Optimization

- Unconstrained
- Constrained with equality constraints
- Constrained with inequality constraints

1

Unconstrained with a function of a single variable

- Given a real valued function, y = f(x) we will be concerned with the existence of extreme values of the dependent variable y and the values of x which generate these extrema.
- f(x) is called the *objective function* and the independent variable x is called the *choice variable*.
- In order to avoid boundary optima, we will assume that $f : X \rightarrow R$, where X is an open interval of R.

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Local and global extrema

- *Def 1:* f has a *local maximum* at a point $x^o \in X$ if $\exists$ $N(x^o)$ such that $f(x) - f(x^o) < 0$ for all $x \in N(x^o)$, $x \neq x^o$.
- *Def 1:* f has a *local minimum* at a point $x^o \in X$ if $\exists$ $N(x^o)$ such that $f(x) - f(x^o) > 0$ for all $x \in N(x^o)$, $x \neq x^o$.
- *Def 3*: A real valued function f(x), $f: X \rightarrow R$, has an *absolute or global maximum (minimum)* at a point $x^o \in X$, if $f(x) < f(x^o)$ ($f(x) > f(x^o)$), for all $x \neq x^o$, such that $x \in X$.

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Illustrations: global

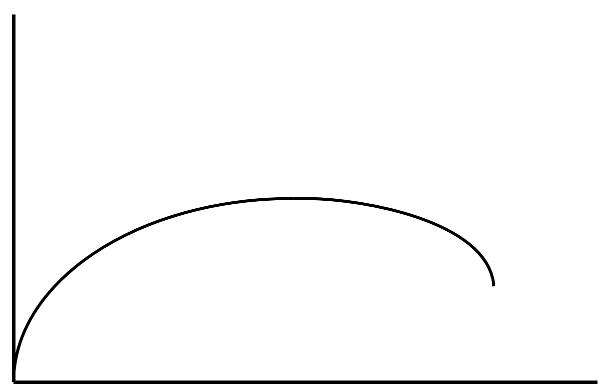

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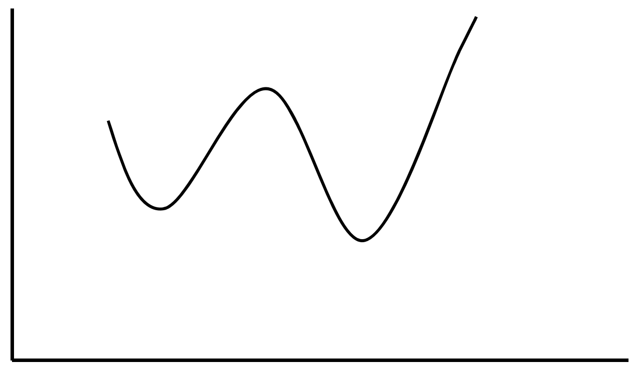
Illustration: local



Optimization and the First Derivative Test

- *Proposition 1.* If f has a local maximum or minimum at $x^o \in X$, then $f'(x^o) = 0$.
- Points at which f ' = 0 are called *critical values*. The image $f(x^o)$ is called a critical value of the function and x^o is called a critical value of the independent variable.
- To distinguish maxima and minima we must study the curvature of the function.

Curvature

- In a neighborhood of a maximum, a function has the property that it is locally concave, and in a neighborhood of a minimum it is locally convex.
- Concavity is implied by a nonpositive second derivative and convexity is implied by a nonnegative second derivative.

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Some definitions

- *Def 1*: A set $X \subset R^n$ is *convex* if for any $x, x' \in X$, $\alpha \in [0, 1]$, $\alpha x + (1-\alpha) x' \in X$.
- *Def 2:* A real-valued fn ,f(x), $f: X \to R$, $X \subset R$ (X convex) is *concave* (*convex*) if for any $x, x' \in X$, $\alpha \in [0, 1]$,

  $f(\alpha x + (1- \alpha) x') \geq (\leq) \alpha f(x) + (1- \alpha) f(x')$.
- *Def 3:* A real-valued fn, f(x), $f: X \to R$, $X \subset R$ (X convex) is *strictly concave (strictly convex)* if for any $x \neq x'$, $x,x' \in X$, $\alpha \in (0, 1)$,

  $f(\alpha x + (1 - \alpha) x') > (<) \alpha f(x) + (1 - \alpha) f(x')$.

