Lecture 7: Integration Techniques

Antiderivatives and Indefinite Integrals

1. In differential calculus, we were interested in the derivative of a given real-valued function, whether it was algebraic, exponential or logarithmic. Here we are concerned with the inverse of the operation of differentiation. That is, the operation of searching for functions whose derivatives are a given function.

2. Consider any arbitrary real-valued function

$$f: X \to R$$

defined on a subset x of the real line, i.e. $X \subset R$. By the antiderivative (primitive) of f; we mean any differentiable function

$$F: X \to R$$

whose derivative is the given function f.   Hence,

$$\frac{dF}{dx} = F'(x) = f(x).$$

3. Let c be any real number $c \in R$, (which is constant). Now since the derivative of c with respect to x or $dc/dx = 0$, we have:

Theorem 1. If a function $F: X \to R$ is an antiderivative of $f: X \to R$, so is the function $F + c: X \to R$ defined by

$$(F + c)(x) = F(x) + c,$$

for every $x \in X \subset R$.

Remark 1: Hence, the derived function f(x) is traceable back to an infinite number of possible primary, antiderivative or primitive functions of the form $F(x) + c$.

Remark 2: However, here there is a subtle point. Every antiderivative fn of f is of the form $F(x) + c$ if $X \subset R$ is a connected subset of the real line R.

Def 1:   A set $X \subset R$ is connected if and only if, for any two real numbers a, b $\in$ X with a < b, X

contains the closed interval [a, b].

Example: Consider the function $f(x) = x^2$, where f: X $\rightarrow$ R, X = [0, ∞). It has an antiderivative

function defined by

$$F(x) = \frac{1}{3} x^3 + 10,$$

since $F'(x) = dF/dx = x^2$. However, it also has an antiderivative function defined by

$$G(x) = 1/3 \ x^3 + 100,$$

since $\frac{dG(x)}{dx} = x^2$. In fact, any function of the form

$$F(x) = \frac{1}{3} x^3 + c,$$

for any c $\in$ R is an antiderivative function of $f(x) = x^2$.

Theorem 2. If F: X $\rightarrow$ R is an antiderivative function of a real-valued function f: X $\rightarrow$ R, where

X $\subset$ R is connected, then {F + c: c $\in$ R} is the set of all antiderivatives of f.

Def. 2. The set of all antiderivative functions of a real valued function f: X $\rightarrow$ R, X $\subset$ R, is

called the indefinite integral of f and is denoted by

$$\int f(x) dx.$$

Remark 3: The symbol $\int$ is the integral sign. If f has no antiderivative (nonintegrable), then its

indefinite integral is $\varnothing$.   The f(x) part is the integrand. The f(x) dx may be taken as the

differential dF of a primary or antiderivative function.

Remark 4: The process of determining the indefinite integral of a given real-valued function say

f: X $\rightarrow$ R is termed the indefinite integration of f. We have that

$$\frac{d}{dx} F(x) = f(x) \text{ implies } \int f(x) dx = F(x) + c$$

for any real c, where X connected. The constant x will be referred to as the arbitrary constant of integration. If X is not connected, then

$$\int f(x)dx = F + c$$

means only that F + c is an antiderivative of f for some x.

Counterexample.

Let x denote the subset of R which consists of all non-zero real numbers:

$$X = \{x \in R: x \neq 0\}$$

(1)   Consider the fn $f(x) = 1/x^2$, f: X $\rightarrow$ R.

Then the fn. F(x)

$$F(x) = -x^{-1}, F: X \rightarrow R$$

is an antiderivative of f.

(2)   Consider the function G(x), G: X $\rightarrow$ R,

$$G(x) \begin{cases} -x^{-1} + 1 & \text{if} \quad x > 0 \\ -x^{-1} - 1 & \text{if} \quad x < 0 \end{cases}$$

Note dG/dx = G′(x) is given by:

$$G'(x) = 1/x^2 = f(x).$$

$\therefore$ G(x) is also an antiderivative of f.

However,

$$G - F = -\frac{1}{x} + 1 + \frac{1}{x}; \text{ if } x > 0$$

$$G - F = \left(-\frac{1}{x} - 1\right) + \frac{1}{x}; \text{ if } x < 0$$

Hence

$$G - F = \begin{cases} +1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

Thus, G - F $\neq$ c, where c represents a constant function, for all x. Hence, G is not of form F(x) + c, for all x.

