Lecture 7: Integration Techniques

- Antiderivatives and Indefinite Integrals
- Basic Rules of Integration
- The Methods of Integration by Substitution and by Parts
- Definite Integrals
- Improper Integrals
- Differentiation of an Integral
- Some Notes on Multiple Integrals

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Antiderivatives and Indefinite Integrals

- Here we are interested in the operation of searching for functions whose derivatives are a given function.
- Let f(x) be a function defined on interval of real numbers. An antiderivative function of f is any function F(x) such that

$$F'(x) = f(x).$$

Clearly if F is an antiderivative of f then so is F + c, where c is a constant.

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Antiderivatives and Indefinite Integrals

- The indefinite integral is written as

$$F(x) + c = \int f(x)\,dx.$$

- The symbol $\int$ is the integral sign. If f has no antiderivative (nonintegrable), then its indefinite integral is $\varnothing$. The f(x) part is the integrand. The f(x) dx may be taken as the differential dF of a primary or antiderivative function.

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Basic Rules of Integration

- R1 $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx.$
- R2 $\int cf(x)\,dx = c\int f(x)\,dx.$
- R3 For every $N \neq -1$,

$$\int x^N\,dx = \frac{1}{N+1}x^{N+1} + c.$$

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Examples

#1 Let $f(x) = x^4$, then

$$\int x^4 dx = \frac{x^5}{5} + c.$$

check:

$$\frac{d\left(x^5/5\right)}{dx} = x^4.$$

#2 Let $f(x) = x^3 + 5x^4$, then

$$\int [x^3 + 5x^4]\, dx = \int x^3\, dx + 5\int x^4\, dx$$

$$= x^4/4 + 5(x^5/5) + c$$

$$= x^4/4 + x^5 + c$$

check.

$$\frac{d}{dx}(x^4/4 + x^5) = x^3 + 5x^4.$$

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Examples

#3 Let $f(x) = x^2 - 2x$

$$\int [x^2 - 2x]\, dx = \int x^2\, dx + \int -2x\, dx$$

$$= x^3/3 - 2\int x\, dx + c'$$

$$= x^3/3 - 2(x^2/2) + c$$

$$= x^3/3 - x^2 + c.$$

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Basic Rules of Integration

- R4 The antiderivative of e^x is given by
  $$\int e^x \, dx = e^x + c.$$
- R5 The antiderivative of x^{-1}, $x \neq 0$ is given by
  $$\int x^{-1} \, dx = \ln|x| + c.$$

(Note ln is only defined on positive reals)

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Example:

Let $f(x) = 2e^x - x^{-1}$.

$$\int f(x) \, dx = 2\int e^x \, dx - \int x^{-1} \, dx = 2e^x - \ln|x| + c$$

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The Methods of Integration by Substitution and by Parts

- Substitiuton:

$$\int f(x)\, dx = \int g(u)\, u'(x)\, dx = \int g(u)\, du.$$

- Requires us to find g and u such that

$$f(x) = g(u)\, u'(x)$$

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Example #1

Let $f(x) = \dfrac{x}{(x^2+1)^{1/2}}$, find $\int f(x)\, dx$.

Choose $u = u(x) = x^2 + 1$, then we have that $u'(x) = 2x$, such that

$$du = u'(x)\, dx = 2x dx$$

Now

$$f(x) = g(u)\, u'(x)$$

$$f(x) = \frac{x}{(x^2+1)^{1/2}} = \frac{x}{u^{1/2}} = \frac{1}{2}\frac{1}{u^{1/2}}u'(x)$$

Here, $g(u) = (1/2)(u^{-1/2})$.

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Example #1

$\int f(x)\,dx = \frac{1}{2}\int u^{-1/2}du$

$$= \tfrac{1}{2}\left(\frac{1}{-\frac{1}{2}+1}\right)u^{1/2}+c$$

$$= \tfrac{1}{2}\left(\frac{1}{\frac{1}{2}}\right)u^{1/2}+c$$

$$= u^{1/2}+c$$

Substitute for $u = x^2 + 1$, then

$$\int \frac{x}{\left(x^2+1\right)^{1/2}}\,dx = (x^2+1)^{1/2}+c.$$

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Example #2

Let $f(x) = \dfrac{2x^3 - x}{\left(x^4 - x^2 + 1\right)^{1/3}}$. Find $\int f\,dx$.

(1) Choose $u = (x^4 - x^2 + 1)$

$\therefore\ u'(x) = 4x^3 - 2x$, such that

$du = (4x^3 - 2x)\,dx$.

