0) = -3$.

Thus the slope at x = 0 is -3.

Examples

#2 Find $f'(x)$ if $f(x) = \frac{x^3}{x^2} - 7x^{-1/2} + 5$.

Clearly we may write f(x) as

$$f(x) = x - 7x^{-1/2} + 5$$

Hence

$$f'(x) = \frac{d}{dx}x - 7\frac{d}{dx}x^{-1/2} + \frac{d}{dx}5$$

$$f'(x) = 1 + \frac{7}{2}x^{-3/2}$$

Product

- *Def* By the *product* of any two real valued functions f(x) and g(x), where $f: D \to R$ and $g: E \to R$, we mean the real function

  $$fg : D \cap E \to R$$

  whose value at any $x \in D \cap E$ is the product of the two real numbers f(x) and g(x).

Product Rule

R5 (product) $h(x) = f(x)g(x) \Rightarrow$
$h'(x) = f'(x)g(x) + f(x)g'(x)$

Remark: This rule can be generalized as

$$\frac{d}{dx}[f_1(x) \cdot f_2(x) \ldots f_N(x)] = f'_1(x)[f_2(x) \cdot f_3(x) \ldots f_N(x)] + f'_2(x)[f_1(x) \cdot f_3(x) \ldots f_N(x)]$$

$$+ \ldots + f'_N(x)[f_1(x) \cdot f_2(x) \ldots f_{N-1}(x)].$$

Example

Let $f(x) = \left(\frac{x^5}{5} + x^2\right)(5x + 6)$

Find $\frac{d}{dx} f(x)$. Here we may consider $f(x)$ as the product of two functions, say

$h(x) = \left(\frac{x^5}{5} + x^2\right)$, and $g(x) = (5x + 6)$.

Hence $\frac{d}{dx} f(x) = \frac{d\left(h(x) \cdot g(x)\right)}{dx}$. Using the product rule we obtain

$f'(x) = (x^4 + 2x)(5x + 6) + x^5 + 5x^2$

$= x^4 5x + 6x^4 + 10x^2 + 12x + x^5 + 5x^2$

$= x^5 5 + 6x^4 + 15x^2 + 12x + x^5$

$f'(x) = 6x^5 + 6x^4 + 15x^2 + 12x.$

Quotient

- *Def*. The *quotient* of any two real-valued functions f(x) and g(x), where $f: D \rightarrow R$, $g: E \rightarrow R$, is the real-valued function $f/g: D \cap E \rightarrow R$, defined for $g(x) \neq 0$, whose value at any $x \in D \cap E$ is the quotient of the two real numbers $f(x) / g(x)$, $g(x) \neq 0$.

Quotient Rule

- R6 (quotient) $h(x) = f(x)/g(x) \Rightarrow$

$$h'(x) = [f'(x)g(x) - g'(x)f(x)]/[g(x)]^2$$

Examples

#1 Find f′(x), where $f(x) = \frac{x^3}{g(x)}$, g(x) differentiable fn of x.

$$\frac{3x^2 g(x) - g'(x)x^3}{[g(x)]^2}$$

#2 Find $\frac{d}{dx}\left[\frac{(x^2+1)(x^3+4)}{x^2}\right]$, where $x \neq 0$

Here we must use the product rule and the quotient rule.

$$\frac{\left[2x(x^3+4) + 3x^2(x^2+1)\right]x^2 - 2x\left[(x^2+1)(x^3+4)\right]}{x^4}$$

Composite functions

Def The *composite function* of any two functions z = f(y) and y = g(x), where

$$X \xrightarrow{g} Y \xrightarrow{f} Z$$

such that the domain of f coincides with the range of g, is a function h(x), where

$$h : X \to Z,$$

defined by h(x) = f(g(x)) for every member $x \in X$.

Remark: The notation f o g is used to denote the composite function f(g(x)).