Basic Rules of Integration

1. *Theorem 1.* If F is an antiderivative of f and G is an antiderivative of g, then F + G is an antiderivative of f + g. Hence,

$$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx.$$

(holds for finite sum)

2. *Theorem 2.* If F is an antiderivative of f, then cF is an antiderivative of cf for every c $\in$ R. Thus,

$$\int cf(x)\, dx = c\int f(x)\, dx.$$

3. *Theorem 3.* For every N $\neq$ -1, $x^{N+1}/N + 1$ is the antiderivative of x^N. Thus,

$$\int x^N\, dx = \frac{1}{N+1} x^{N+1} + c.$$

4. Examples:

   #1   Let f(x) = x^4, then

   $$\int x^4 dx = \frac{x^5}{5} + c.$$

   check:

   $$\frac{d(x^5/5)}{dx} = x^4.$$

   #2   Let f(x) = $x^3 + 5x^4$, then

   $$\int [x^3 + 5x^4]\, dx = \int x^3\, dx + 5\int x^4\, dx$$

   $$= x^4/4 + 5(x^5/5) + c$$

   $$= x^4/4 + x^5 + c$$

   check.

$$\frac{d}{dx}(x^4/4 + x^5) = x^3 + 5x^4.$$

#3   Let $f(x) = x^2 - 2x$

$$\int [x^2 - 2x]\, dx = \int x^2\, dx + \int -2x\, dx$$

$$= x^3/3 - 2\int x\, dx + c'$$

$$= x^3/3 - 2(x^2/2) + c$$

$$= x^3/3 - x^2 + c.$$

5. *Theorem 4.* The antiderivative of e^x is given by

$$\int e^x\, dx = e^x + c.$$

Theorem 5. The antiderivative of x^{-1}, $x \neq 0$ is given by

$$\int \frac{1}{x}\, dx = \ln|x| + c.$$

Remark 1: (Why $\ln|x|$ instead of $\ln x$.) Note that the function $f(x) = x^{-1}$ is such that

$$f: R - \{0\} \rightarrow R.$$

That is, $f(x) = x^{-1}$ is defined on the set of non-zero real numbers. Now, the natural logarithmic

function

$G(x) = \ln x$, is such that $G: P \rightarrow R$, where $P \equiv \{x: x \in R, x > 0\}$. That is, $\ln x$ is defined only on

the positive real numbers. Thus, $G(x) = \ln x$, $G: P \rightarrow R$, can not be an antiderivative function of

$f(x) = x^{-1}$, where $f: R - \{0\} \rightarrow R$. This is true since the antiderivative of f must map from the

same domain. In this case, clearly, $R- \{0\} \neq P$. To be able to define an antiderivative function of

x^{-1}, we must extend the function $\ln x$ to the set $R - \{0\}$ somehow. Thus, we define the function

$F(x) = \ln(|x|)$, where, of course $F: R-\{0\} \rightarrow R$. That is, $\ln(|x|)$ is defined on all non-zero real

numbers.

Example: Let $f(x) = 2e^x - x^{-1}$.

$$\int f(x)\, dx = 2\int e^x\, dx - \int x^{-1}\, dx = 2e^x - \ln|x| + c.$$

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The Methods of Integration by Substitution and by Parts.

1. *Method of Substitution*: To determine the indefinite integral of a function f(x), we choose by inspection a suitable differentiable function u(x) such that the given function f(x) can be expressed as the product g(u(x)) · u′(x), where g(u(x)) is a function of u and constants. Then we have

$$\int f(x)\, dx = \int g(u)\, u'(x)\, dx = \int g(u)\, du.$$

Remark: We have that u = u(x); hence, du = u′(x) dx.

2. Examples

#1   Let $f(x) = \dfrac{x}{\left(x^2 + 1\right)^{1/2}}$, find ∫ f(x) dx.

Choose u = u(x) = x² + 1, then we have that u′(x) = 2x, such that

$$du = u'(x)\, dx = 2x\, dx$$

Now

$$f(x) = g(u)\, u'(x)$$

$$f(x) = \frac{x}{\left(x^2 + 1\right)^{1/2}} = \frac{x}{u^{1/2}} = \frac{1}{2}\frac{1}{u^{1/2}} u'(x)$$

Here, g(u) = (1/2)(u$^{-1/2}$). Thus

$$\int f(x)\, dx = \int \frac{x}{\left(x^2 + 1\right)^{1/2}}\, dx = \int \frac{1}{2}\frac{1}{u^{1/2}} u'(x)\, dx$$

but du = u′(x) dx, hence

$$\int f(x)\, dx = \int g(u)\, du = \int \frac{1}{2}\frac{1}{u^{1/2}}\, du.$$

Now integrate the last result.