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A result

- *Theorem 1:* If a function, f(x), f: X $\rightarrow$ R, such that X is an open interval X $\subset$ R, is differentiable over X,

(i) it is *strictly concave (strictly convex)* if and only if f′(x) is decreasing (increasing) over x,

(ii) if f '' exists and is negative (positive), for all x, then f is strictly concave (convex).

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Functions of n variables

- The definition of strict concavity for a function of n variables is analogous to the above definition. In this case f: X $\rightarrow$ R, where X is a subset of R^n. We have
- *Def 4.* A real-valued fn, f(x), f: X $\rightarrow$ R, X $\subset$ R^n (X convex) is *strictly concave (convex)* if for any x $\neq$ x′, x,x' $\in$ X, $\alpha \in (0, 1)$, f(α x + (1 - α) x′) > (<) α f(x) + (1 - α) f(x′).

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Derivative characterization

- *Theorem 2.* Let f: f: $X \rightarrow R$, $X \subset R^n$ (X convex). Suppose that f is twice continuously differentiable on X. If $d^2f = dx'Hdx$ is negative (positive) definite for all $x \in X$, then f is strictly concave (convex). If d^2f is negative (positive) semidefinite for all $x \in X$, then f is concave (convex).

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A sufficiency result

- *Proposition 2.* Let f be twice differentiable. Let there exists an $x^o \in X$ such that $f'(x^o) = 0$.

(i) If $f''(x^o) < 0$, then f has a local maximum at x^o. If, in addition, $f'' < 0$ for all x or if f is strictly concave, then the local maximum is a unique global maximum.

(ii) If $f''(x^o) > 0$, then f has a local minimum at x^o. If, in addition, $f'' > 0$ for all x or if f is strictly convex, then the local minimum is a unique global minimum.

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Remark: terminology

- The zero derivative condition is called the *first order condition (FOC)* and the second derivative condition is called the *second order condition (SOC)*.

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Examples

#1 Let $f(x) = x + x^{-1}$ we know that $f'(x) = 1 - x^{-2}$ and $x = \pm 1$. Now calculate $f''(x)$

$f''(x) = 2x^{-3}$. At $x = 1$, $f''(1) = 2 > 0$, so that 1 is a local min. At $x = -1$, $f'' = -2$, so that -1 is a local max.

#2. Let $f = ax - bx^2$, $a,b > 0$ and $x > 0$. Here $f' = 0$ implies $x = a/2b$. Moreover, $f'' = -2b < 0$, for all x. Thus, we have a global max.

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Existence

- In the above discussion, we have assumed the existence of a maximum or a minimum. In this section, we wish to present sufficient conditions for the existence of an extremum.
- *Proposition.* Let f: $X \rightarrow R$, $X \subset R$, where X is a closed interval of R. Then if f is continuous, it has a maximum and a minimum on X. If f is strictly concave, then there is a unique maximum on X. If f is strictly convex, then it has a unique minimum on X.

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Remarks

- This proposition includes boundary optima in its assertion of existence.
- To insure existence of interior optima, one must show that boundary points are dominated by some interior point.
- If those boundary points are not part of the domain of the function, then one can take the limit of the function at boundary points and show that those limit points are dominated by some interior point.

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An n-Variable Generalization

- A set X is an open subset of R^n if $\forall$ x $\in$ X $\exists$ N(x) $\subset$ X. In this case N(x^o) is defined as the set of points within an ε distance of x^o:

$$N(x^o) \equiv \{x \mid [\sum_{i=1}^{n}(x_i - x_i^o)^2]^{1/2} < \varepsilon, \varepsilon > 0\}.$$

- Let y = f(x), where x $\in$ R^n and f : X $\rightarrow$ R, where X is an open subset of R^n.

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A local extremum

- *Def*. f has a local maximum (minimum) at a point x^o $\in$ X if $\exists$ N(x^o) such that for all x$\in$N(x^o), f(x) - f(x^o) < 0. (f(x) - f(x^o) > 0 for a minimum.)