Example

- max $u(x_1,x_2)$ st $I = p_1 x_1 + p_2 x_2$
- $x_2 = I/p_2 - (p_1/p_2) x_1 = g(x_1)$
- $u(x_1, I/p_2 - (p_1/p_2) x_1) = u(x_1,g(x_1))$
- max $u(x_1,g(x_1))$

Example

Consider the two functions g: $R \rightarrow R$ and f: $R \rightarrow R$ defined by

$y = g(x) = 3x - 1$

$z = f(y) = 2y + 3$

for any real numbers x, y. From our definition we may form the function f o g or f[g(x)]:

$f \circ g = f[g(x)] = 2(3x-1) + 3 = z$,

where f o g is defined for every real x.

Composite function rule

- R7 (chain) If z = f(y) is a differentiable function of y and y = g(x) is a differentiable function of x, then the composite function f o g or h(x) = f[g(x)] is a differentiable function of x and

$$h'(x) = f'[g(x)] \cdot g'(x).$$

Remark: This rule can be extended to any finite chain. e.g., h = f(g(r(x))) ⇒
h' = f'(g)g'(r)r'(x)

Examples

#1 Let $z = 6y + 1/2y^2$ and $y = 3x$. Then

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

$$=(6 + y)3$$

but y = 3x

$$\frac{dz}{dx} = (6 + 3x)\,3 = 9x + 18$$

Examples

#2 Let $z = \frac{y+1}{y^2}$, $y = 5x$

$$\frac{dz}{dx} = \frac{1(y^2) - 2y(y+1)}{(y^2)^2}(5)$$

$$= \frac{5(y^2 - 2y^2 - 2y)}{y^4} = \frac{5(-y^2 - 2y)}{y^4} = \frac{-5y^2 - 10y}{y^4} = \frac{y(-5y - 10)}{y^4}$$

$$\frac{dz}{dx} = \frac{-(5y+10)}{y^3}$$

but $y = 5x$

$$\frac{dz}{dx} = \frac{-5(5x) - 10}{(5x)^3}$$

Inverse function

- *Def.* The *inverse function* of a function $y = f(x)$, where $f: X \rightarrow Y$, is a function $x = f^{-1}(y)$, where $f^{-1}: f[X] \rightarrow X$. We have that $y = f(x)$ if and only if $x = f^{-1}(y)$ for all $(x,y) \in Gr(f)$.

- *Proposition.* The function $y = f(x)$ is one-to-one if and only if the inverse function $x = f^{-1}(y)$ exists.

Inverse function

- A one-to-one function defined on real numbers is called *monotonic*. A monotonic function is either increasing or decreasing.
- *Def.* A monotonic function f(x), f: $R \rightarrow R$ is *monotonically increasing* iff for any $x', x'' \in R$, $x' > x''$ implies $f(x') > f(x'')$.
- *Def.* A monotonic function f(x), f: $R \rightarrow R$ is a *monotonically decreasing* function iff for any $x', x'' \in R$, $x' > x''$ implies $f(x') < f(x'')$.

Inverse function

- Remark: A practical method of determining whether a particular differentiable function is monotonic is to see if its derivative never changes sign. (> 0 if ↑, < 0 if ↓)

Example

- Direct demand Q = D(p)
- Inverse demand $p = D^{-1}(Q) = p(Q)$
- Q = 5-p. What is inverse demand?

Inverse function rule

- R8 (inverse function) Given $y = f(x)$ and $x = f^{-1}(y)$, we have

$$f^{-1\prime}(y) = 1/f'(x).$$

- *Proposition.* If a function $y = f(x)$, $f : X \rightarrow Y$, is one-to-one, then

(i) the inverse function $x = f^{-1}(y)$ exists,

(ii) $f^{-1}(y)$ is one-to-one,

(iii) $(f^{-1})^{-1} = f$.