(2) Construct the product $g(u)\,u'(x)$ such that $f(x) = g(u)\,u'(x)$

$$f(x) = \frac{2x^3 - x}{\left(x^4 - x^2 + 1\right)^{1/3}} = \frac{2x^3 - x}{u^{1/3}}$$

$$= \frac{(\frac{1}{2})u'(x)}{u^{\frac{1}{3}}}$$

$$= \frac{u'(x)}{2u^{\frac{1}{3}}}$$

Here $g(u) = \dfrac{1}{2u^{\frac{1}{3}}}$

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Example #2

(3) Thus we have that

$$\int f(x)\,dx = \int g(u)\,u'(x)\,dx$$

$$= \int \frac{1}{2u^{1/3}}\,u'(x)\,dx = \int \frac{1}{2u^{1/3}}\,du$$

Take last integral:

$$\int f(x)\,dx = \frac{1}{2}\int u^{-1/3}\,du$$

$$= \frac{1}{2}\left(\frac{1}{2/3}\right)u^{2/3} + c$$

$$= \frac{3}{4}u^{2/3} + c$$

$$= \frac{3}{4}\left(x^4 - x^2 + 1\right)^{2/3} + c$$

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By parts

- To determine the indefinite integral of the function f(x), we choose by inspection two differentiable functions u(x), v(x) such that f(x) = u(x)v'(x). Then

$$\int f(x)\,dx = \int u\,dv = uv - \int v\,du.$$

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By parts

- If we can find u and v such that $f(x) = u\, v'(x)$, then $dv = v'(x)\, dx$ and
  $\int f(x)\, dx = \int u\, v'(x)\, dx = \int u\, dv$.
- To see that $\int u\, dv = uv - \int v\, du$, let $z \equiv vu$, then
  $dz = u\, dv + v\, du$
  Integrate both sides
  $\int dz = \int u\, dv + \int v\, du$
  $z = \int u\, dv + \int v\, du$
  $\int u\, dv = z - \int v\, du$
  $\therefore \int u\, dv = uv - \int v\, du.$ ||

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Examples

#1 Let $f(x) = xe^{ax}$

(1) Choose dv so that it is the most complicated expression, but is easy to integrate

Let $dv = e^{ax}\, dx$

Let $u = x$ such that $du = dx$

(2) $\int u\, dv = x \frac{1}{a} e^{ax} - \int v\, du$

Now since

$$dv = e^{ax}\, dx$$

$$\int dv = v = \int e^{ax}\, dx$$

$$v = \frac{1}{a} e^{ax}$$

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#1 continued

Also since

$$u = x$$

$$du = 1\,dx$$

$$\int u\,dv = \frac{x}{a}e^{ax} - \int v\,du$$

$$= \frac{x}{a}e^{ax} - \int \frac{1}{a}e^{ax}\,dx$$

$$= \frac{x}{a}e^{ax} - \frac{1}{a}\int e^{ax}\,dx$$

$$= \frac{x}{a}e^{ax} - \frac{1}{a^2}e^{ax} + c$$

$$\therefore \int x\,e^{ax}\,dx = \frac{x}{a}e^{ax} - \frac{1}{a^2}e^{ax} + c.$$

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Example

#2 Let $f(x) = 6xe^{x^2+2}$. Integrate by parts.

(1) Let $v = e^{x^2+2}$

then $dv = 2x\,e^{x^2+2}\,dx$

Hence, let

$$u = 3$$

$$du = 0\,dx = 0$$

(2) Then $\int f(x)\,dx = \int u\,dv$

$$= 3\,e^{x^2+2} - \int e^{x^2+2}\,du.$$

However, $du = 0$.

Hence,

$$\int 6x\,e^{x^2+2}\,dx = 3\,e^{x^2+2} + c$$

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Definite Integrals

- Let f(x) be continuous on an interval $X \subset R$, where $f: X \to R$. Let F(x) be an antiderivative of f, then $\int f(x)\,dx = F(x) + c$.
- Now choose $a, b \in X$ such that $a < b$. Form the difference

  $[F(b) + c] - [F(a) + c] = F(b) - F(a)$.
- This difference F(b) - F(a) is called the *definite integral of f from a to b*. The point a is termed the *lower limit of integration* and the point b, the *upper limit of integration.*

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Notation

$$\int_a^b f(x)dx \equiv F(x)\Big]_a^b \equiv F(x)\Big|_a^b = F(b) - F(a)$$

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Examples

#1 Let $f(x) = x^3$, find

$$\int_0^1 x^3 dx = \frac{1}{4}x^4\Big|_0^1 = \frac{1}{4}(1)^4 - \frac{1}{4}(0)^4$$

$$= \frac{1}{4}.$$

#2 $f(x) = 2x\,e^{x^2}$, find $\int_3^5 f(x)dx$,

$$\int_3^5 2xe^{x^2} = e^{x^2}\Big|_3^5 = e^{25} - e^9$$

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Illustration

- The absolute value of the definite integral represents the area between f(x) and the x-axis between the points a and b.