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The FOC

- *Proposition 1*. If a differentiable function f has a maximum or a minimum at $x^o \in X$, then $f_i(x^o) = 0$, for all i.

- Remark: The n equations generated by setting each partial derivative equal to zero represent the first order conditions. If a solution exists, then they may be solved for the n solution values x_i^o.

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The SOC

- The SOC for a max is

$$(*) \qquad d^2f(x^o) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(x^o)\, dx_i dx_j < 0 \text{ for all } (dx_{1,\ldots,} dx_j) \neq 0.$$

This condition is that the quadratic form $\sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(x^o)\, dx_i dx_j$ is *negative definite*.

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SOC for a max

- In (*), the discriminate is the Hessian matrix of f (the objective function).
- As discussed above, the rather cumbersome (*) condition is equivalent to a fairly simple sign condition.

$$\text{(SOC) (max)} \quad |PM_i| \quad \text{of } H = \begin{bmatrix} f_{11} & \cdot & \cdot & \cdot & f_{1n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ f_{n1} & \cdot & \cdot & \cdot & f_{nn} \end{bmatrix}, \text{ evaluated at } x^o \text{, have signs } (-1)^i.$$

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The SOC for a min

- The analogous conditions for a minimum are that

$$(**) \qquad d^2 f(x^o) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(x^o)\, dx_i dx_j > 0 \text{ for all } (dx_{1,\ldots,} dx_n) \neq 0$$

- The matrix condition is

$$\text{(SOC) (min)} \quad |PM_i| \quad \text{of } H = \begin{bmatrix} f_{11} & \cdot & \cdot & \cdot & f_{1n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ f_{n1} & \cdot & \cdot & \cdot & f_{nn} \end{bmatrix}, \text{ evaluated at } x^o \text{, have positive signs}$$

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SOC

- If f satisfies the SOC for a maximum globally, then f is strictly concave. If it satisfies the SOC for a minimum globally, then f is strictly convex.

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A sufficiency result

Proposition 2. If at a point x^o we have

(i) $f_i(x^o) = 0$, for all i, and

(ii) SOC for a maximum (minimum) is satisfied at x^o,

Then x^o is a local maximum (minimum). If in addition the SOC is met for all $x \in X$ or if f is strictly concave (convex), then x^o is a unique global maximum (minimum).

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Examples

#1 Maximizing a profit function over two strategy variables. Let profit be a function of the two variables x_i, i =1,2. The profit function is $\pi(x_1,x_2) = R(x_1,x_2) - \Sigma r_i x_i$, where r_i is the unit cost of x_i and R is revenue. We wish to characterize a profit maximal choice of x_i. The problem is written as

$$\underset{\{x_1,x_2\}}{\text{Max}}\ \pi(x_1,x_2).$$

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#1

- Show FOC and SOC
- Characterize , $\partial x_i / \partial r_1$ using Cramer's rule.

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#1

The FOC are

$\pi_1(x_1,x_2) = 0$

$\pi_2(x_1,x_2) = 0.$

The second order conditions are

$\pi_{11} < 0$, $\pi_{11}\pi_{22} - \pi_{12}^2 > 0$ (recall Young's Theorem $\pi_{ij} = \pi_{ji}$).

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#1

$$H\begin{bmatrix}\partial x_1/\partial r_1\\ \partial x_2/\partial r_1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix},$$ where H is the relevant Hessian.

Using Cramer's rule,

$$\partial x_1/\partial r_1 = \frac{\begin{vmatrix}1 & \pi_{12}\\ 0 & \pi_{22}\end{vmatrix}}{|H|} = \pi_{22}/|H| < 0.$$

Likewise

$$\partial x_2/\partial r_1 = \frac{\begin{vmatrix}\pi_{11} & 1\\ \pi_{21} & 0\end{vmatrix}}{|H|} = -\pi_{21}/|H|.$$

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Remark

- The comparative static derivative

$$\partial x_2 / \partial r_1$$

has a sign which depends on whether 1 and 2 are complements or substitutes. It is negative in the case of complements and positive in the case of substitutes.