Examples

Let $y = f(x) = 5x + 4$, $f: R \rightarrow R$.

f is bijective $\therefore \exists f^{-1}(y)$.

$$f^{-1}(y) = \frac{y-4}{5} = x$$

Now find $\frac{d}{dx} f(x)$ and $\frac{d}{dy} f^{-1}(y)$ and compare.

$\frac{d}{dx} f(x) = 5$, then by inverse fn rule $\frac{d}{dy} f^{-1}(y)$ should $= 1/5$.

$$\frac{d}{dy} f^{-1}(y) = \frac{(1)5 - 0}{25} = \frac{5}{25} = \frac{1}{5}$$

From above we see that

$y = 5x + 4$

$x = \frac{1}{5} y - 4/5$

are both bijective.

Higher Order Derivatives

- If a function is differentiable, then its derivative function is itself a function which may possess a derivative.
- If this is the case, then the derivative function may be differentiated. This derivative is called the *second derivative*.
- If a derivative of the second derivative function exists, then the resultant derivative is called the *third derivative*.

Higher Order Derivatives

- Generally, if successive derivatives exist, a function may have any number of higher order derivatives.
- The second derivative is denoted f''(x) or d^2f/dx^2 and the nth order derivative is given by d^nf/dx^n.
- Example: Let $f(x) = 3x^5 + 10x$. We have that $f' = 15x^4 + 10$, $f'' = 60x^3$, and $d^3f/dx^3 = 180x^2$.

Partial derivatives

- Here we consider a function of the form

  $y = f(x_1, \ldots, x_n)$, $f: R^n \rightarrow R$.

- Short-hand notation: $y = f(x)$, where $x \in R^n$.

Partial derivative: definition

- *Def.* The *partial derivative* of the function $f(x_1,x_2,\ldots, x_n)$, $f: R^n \to R$, at a point $(x_1^o ,x^o_2,\ldots, x^o_n)$ *with respect to* x_i is given by

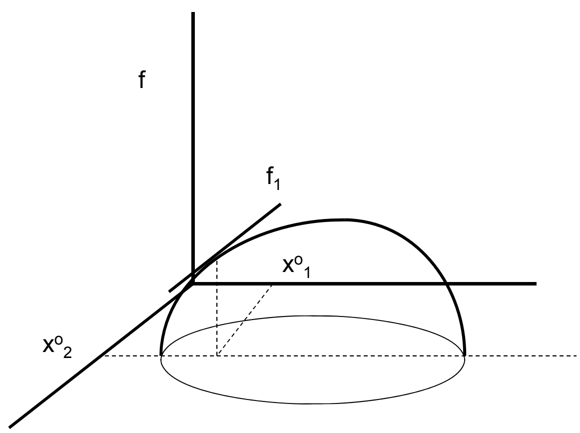
$$\lim_{\Delta x_i \to 0} \Delta y / \Delta x_i = \frac{f(x_1^o, \ldots, x_i^0 + \Delta x_i, \ldots, x_N^0) - f(x_1^0, \ldots, x_N^0)}{\Delta x_i}$$

Notation

Notation: We denote the partial derivative, $\lim_{\Delta x_i \to 0} \Delta y / \Delta x_i$, in each of the following ways.

$f_i(x)$, $\partial f(x)/\partial x_i$ or $\frac{\partial}{\partial x_i} f(x)$.

Illustration



Summary

The mechanics of differentiation are very simple:

- When differentiating with respect to x_i, regard all other independent variables as constants
- Use the simple rules of differentiation for x_i.