$$\int f(x)\, dx = \tfrac{1}{2}\int u^{-1/2} du$$

$$= \tfrac{1}{2}\left(\frac{1}{-\tfrac{1}{2}+1}\right) u^{1/2} + c$$

$$= \tfrac{1}{2}\left(\frac{1}{\tfrac{1}{2}}\right) u^{1/2} + c$$

$$= u^{1/2} + c$$

Substitute for $u = x^2 + 1$, then

$$\int \frac{x}{\left(x^2 + 1\right)^{1/2}}\, dx = (x^2+1)^{1/2} + c.$$

To check, find

$$\frac{d}{dx}(x^2 + 1)^{1/2} = \tfrac{1}{2}(x^2+1)^{-1/2}\, 2x$$

$$= \frac{x}{(x^2 + 1)^{1/2}}$$

#2   Let $f(x) = \dfrac{2x^3 - x}{\left(x^4 - x^2 + 1\right)^{1/3}}$. Find $\int f\, dx$.

(1)   Choose $u = (x^4 - x^2 + 1)$

∴ $u'(x) = 4x^3 - 2x$, such that

$$du = (4x^3 - 2x)\, dx.$$

(2)   Construct the product $g(u)\, u'(x)$ such that $f(x) = g(u)\, u'(x)$

$$f(x) = \frac{2x^3 - x}{\left(x^4 - x^2 + 1\right)^{1/3}} = \frac{2x^3 - x}{u^{1/3}}$$

$$= \frac{\left(\tfrac{1}{2}\right) u'(x)}{u^{1/3}}$$

$$= \frac{u'(x)}{2u^{1/3}}$$

Here $g(u) = \dfrac{1}{2u^{1/3}}$

(3)   Thus we have that

$$\int f(x)\, dx = \int g(u)\, u'(x)\, dx$$

$$= \int \frac{1}{2u^{1/3}}\, u'(x)\, dx = \int \frac{1}{2u^{1/3}}\, du$$

Take last integral:

$$\int f(x)\, dx = \frac{1}{2} \int u^{-1/3}\, du$$

$$= \frac{1}{2}\left(\frac{1}{2/3}\right) u^{2/3} + c$$

$$= \frac{3}{4} u^{2/3} + c$$

$$= \frac{3}{4}\left(x^4 - x^2 + 1\right)^{2/3} + c$$

To check take d/dx

$$= \frac{1}{2}(x^4 - x^2 + 1)^{-1/3}(4x^3 - 2x)$$

$$= \frac{4x^3 - 2x}{2\left(x^4 - x^2 + 1\right)^{1/3}}$$

$$= \frac{2x^3 - x}{\left(x^4 - x^2 + 1\right)^{1/3}}$$

2. *Method of Integration by Parts*. To determine the indefinite integral of the fn f(x), we choose by inspection two differentiable functions u(x), v(x) such that

$$\int f(x)\, dx = \int u\, dv = uv - \int v\, du.$$

Remark. If we can find u and v such that $f(x) = u\, v'(x)$, then $dv = v'(x)\, dx$ and

$$\int f(x)\, dx = \int u\, v'(x)\, dx = \int u\, dv.$$

To see that $\int u\, dv = uv - \int v\, du$, let $z \equiv vu$, then

$$dz = u\, dv + v\, du$$

Integrate both sides

$\int dz = \int u\ dv + \int v\ du$

$z = \int u\ dv + \int v\ du$

$\int u\ dv = z - \int v\ du$

$\therefore \int u\ dv = uv - \int v\ du.$

Examples.

#1   Let $f(x) = xe^{ax}$

(1)   Choose dv so that it is the most complicated expression,

but is easy to integrate

Let $dv = e^{ax}\ dx$

Let $u = x$ such that $du = dx$

(2) $\int u\ dv = x\dfrac{1}{a}e^{ax} - \int v\ du$

Now since

$dv = e^{ax}\ dx$

$\int dv = v = \int e^{ax}\ dx$

$v = \dfrac{1}{a}e^{ax}$

Also since

$u = x$

$du = 1\ dx$

$\int u\ dv = \dfrac{x}{a}e^{ax} - \int v\ du$

$= \dfrac{x}{a}e^{ax} - \int \dfrac{1}{a}\ e^{ax}\ dx$

$$= \frac{x}{a}e^{ax} - \frac{1}{a}\int e^{ax}\,dx$$

$$= \frac{x}{a}e^{ax} - \frac{1}{a^2}e^{ax} + c$$

$$\therefore \int x\,e^{ax}\,dx = \frac{x}{a}e^{ax} - \frac{1}{a^2}e^{ax} + c.$$

check

$$\frac{d}{dx}\left(\frac{x}{a}e^{ax} - \frac{1}{a^2}e^{ax}\right) = \frac{1}{a}e^{ax} + xe^{ax} - \frac{1}{a}e^{ax} = xe^{ax}$$