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Examples

#2. $\underset{\{x,y\}}{\text{Min}}\ x^2 + xy + 2y^2$. The FOC are

$$2x + y = 0,$$

$$x + 4y = 0.$$

Solving for the critical values x = 0 and y = 0. $f_{11} = 2$, $f_{12} = 1$ and $f_{22} = 4$. The Hessian is

$$H = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, \text{ with } f_{11} = 2 > 0 \text{ and } |H| = 8 - 1 = 7 > 0.$$

Thus, (0,0) is a minimum. Further, it is global, because the Hessian sign conditions are met for any x,y.

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Existence with n Choice Variables

- *Def. 1*. A set $X \subset R^n$ is said to be *open* if for all $x \in X$ $\exists$ $N(x)$ such that $N(x) \subset X$. The set X is said to be *closed* if its complement is open.
- *Def. 2.* A set $X \subset R^n$ is said to be *bounded* if the distance between any two of its points is finite. That is,

$$\sum_{i=1}^{n} [(x_i - x_i')^2]^{1/2} < \infty$$

for all $x, x' \in X$.

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Existence

- *Def. 3* A set $X \subset R^n$ is said to be *compact* if it is both closed and bounded.

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Existence result

- *Proposition.* Let f: X$\rightarrow$ R, where X is a subset of R^n. If X is compact and f is continuous, then f has a maximum and a minimum on X. If X is both compact and convex and f is strictly concave, then f has a unique maximum on X. If X is both compact and convex and f is strictly convex, then f has a unique minimum on X.

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Existence

- Remark: This proposition does not distinguish between boundary optima and interior optima.
- The results presented here can be used to show the existence of interior optima by showing that boundary optima are dominated. The technique is as described above.

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Constrained Optimization

- The basic problem is to maximize a function of at least two independent variables subject to a constraint.
- We write the objective function as f(x) and the constraint as g(x) = 0.
- The constraint set is written as C = { x | g(x) = 0} .
- f maps a subset of R^n into the real line.

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Examples

- Max $u(x_1, x_2)$ subject to $I - p_1x_1 – p_2x_2 = o$
- Min $r_1x_1 + r_2 x_2$ subject to $q – f(x_1, x_2) = 0$

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The basic problem

- The optimization problem is written as

$$\underset{\{x_1,\ldots,x_n\}}{\text{Max}}\ f(x) \text{ subject to } g(x) = 0.$$

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A local constrained extremum

- *Def.* x^o is a local maximum (minimum) of f(x) subject to g(x) = 0 if there exists $N(x^o)$ such that $N(x^o) \cap C \neq \emptyset$ and $\forall\ x \in N(x^o) \cap C$, $f(x) < (>)\ f(x^o)$.

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The FOC

Proposition 1. Let f be a differentiable function whose n independent variables are restricted by the differentiable constraint $g(x) = 0$. Form the function $L(\lambda,x) \equiv f(x) + \lambda g(x)$, where λ is an undetermined multiplier. If x^o is an interior maximizer or minimizer of f subject to $g(x) = 0$, then there is a λ^o such that

(1) $\partial L(\lambda^o, x^o)/\partial x_i = 0$, for all i, and

(2) $\partial L(\lambda^o, x^o)/\partial \lambda = 0$.

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The SOC

- The relevant SOC for a maximum is that

(*) $$d^2f(x^o) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(x^o)\, dx_i dx_j < 0 \text{ for all } (dx_{1,\ldots,} dx_n) \neq 0 \text{ such that } dg = 0.$$

- Condition (*) says that the quadratic form d^2f is negative definite subject to the constraint that dg = 0. This is equivalent to $d^2L = d^2f$ being negative definite subject to dg = 0, because $dL = df + \lambda dg + g d\lambda = df$, if dg = g = 0.