Examples

#1 Let $y = f(x_1,x_2) = x_1^2 + x_2 + 3$

then $\frac{\partial f}{\partial x_1} = 2x_1$ and $\frac{\partial f}{\partial x_2} = 1$

#2 Let $y = f(x_1, x_2, x_3) = (x_1 + 3)(x_2^2 + 4)(x_3)$

$$\frac{\partial f}{\partial x_1} = 1 (x_2^2 + 4)(x_3)$$

#3 Let $y = f(x_1,x_2) = x_1^2 x_2^4 + 10x_1$

$$\frac{\partial f}{\partial x_1} = 2x_1 x_2^4 + 10$$

Extensions of the chain rule

a. Let $y = f(x_1,\ldots,x_n)$, where $x_i = x_i(x_1)$, for $i = 2,\ldots,n$. We have that

$$dy/dx_1 = \partial f/\partial x_1 + \sum_{i=2}^{n} \frac{\partial f}{\partial x_i} \frac{dx_i}{dx_1}.$$

b. Let $y = f(x_1,\ldots,x_n)$, where $\forall\ i\ \ x_i = x_i(u)$. Then we have that

$$dy/du = \sum_{i=1}^{n} f_i \frac{dx_i}{du}.$$

Extensions of the chain rule

c. Let $y = f(x_1,...,x_n)$, where $x_i = x_i(v_1,...,v_m)$, for all i. Then we have that

$$\partial y/\partial v_j = \sum_{i=1}^{n} f_i \frac{\partial x_i}{\partial v_j}.$$

Example

- Let $y = f(x_1,x_2) = 3x_1 + x_2^2$, where $x_1 = v^2 + u$ and $x_2 = u + 5v$. Find $\partial y / \partial u, \partial y / \partial v$

Higher order partial derivatives

- A first order partial derivative function is itself a function of the n independent variables of the original function. Thus, if this function has partial derivatives, then we can define higher order partial derivative functions.
- The most interesting are the second order partial derivative functions denoted

$$f_{ij}(x) = \partial^2 f / \partial x_i \partial x_j \quad x \in R^n.$$

Higher order partial derivatives

- If i and j are equal, then f_{ii} is called a *direct second order partial derivative* and if $i \neq j$, f_{ij} is called a *cross second order partial derivative*.
- We can associate a matrix of partial derivatives to each point $(x_1,...,x_n)$ in the domain of f. The matrix is defined by

$$H(x^o) = [f_{ij}(x^o)]_{i,j = 1,\ldots,n}$$

and it is called the *Hessian* of f at the arbitrary point x^o.

Examples

#1. Let $f = (x_1 + 3)(x_2^2 + 4)x_3$. We have that

$f_1 = (x_2^2 + 4)x_3$, $f_{11} = 0$, $f_{12} = 2x_2x_3$, $f_{13} = (x_2^2 + 4)$

$f_2 = (x_1 + 3)(x_3)(2x_2)$, $f_{22} = 2(x_1 + 3)x_3$, $f_{21} = 2x_2x_3$, $f_{23} = (x_1 + 3)(2x_2)$.

$f_3 = (x_1 + 3)(x_2^2 + 4)$, $f_{32} = (x_1 + 3)(2x_2)$, $f_{31} = (x_2^2 + 4)$, $f_{33} = 0$.

#2. Let $f = x_1x_2$.

$f_1 = x_2$, $f_{12} = 1$, $f_{11} = 0$,

$f_2 = x_1$, $f_{21} = 1$, $f_{22} = 0$.

The Hessian at any point (x_1,x_2) is given by

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}.$$

A result

- Young's Theorem. If $f(x)$, $x \in R^n$, possesses continuous partial derivatives at a point, then $f_{ij} = f_{ji}$ at that point.
- Example: $f = x^2y^2$. Show $f_{xy} = f_{yx}$.

Differentials

- Given that y = f(x), a Δx will generate a Δy as discussed above. When the Δx is infinitesimal we write dx, which, thus, generates an infinitesimal change in y, dy.
- The first order differential of y = f(x) is

$$dy = df = f'(x)dx.$$

Example

$f(x) = 3x + x^2$, find dy:

$$dy = f'(x)\,dx$$

$$dy = (3 + 2x)\,dx.$$

Functions of Many Variables

- Given $y = f(x_1,\ldots x_n)$, the differential of f is given by

$$df = \Sigma_i f_i (x)\, dx_i.$$

Some rules

Proposition 1. If $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$ are differentiable functions of the variables x_i ($i = 1, \ldots, n$), then

(i) $d\,(f^n) = nf^{n-1}df$

(ii) $d\,(f \pm g) = df \pm dg$

(iii) $d\,(f \cdot g) = gdf + fdg$

(iv) $d\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}$, $g \neq 0$.