#2   Let $f(x) = 6xe^{x^2+2}$. Integrate by parts.

(1)   Let $v = e^{x^2+2}$

then $dv = 2x\,e^{x^2+2}\,dx$

Hence, let

$$u = 3$$

$$du = 0\ dx = 0$$

(2)   Then $\int f(x)\,dx = \int u\,dv$

$$= 3\,e^{x^2+2} - \int e^{x^2+2}\,du.$$

However, $du = 0$.

Hence,

$$\int 6x\,e^{x^2+2}\,dx = 3\,e^{x^2+2} + c$$

To check:

$$\frac{d}{dx}(3\,e^{x^2+2}) = 6x\,e^{x^2+2}.$$

Definite Integrals

1. Let f(x) be continuous on an interval $X \subset R$, where f: $X \to R$. Let F(x) be an antiderivative of

f, then $\int f(x) \, dx = F(x) + c$.

2. Now choose a, b $\in$ X such that a < b. Form the difference

$$[F(b) + c] - [F(a) + c] = F(b) - F(a).$$

(Note that this is independent of the constant c)

This difference F(b) - F(a) is called the *definite integral of f from a to b*. The point a is termed

the *lower limit of integration* and the point b, the *upper limit of integration.*

3. Notation: We would write

$$\int_a^b f(x) \, dx \equiv F(x)\big]_a^b \equiv F(x)\big|_a^b = F(b) - F(a)$$

4. All that we have described thus far is known as the fundamental theorem of calculus. We

state these results formally as follows:

Theorem 1: For any antiderivative function F(x), F: $X \to R$, $X \subset R$, of a continuous function

f(x), f: $X \to R$ defined on a closed interval I = [a, b], we have

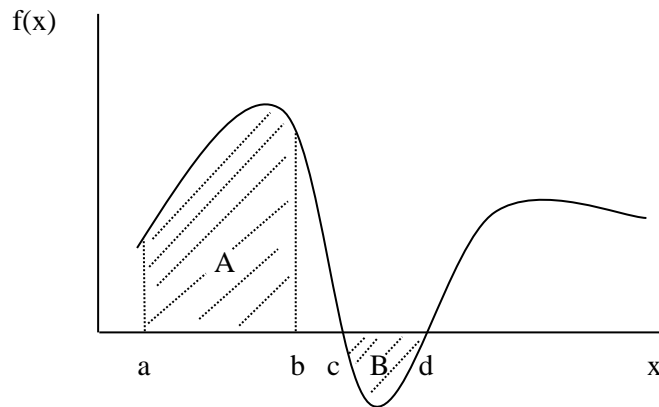$$\int_a^b f(x) \, dx = F(b) - F(a).$$

<u>Examples</u>

        #1   Let $f(x) = x^3$, find

$$\int_0^1 x^3 dx = \frac{1}{4} x^4 \big|_0^1 = \frac{1}{4}(1)^4 - \frac{1}{4}(0)^4$$

$$= \frac{1}{4}.$$

        #2 $f(x) = 2x \, e^{x^2}$, find $\int_3^5 f(x) \, dx$,

$$\int_3^5 2xe^{x^2} = e^{x^2} \big|_3^5 = e^{25} - e^9$$

5. The absolute value of the definite integral represents the area between f(x) and the x-axis between the points a and b.

The area $A = \int_a^b f(x)dx$ and the area $B = (-1)\int_c^d f(x)dx$.