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SOC

- Condition (*) is equivalent to a rather convenient condition involving the bordered Hessian matrix. The bordered Hessian is given by

$$\overline{H}(\lambda^o, x^o) = \begin{bmatrix} 0 & g_1 & . & . & . & g_n \\ g_1 & L_{11} & . & . & . & L_{1n} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ g_n & L_{n1} & . & . & . & L_{nn} \end{bmatrix}.$$

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Remark

- The bordered Hessian is the Jacobian of the FOC in variables (λ, x_i)

$$\partial L / \partial \lambda = 0$$

$$\partial L / \partial x_i = 0$$

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SOC

- The sign condition for a max is

SOC (max) $|PM_i|$ of $|\overline{H}|$ of order $i \geq 3$, evaluated at (λ^o, x^o), has sign $(-1)^{i+1}$

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SOC

- For a minimum, the second order condition is

$$(^{**})\ d^2f(x^o) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(x^o)\, dx_i dx_j > 0 \text{ for all } (dx_{1,\ldots,}dx_n) \neq 0 \text{ such that } dg = 0$$

- The equivalent sign condition is

SOC (min) $|PM_i|$ of $|\overline{H}|$ of order $i \geq 3$, evaluated at (λ^o, x^o), are all negative.

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Sufficiency Result

Proposition 2. Let f be a differentiable function whose n independent variables are restricted by the differentiable constraint $g(x) = 0$. Form the function $L(\lambda,x) \equiv f(x) + \lambda g(x)$, where λ is an undetermined multiplier. Let there exist an x^o and a λ^o such that

(1) $\partial L(\lambda^o,x^o)/\partial x_i = 0$, for all i, and

(2) $\partial L(\lambda^o,x^o)/\partial\lambda = 0$.

Then x^o is a local maximum (minimum) of $f(x)$ subject to $g(x) = 0$, if, in addition to (1) and (2) the SOC for a maximum (minimum) is satisfied. If SOC is met for all $x \in C$, then x^o is a unique global maximizer (minimizer).

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Curvature conditions and the SOC

- In the above maximization problem, as long as the relevant constraint set is convex, the maximum will be a global maximum if the objective function is *strictly quasi-concave*.
- The latter property means that the sets $\{x : f(x) \geq f(x')\} = U(x')$ are strictly convex. These set are called upper level sets.
- A set X in R^n is strictly convex if $\alpha x + (1-\alpha)x' \in$ interior X for all $\alpha \in (0,1)$ and all $x \neq x'$ with $x, x' \in X$.

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Curvature conditions and the SOC

- If the lower level sets ($\{x : f(x) \leq f(x')\} = L(x')$) of a function are strictly convex sets, then the function is said to be strictly quasi-convex.
- When the relevant constraint set is convex and the objective function is strictly quasi-convex (strict quasi-concave), then the minimum (maximum) is unique.

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An operational check

- If a function is defined on the nonnegative orthant of R^n ($R^n_+ = \{x \in R^n \mid x_i \geq 0 \text{ for all } i\}$) and is twice continuously differentiable, then there is an easy operational check for quasi-concavity or quasi-convexity. Let $f(x_1,\ldots,x_n)$ be such a function.

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A Bordered Hessian Condition

- The function f is *strictly quasi-concave* if

$$B = \begin{bmatrix} 0 & f_1 & f_2 & . & f_N \\ f_1 & f_{11} & f_{12} & . & f_{1N} \\ f_2 & f_{12} & f_{22} & . & f_{2N} \\ . & . & . & . & . \\ f_N & f_{N1} & f_{N2} & . & f_{NN} \end{bmatrix} \text{ has } |PM_i| \text{ with sign } (-1)^{i+1} \text{ for } i \geq 2.$$

- For *strict quasi-convexity*, the condition is that $|PM_i|$ are all negative for $i \geq 2$.

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A Bordered Hessian Condition

- It is important to note that the condition pertains only to functions defined on $R_+^n = \{x \in R^n : x_i \geq 0 \text{ for all } i\}$.
- The condition on B is equivalent to the statement that d^2f is negative definite (or positive definite) subject to $df = 0$.

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Example

- Display the 2 x 2 condition and show that it is equivalent to an iso-surface being strictly convex.

Existence

- The conditions for existence of a constrained optimum are the same as those for the unconstrained problem except that the set $X \cap C$ is the relevant set for which we assume boundedness and closedness (compactness).
- Further, the objective function is assumed to be continuous over this set. Under these conditions, a constrained optimum will exist.