Example

- Let f = $(x_1^2 + 3x_2)/x_1x_2$, find df.

$$df = \frac{x_1^2 x_2 - 3x_2^2}{(x_1 x_2)^2} dx_1 - \frac{x_1^3}{(x_1 x_2)^2} dx_2 .$$

2nd order differential

- It is also possible to take a differential of a first order differential so as to define a second order differential.

$$d^2 f = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} dx_i dx_j .$$

- We have that d^2f can be expressed as a quadratic form in dx:

$$dx' H dx,$$

Derivatives of Implicit Functions

- Implicit relationships between two variables x and y are often expressed as equations of the form

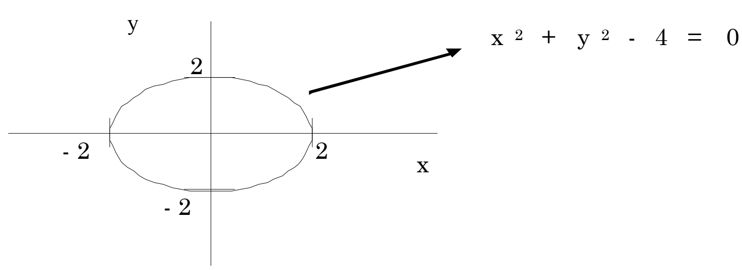
  $$F(x, y) = 0.$$

  If possible, we would like to define explicit functions say $x = x(y)$ and $y = y(x)$ from this relation.
- We would also like to define the derivatives of such functions.

Example

- A circle with radius 2 and center 0:

  $$F(x, y) = x^2 + y^2 - 4 = 0.$$



Example

- Here every $x \in (-2, 2)$ is associated with two values of y.
- It is possible to define two functions:

$y = f(x) = \sqrt{4 - x^2}$, where

$$f: [-2, +2] \rightarrow [0, 2]$$

$y = g(x) = -\sqrt{4 - x^2}$, where

$$g: [-2, +2] \rightarrow [-2, 0].$$

A result: basic IFT

Proposition 1. The relation $F(x, y) = 0$ defines one or more differentiable implicit functions of the form $y = f(x)$, at some point x, if $\partial F / \partial y \neq 0$. We have that

$$\frac{dy}{dx} = \frac{-\partial F / \partial x}{\partial F / \partial y}.$$

Our Example

$y = f(x) = \sqrt{4 - x^2}$; f: [-2, +2] $\rightarrow$ [0, 2]

$y = g(x) = -\sqrt{4 - x^2}$; g: [-2, +2] $\rightarrow$ [-2, 0]

$$f'(x) = -x(4 - x^2)^{-1/2} = -x/y$$

$$g'(x) = +x\,(4 - x^2)^{-1/2} = -x/y$$

Use IFT on $x^2 + y^2 - 4 = 0$.

$$\frac{dy}{dx} = \frac{-\partial F / \partial x}{\partial F / \partial y} = \frac{-2x}{2y} = \frac{-x}{y},$$

A generalization

Proposition 2. The relation $F(x_1, x_2, x_3, \ldots, x_N) = 0$ defines one or more differentiable implicit functions of the form $x_i = f_i(x_j)$, at x_j, if $\partial F / \partial x_i \neq 0$, i, j=1, …, N, $i \neq j$. We have that

$$\frac{\partial x_i}{\partial x_j} = \frac{-\partial F / \partial x_j}{\partial F / \partial x_i} .$$