6. Properties of Definite Integrals.

<u>P. 1</u>   If f(x) is such that $\exists \int_a^b f(x)\,dx$ over [a, b], then

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

<u>P. 2</u>   If f(x) is defined and continuous at the point a, then

$$\int_a^a f(x)dx = 0.$$

<u>P. 3</u>   If f(x) is defined and continuous on each of the closed intervals $[x_1, x_2]$, …, $[x_N, x_{N+1}]$, where N, the number of subintervals, is finite and

$$\bigcup_i \left[x_i, x_{i+1}\right] = \left[x_1, x_{N+1}\right], \text{ then}$$

$$\int_{x_1}^{x_{N+1}} f(x)dx = \int_{x_1}^{x_2} f(x)dx + \int_{x_2}^{x_3} f(x)dx + K + \int_{x_N}^{x_{N+1}} f(x)dx.$$

<u>P. 4</u>   If f(x) and g(x) are such that $\exists \int_a^b f(x)\,dx$ and $\exists \int_a^b g(x)\,dx$, then

(i) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$, for any $k \in R$

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$$(ii) \qquad \int_a^b \left[f(x) + g(x) \right] dx = \int_a^b f(x) dx + \int_a^b g(x) dx .$$

Improper Integrals

1. Consider a function f(x) and the definite integral $\int_a^b f(x) dx$. If a or b or both are infinite or if

f(x) is undefined for some $x \in [a, b]$, the above expression is termed *improper.*

Def 1. If f(x) is defined for $x \in [a, +\infty)$, then the expression $\int_a^\infty f(x) dx$ is defined as

$\lim\limits_{b \to \infty} \int_a^b f(x) dx$. If f(x) is defined for $x \in (-\infty, b]$, then $\int_{-\infty}^b f(x) dx$ is defined as $\lim\limits_{a \to -\infty^+} \int_a^b f(x) dx$.

Def 2. If f(x) is defined for $x \in [a, b]$ then the expression $\int_{-\infty}^{+\infty} f(x) dx$ is defined as

$\lim\limits_{\substack{b \to +\infty^- \\ a \to -\infty^+}} \int_a^b f(x) dx$.

Def 3. If f(x) is defined for $x \in [a, b)$, $b \in R$, but not defined for $x = b$, then $\int_a^b f(x) dx$ is

defined as $\lim\limits_{c \to b^-} \int_a^c f(x) dx$. If f(x) is defined for $x \in (a, b]$, $a \in R$, but not defined for $x = a$, then

the expression $\int_a^b f(x) dx$ is defined as $\lim\limits_{c \to a^+} \int_c^b f(x) dx$.

Def 4. If the limit of the improper integrals called for in Def. 1, 2 or 3 exists, the improper

integral is said to be *convergent*, otherwise *divergent.*

2. Examples:

#1   Evaluate $\int_2^\infty x^{-7} dx$. Hence, by Definition 1,

$$\int_2^\infty x^{-7} dx = \lim\limits_{b \to \infty^-} \int_2^b x^{-7} dx$$

$$= \lim\limits_{b \to \infty^-} \left[-\frac{1}{6} x^{-6} \Big|_2^b \right] = \lim\limits_{b \to \infty^-} \left[-\frac{1}{6} \frac{1}{b^6} + \frac{1}{6} \frac{1}{2^6} \right]$$

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$$= \frac{1}{6}\left(2^{-6} - \lim_{b\to\infty} b^{-6}\right)$$

$$= \frac{1}{6}2^{-6}$$

$\therefore$ our integral is convergent and $= \dfrac{1}{(6)2^6}$

#2   Evaluate $\displaystyle\int_0^1 x^{-2}dx = \lim_{a\to 0^+}\int_a^1 x^{-2}dx$

$$\lim_{a\to 0}\int_a^1 x^{-2}dx = \lim_{a\to 0}\left[-x^{-1}\Big|_a^1\right] = \lim_{a\to 0}\left[-1+\frac{1}{a}\right]$$

$$\lim_{a\to 0}\left(\frac{1}{a}-1\right) = \infty$$

Hence our integral is divergent and has no value.

#3   Compute #2 for $\displaystyle\int_{-1}^{+1} x^{-2}dx$

$$\int_{-1}^{+1} x^{-2}dx = \int_{-1}^{0} x^{-2}dx + \int_0^1 x^{-2}dx,$$

but we found above that

$$\lim_{a\to 0}\int_a^1 x^{-2}dx = \lim_{a\to 0}\left(\frac{1}{a}-1\right) = \infty$$

$\therefore$ it diverges.