Examples

- Max f(L,K) st C-wL-rK = 0

- Min wL+rK st q- f(L,K) = 0

Extension to Multiple Constraints

- Suppose that there are m < n equality constraints $g_j(x) = 0$, j = 1 , ..., m.
- The Lagrangian is written as $L(\lambda_1, \ldots, \lambda_m, x_1, \ldots, x_n) = f(x) + \Sigma\lambda_j g_j(x)$.
- The FOC are that the derivatives of L in x_i and λ_j vanish:

$$f_i + \Sigma\lambda_j \partial g_j/\partial x_i = 0, \text{ for all } i,$$

$$g_j(x) = 0, \text{ for all } j.$$

Multiple Constraints

- The bordered Hessian becomes

$$|\overline{H}| = \begin{bmatrix} \underset{m\times m}{0} & J_g \\ J_g^{'} & L_{ij} \end{bmatrix},$$

- where J_g is the Jacobian of the constraint system in x, and $[L_{ij}]$ is the Hessian of the function L in x.

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Remark

- The bordered Hessian is the Jacobian of the FOC in variables (λ_j, x_i)

$$\partial L / \partial \lambda_j = 0$$

$$\partial L / \partial x_i = 0$$

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SOC

- For a maximum, the condition is

(SOC)(max) PM_i of $|\overline{H}|$ of order $i > 2m$ has sign $(-1)^r$, where r is the order of the largest order square $[L_{ij}]$ embedded in $|PM_i|$.

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SOC

- For a minimum, the condition is

(SOC)(min) $|PM_i|$ of $|\overline{H}|$ of order $i > 2m$ has sign $(-1)^m$.

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Example

- Presented in class

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Inequality Constrained Problems

- In many problems the side constraints are best represented as inequality constraints:

$$g_j(x) \geq 0.$$

- We wish to characterize the FOC to this extended problem:

$$\underset{\{x\}}{\text{Max}}\ f(x) \text{ subject to } g_j(x) \geq 0, j = 1,\ldots m.$$

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Inequality Constrained Problems

- The first step is to form the Lagrangian

$$L(\lambda, x) = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x).$$

FOC

(1) $\partial L/\partial x_i = f_i + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(x)}{\partial x_i} = 0$ for all i,

(2) $\partial L/\partial \lambda_j = g_j \geq 0$, $\lambda_j \geq 0$ and $g_j \bullet \lambda_j = 0$, $j = 1, \ldots, M$ and

(3) The Constraint Qualification holds.

A Min

- These same conditions are used for a minimum. If one were to solve Min f(x) subject to $g_j(x) \leq 0$, then to use the above conditions one would rewrite the problem as Max -f(x) subject to $-g_j(x) \geq 0$.

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Constraint qualification

- The FOC (1) and (2) are necessary conditions only if the Constraint Qualification holds. This rules out particular irregularities by imposing restrictions on the boundary of the feasible set.
- These irregularities would invalidate the FOC (1) and (2) should the solution occur there.

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Constraint qualification

- Let x^o be the point at which (1) and (2) hold and let index set k = 1,…, K represent the set of g_j which are satisfied with equality at x^o.

Constraint qualification

- The matrix

$$
J = \begin{vmatrix}
\frac{\partial g_1(x^o)}{\partial x_1} & . & . & . & \frac{\partial g_1(x^o)}{\partial x_n} \\
. & . & & & . \\
. & & . & & . \\
. & & & . & . \\
\frac{\partial g_K(x^o)}{\partial x_1} & & & & \frac{\partial g_K(x^o)}{\partial x_n}
\end{vmatrix}
$$

- has rank $K \le n$. That is the gradient vectors of the set of equality constraints are linearly independent.

Example: #1

Nonnegativity constraints on x_i

$$L = f + \lambda_1 x_1 + \lambda_2 x_2.$$

The FOC are

$$f_i + \lambda_i = 0$$

$$x_i \geq 0,\ \lambda_i \geq 0 \text{ and } \lambda_i x_i = 0.$$

If all $x_i > 0$, then $\lambda_i = 0$ and we have the previous conditions.