Example

- $x_1^3\, x_2 + x_3\, x_2 = 0$. Find $\partial x_2 / \partial x_3$

$$\frac{\partial x_2}{\partial x_3} = \frac{-\partial F / \partial x_3}{\partial F / \partial x_2} = \frac{-x_2}{(x_1^3 + x_3)}, \text{ for } (x_1^3 + x_3) \neq 0.$$

A Generalized Implicit Function Theorem

- Consider a system of n equations

$$0 = F^i(y_1,...y_n,x_1,...,x_m),\ i = 1,...,n.$$

- The variables $y_1,...,y_n$ represent n independent variables and the $x_1,...,x_m$ represent m parameters. Write as $y \in R^n$ and $x \in R^m$.

The Jacobian of the System

$$J_y = \begin{bmatrix} \partial F^1/\partial y_1 & \bullet\ \bullet\ \bullet & \partial F^1/\partial y_n \\ \bullet & & \bullet \\ \bullet & & \bullet \\ \bullet & & \bullet \\ \partial F^n/\partial y_1 & \bullet\ \bullet\ \bullet & \partial F^n/\partial y_n \end{bmatrix}$$

General IFT

Proposition 3. Let $F^i(y,x) = 0$ possess continuous partial derivatives in y and x. If at a point (y^o,x^o), $|J_y| \neq 0$, then $\exists$ n functions $y_i = f^i(x_1,\ldots,x_m)$ which are defined in a neighborhood of x^o. We have

(i) $F^i(f^1,\ldots f^n,x^o) = 0, \forall i$,

(ii) $y_i = f^i(\cdot)$ are continuously differentiable in x locally

(iii) $$J_y \begin{bmatrix} \partial y_1 / \partial x_k \\ \bullet \\ \bullet \\ \bullet \\ \partial y_n / \partial x_k \end{bmatrix} = \begin{bmatrix} -\partial F^1 / \partial x_k \\ \bullet \\ \bullet \\ \bullet \\ -\partial F^n / \partial x_k \end{bmatrix} \text{ and } \begin{bmatrix} \partial y_1 / \partial x_k \\ \bullet \\ \bullet \\ \bullet \\ \partial y_n / \partial x_k \end{bmatrix} = J_y^{-1} \begin{bmatrix} -\partial F^1 / \partial x_k \\ \bullet \\ \bullet \\ \bullet \\ -\partial F^n / \partial x_k \end{bmatrix}.$$

Cramer's Rule and IFT

Remark: Using Cramer's Rule, we have that

$$\partial y_i/\partial x_k = |J_{yi}|/|J_y|.$$

Example

Example: Solve the following system for system $\partial y_1/\partial x$ and $\partial y_2/\partial x$.

$2y_1 + 3y_2 - 6x = 0$

$y_1 + 2y_2 = 0.$

Exponential and Logarithmic Functions

- An exponential function is a function in which the independent variable appears as an exponent:

  $y = b^x$, where $b > 1$

We take $b > 0$ to avoid complex numbers. $b > 1$ is not restrictive because we can take $(b^{-1})^x = b^{-x}$ for cases in which $b \in (0, 1)$.

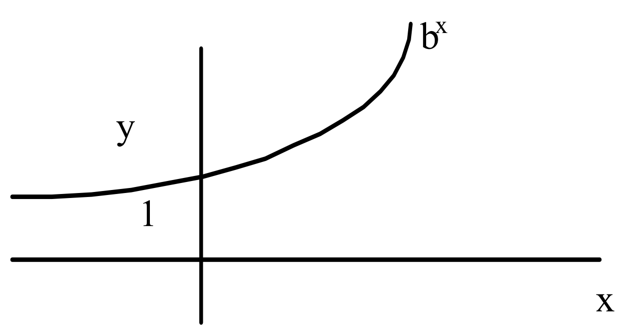
Logarithmic

- A logarithmic function is the inverse function of b^x. That is,

$$x = \log_b y. \quad (\text{iff } y = b^x)$$

A preferred base is the number $e \cong 2.72$. More formally, $e = \lim_{n\to\infty} [1 + 1/n]^n$. e is called the natural logarithmic base. It has the desirable property that $\frac{d}{dx} e^x = e^x$. The corresponding log function is written $x = \ln y$, meaning $\log_e y$.