#4   Let f(x) be a continuous probability density function with

support (domain) (-∞,+∞). We know that the following improper

integral is convergent.

$$\int_{-\infty}^{+\infty} f(x)dx = 1.$$

Both the expectation and the variance of the random variable x

are convergent improper integrals.

$$E(x) = \int_{-\infty}^{+\infty} x f(x) dx,$$

$$Var(x) = \int_{-\infty}^{+\infty} (x - E(x))^2 f(x) dx.$$

Likewise, the distribution function of x is a convergent improper integral. This function is defined by

$$F(x') = \int_{-\infty}^{x'} f(x) dx = Prob(x \leq x').$$

Differentiation of an Integral

1. The following rule applies to the differentiation of an integral.

$$\frac{\partial}{\partial y} \int_{p(y)}^{q(y)} f(x, y) dx = \int_{p}^{q} f_y(x, y) dx + f(q, y) q'(y) - f(p, y) p'(y).$$

2. Example: In Economics, we study a consumer's demand function in inverse form

$$p = p(Q),$$

where Q is quantitiy demanded and p denotes the maximum uniform price that the consumer is willing to pay for a given quantity level Q. We assume that p' is negative. The definite integral

$$\int_{0}^{Q} p(z) dz = TV(Q)$$

is called total value at Q. It gives us the maximum revenue that could be extracted from the consumer for Q units of the product. The dollar amount

$$TV(Q) - p(Q)Q = CS(Q)$$

is called consumer's surplus. Let C(Q) be the cost of supplying Q units. If a firm could extract maximum revenue from the consumer, its profit function would be

$$\int_0^Q p(z)dz - C(Q).$$

The output level maximizing the firm's profit sets the derivative of the previous definite integral equal to zero. This implies

$$p(Q) = C'(Q).$$

The firm should set the last quantity unit sold, so that price is equal to marginal cost. Each unit should have a price equal to the maximum price the consumer is willing to pay. This is called a perfectly discriminating monopolist.

Some Notes on Multiple Integrals

1. In this section, we will consider the integration of functions of more than one independent variable. The technique is analogous to that of partial differentiation. When performing integration with respect to one variable, other variables are treated as constants. Consider the following example:

$$\int_c^d \int_a^b f(x,y)dxdy \ .$$

We read the integral operators from the inside out. The bounds a,b refer to those on x, while the bounds c,d refer to y. Likewise, dx appears first and dy appears second. The integral is computed in two steps:

#1. Compute $\int_a^b f(x,y)dx = g(y).$

#2. Compute $\int_c^d g(y)dy = \int_c^d \int_a^b f(x,y)dxdy \ .$

If there were n variables, you would follow the same recursive steps n times. Each successive integration eliminates a single independent variable.

2. Some Examples.

Example 1: Suppose that $z = f(x,y)$. We wish to compute integrals of the form

$$\int_c^d \int_a^b f(x,y)dxdy.$$

Consider the example $f = x^2y$, where $c = a = 0$ and $d = 2$, $b = 1$. We have

$$\int_0^2 \int_0^1 x^2 y dxdy.$$

Begin by integrating with respect to x, treating y as a constant

$$\int_0^1 yx^2 dx = \frac{1}{3}yx^3 \mid_0^1 = \frac{1}{3}y.$$

Next, we integrate the latter expression with respect to y.

$$\int_0^2 \frac{1}{3}ydy = \frac{1}{3}\frac{1}{2}y^2 \mid_0^2 = \frac{1}{6}4 = \frac{2}{3}.$$

Example 2: Compute

$$\iiint (3xy + 2z)dxdydz,$$

where the limits of integration are 0 and 1, in each case. Begin with x.

$$(\frac{3x^2 y}{2} + 2zx) \mid_0^1 = \frac{3y}{2} + 2z.$$

Next, y

$$(\frac{3y^2}{4} + 2zy) \mid_0^1 = \frac{3}{4} + 2z.$$

Finally, z

$$(\frac{3z}{4} + z^2) \mid_0^1 = \frac{7}{4}.$$

Example 3. Let $f(x,y)$ be a joint probability density function defined on the continuous random variables (x,y) with support R^2. The unconditional expectation of x is defined as the double integral

$$E(x) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xf(x,y)dydx \ .$$

This expectation can be rewritten in terms of the marginal probability density function for x

$$f_1(x) = \int_{-\infty}^{+\infty} f(x,y)dy \ .$$

Thus,

$$E(x) = \int_{-\infty}^{+\infty} xf_1(x)dx \ .$$

The conditional expectation of x for a given value of y is quite different. The conditional density of x, given y, is

$$g_1(x \mid y) = f(x,y)/f_2(y),$$

and the conditional expectation is

$$E(x \mid y) = \int_{-\infty}^{+\infty} xg_1(x \mid y)dx \ .$$