If $\lambda_i > 0$, then it must be true that $x_i = 0$ and $f_i < 0$ at the optimum.

It can be true that $\lambda_i = 0$ and $x_i = 0$.

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Examples #2

Consider the problem where we seek to Max $f(x)$, $x \in R^2$, subject to $p_1x_1 + p_2x_2 \leq c$, where c is a constant and $x_i \geq 0$. Assume that $c, p_i > 0$, We should set up the Lagrangian as

$$L = f(x) + \lambda[c - p_1x_1 - p_2x_2] + \Sigma\gamma_i x_i.$$

The relevant first order conditions are as follows:

$$f_i - \lambda p_i + \gamma_i = 0,\ i = 1,2,$$

$$c - p_1x_1 - p_2x_2 \geq 0,\ \lambda \geq 0,\ \lambda[c - p_1x_1 - p_2x_2] = 0, \text{ and}$$

$$x_i \geq 0,\ \gamma_i \geq 0,\ \gamma_i x_i = 0,\ i = 1,2.$$

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Examples #2

- Note that if the nonnegativity constraints are nonbinding and, at optimum, it is true that [$c – p_1x_1 – p_2x_2$], $x_i > 0$, then the optimum is defined as a free optimum with $f_i = 0$.

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Examples #2

- Next, suppose that the constraint $p_1x_1 + p_2x_2 \leq c$ is binding, $\gamma_2 > 0$, and that $x_1 > 0$. In this case, $x_2 = 0$, $\gamma_1 = 0$, and $p_1x_1 + p_2x_2 - c = 0$ and the first order conditions read

$f_1 = \lambda p_1$

$f_2 = \lambda p_2 - \gamma_2$ which implies that $f_2 < \lambda p_2$.

Thus, we have that

$f_1/f_2 > p_1/p_2$, $c = p_1x_1 \Rightarrow x_1 = c/p_1$ and $x_2 = 0$.

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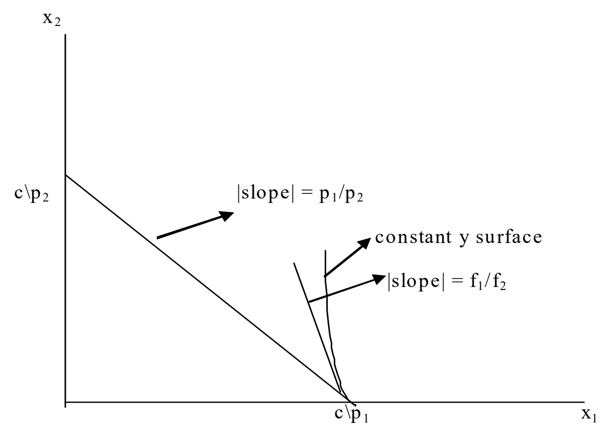
Illustration



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#2

- If it were true that $\gamma_i = 0$ and $\lambda > 0$, then

  $f_1/f_2 = p_1/p_2$, $c = p_1x_1 + p_2x_2$, and $x_1, x_2 > 0$.

- This is an interior optimum. The illustration is as follows:

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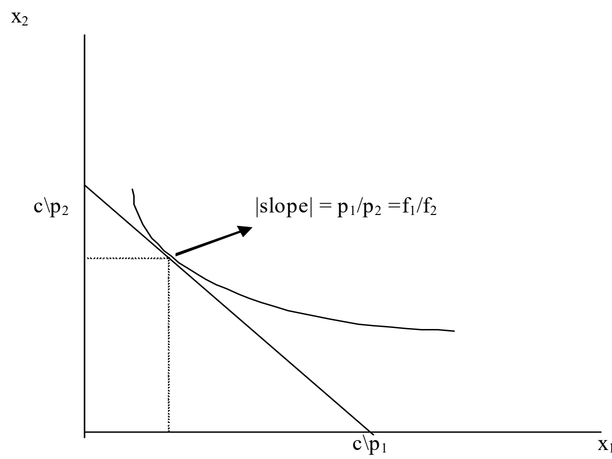
Illustration



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