Illustration



Rules of logarithms

- R. 1: If x, y > 0, then log (yx) = log y + log x.

Proof: $e^{\ln x} = x$ and $e^{\ln y} = y$ so that $xy = e^{\ln x} e^{\ln y} = e^{\ln x+\ln y}$. Moreover $xy = e^{\ln xy}$.

Thus, $e^{\ln x+\ln y} = e^{\ln xy}$. ||

Rules of logarithms

- R. 2: If x, y > 0, then log (y/x) = log y - log x.

Proof: $e^{\ln x} = x$ and $e^{\ln(1/y)} = y^{-1}$ so that $x/y = e^{\ln x} e^{\ln(1/y)}$. Moreover $x/y = e^{\ln x/y}$.

Thus, $e^{\ln x} e^{\ln(1/y)} = e^{\ln x/y}$ and $\ln x + \ln(y^{-1}) = \ln x/y$. Using rule 3 (to be shown) the result holds.

Rules of logarithms

- R. 3: If $x > 0$, then $\log x^a = a \log x$.

Proof: $e^{\ln x} = x$ and $e^{\ln x^a} = x^a$. However, $x^a = (e^{\ln x})^a = e^{a\ln x}$. Thus, $e^{\ln x^a} = e^{a\ln x}$. ||

Conversion and inversion of bases

- conversion

$\log_b u = (\log_b c)(\log_c u)$ ($\log_c u$ is known)

Proof: Let $u = c^p$. Then $\log_c u = p$. We know that $\log_b u = \log_b c^p = p\log_b c$. By definition, $p = \log_c u$, so that $\log_b u = (\log_c u)(\log_b c)$. ||

Conversion and inversion of bases

- inversion

  $\log_b c = 1/(\log_c b)$

Proof: Using the conversion rule, $1 = \log_b b = (\log_b c)(\log_c b)$. Thus, $\log_b c = 1/(\log_c b)$. ||

Derivatives of Exponential and Logarithmic Functions.

- R. 1: The derivative of the log function y = ln f(x) is given by

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

Derivatives of Exponential and Logarithmic Functions

- R. 2: The derivative of the exponential function $y = e^{f(x)}$ is given by

$$\frac{dy}{dx} = f'(x)e^{f(x)}.$$

Derivatives of Exponential and Logarithmic Functions

- R. 3: The derivative of the exponential function $y = b^{f(x)}$ is given by

$$\frac{dy}{dx} = f'(x)b^{f(x)}\ln b.$$

Derivatives of Exponential and Logarithmic Functions

- R. 4: The derivative of the logarithmic function $y = \log_b f(x)$ is given by

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)} \frac{1}{\ln b}$$

Examples

#1 Let $y = e^{2x^2+4}$, then

$$\frac{dy}{dx} = 4x\, e^{2x^2+4}.$$

#2 Let $y = [\ln(16x^2)]x^2$

$$\frac{dy}{dx} = \frac{32x}{16x^2}x^2 + 2x\ln(16x^2)$$

$$= 2x + 2x\ln 16x^2 = 2x + 2x(\ln 16 + 2\ln x)$$

$$= 2x(1 + \ln 16) + 4x\ln x.$$

Examples

#3 Let $y = 6^{x^2+17x}$, then $f' = (2x + 17)\ 6^{x^2+17x} \ln 6.$

The Taylor Series Approximation

- It is sometimes of interest to approximate a function at a point with the use of a polynomial function.
- Linear and quadratic approximations are typically used.
- Let y = f(x).

The Taylor Series Approximation

- A *Taylor's Series Approximation* of f at x^o is given by

$$f(x) \approx f(x^o) + \sum_{i=1}^{k} \frac{1}{i!} \frac{d^i f(x^o)}{dx^i} (x - x^o)^i .$$

Linear Approximation

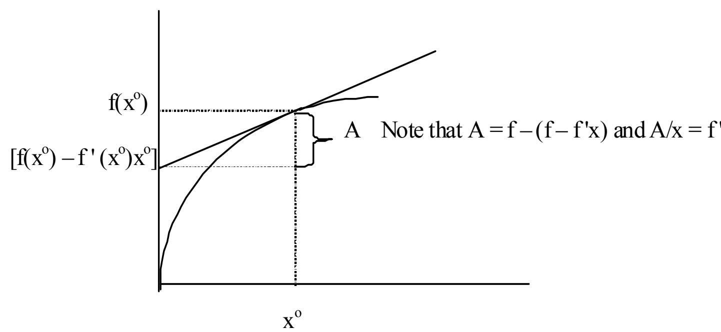
- A *linear approximation* takes k = 1. This is given by

$$f(x) \approx f(x^o) + [df(x^o)/dx](x\text{-}x^o) = [f(x^o) - f'(x^o)x^o] + f'(x^o)x.$$

- Defining, $a \equiv [f(x^o) - f'(x^o)x^o]$ and $b \equiv f'(x^o)$, we have

$$f(x) \approx a + bx.$$

Illustration



Quadratic Approximation

- A quadratic approximation takes k = 2. We have that

$$f(x) \approx f(x^o) + f'(x^o)(x\text{-}x^o) + \frac{f''(x^o)}{2}(x - x^o)^2.$$

- or

$$f(x) \approx [f(x^o) - f'(x^o)x^o + f''(x^o)(x^o)^2/2] + [f'(x^o) - x^o f''(x^o)]x + [f''(x^o)/2]x^2$$

- This is of the form

$$f(x) \approx a + bx + cx^2.$$

Functions of many variables

- General linear

$$f(x) \approx f(x^o) + \sum_{i=1}^{n} f_i(x^o)(x_i - x_i^o).$$

Define $\nabla f(x)' \equiv (f_1(x)\ldots f_n(x))$ and term this the gradient of f. Then we have

$$f(x) \approx f(x^o) + \nabla f(x^o)'(x - x^o),$$

Functions of many variables

- Quadratic

$$f(x) \approx f(x^o) + \nabla f(x^o)'(x - x^o) + \frac{1}{2}\sum_i \sum_j f_{ij}(x^o)(x_i - x_i^o)(x_j - x_j^o).$$

- or

$$f(x) \approx f(x^o) + \nabla f(x^o)'(x - x^o) + \frac{1}{2}(x - x^o)'H(x^o)(x - x^o).$$

Examples

Consider the function $y = 2x^2$. Let us construct a linear approximation at x = 1.

$f \approx 2 + 4(x - 1) = -2 + 4x.$

A quadratic approximation at x = 1 is

$f \approx 2 + 4(x - 1) + (4/2)(x-1)^2 = 2 + 4x - 4 + 2(x^2 - 2x + 1) = 2 + 4x - 4 + 2x^2 - 4x + 2$

$f = 2x^2.$

Given that the original function is quadratic, the approximation is exact.

Examples

. Let $y = x_1^3 x_2$. Construct a linear and a quadratic approximation of f at (1,1). The linear approximation is given by

$f \approx 1 + 3(x_1 - 1) + 1(x_2 - 1).$

Rewriting,

$f \approx -3 + 3x_1 + x_2.$

A quadratic approximation is constructed as follows:

$$f \approx 1 + 3(x_1 - 1) + 1(x_2 - 1) + \frac{1}{2}\,[x_1 - 1 \;\; x_2 - 1]\begin{bmatrix} 6 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}.$$

Expanding terms we obtain the following expression.

$f \approx 3 - 6x_1 - 2x_2 + 3\,x_1x_2 + 3x_1^2.$