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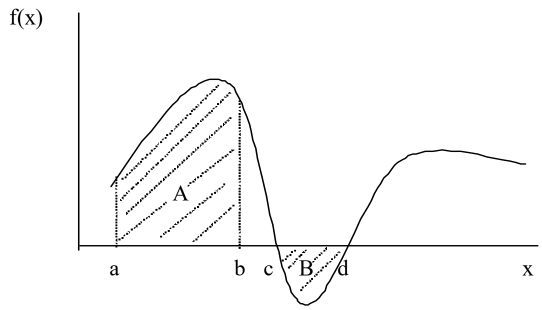
Illustration continued



The area A = $\int_a^b f(x)dx$ and the area B = $(-1)\int_c^d f(x)dx$.

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Properties

P. 1 If f(x) is such that $\exists \int_a^b f(x)\,dx$ over [a, b], then

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

P. 2 If f(x) is defined and continuous at the point a, then

$$\int_a^a f(x)dx = 0.$$

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Properties

P. 3 If f(x) is defined and continuous on each of the closed intervals $[x_1, x_2]$, …, $[x_N, x_{N+1}]$, where N, the number of subintervals, is finite and

$\bigcup_i [x_i, x_{i+1}] = [x_1, x_{N+1}]$, then

$$\int_{x_1}^{x_{N+1}} f(x)dx = \int_{x_1}^{x_2} f(x)dx + \int_{x_2}^{x_3} f(x)dx + 5 + \int_{x_N}^{x_{N+1}} f(x)dx .$$

Properties

P. 4 If f(x) and g(x) are such that $\exists \int_a^b f(x)\,dx$ and $\exists \int_a^b g(x)\,dx$, then

(i) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$, for any $k \in R$

(ii) $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

Improper Integrals

- Consider a function f(x) and the definite integral

$$\int_a^b f(x)\,dx$$

- If a or b or both are infinite or if f(x) is undefined for some x $\in$ [a, b], the above expression is termed *improper.*

Improper Integrals

Def 1. If f(x) is defined for x $\in$ [a, +∞), then the expression $\int_a^{\infty} f(x)\,dx$ is defined as $\lim_{b\to\infty}\int_a^b f(x)\,dx$. If f(x) is defined for x $\in$ (-∞, b], then $\int_{-\infty}^b f(x)\,dx$ is defined as $\lim_{a\to-\infty^+}\int_a^b f(x)\,dx$.

Improper Integrals

Def 2. If f(x) is defined for x $\in$ [a, b] then the expression $\int_{-\infty}^{+\infty} f(x)\,dx$ is defined as $\lim\limits_{\substack{b\to+\infty^- \\ a\to-\infty^+}} \int_a^b f(x)\,dx$.

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Improper Integrals

Def 3. If f(x) is defined for x $\in$ [a, b), b $\in$ R, but not defined for x = b, then $\int_a^b f(x)dx$ is defined as $\lim\limits_{c\to b^-} \int_a^c f(x)dx$. If f(x) is defined for x $\in$ (a, b], a $\in$ R, but not defined for x = a, then the expression $\int_a^b f(x)dx$ is defined as $\lim\limits_{c\to a^+} \int_c^b f(x)dx$.

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Improper Integrals

Def 4. If the limit of the improper integrals called for in Def. 1, 2 or 3 exists, the improper integral is said to be *convergent*, otherwise *divergent.*

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Examples

#1 Evaluate $\int_2^{\infty} x^{-7}dx$. Hence, by Definition 1,

$$\int_2^{\infty} x^{-7}dx = \lim_{b\to\infty^-} \int_2^{b} x^{-7}dx$$

$$= \lim_{b\to\infty^-}\left[-\frac{1}{6}x^{-6}\Big|_2^b \right] = \lim_{b\to\infty^-}\left[-\frac{1}{6}\frac{1}{b^6} + \frac{1}{6}\frac{1}{2^6} \right]$$

$$= \frac{1}{6}\left(2^{-6} - \lim_{b\to\infty} b^{-6}\right)$$

$$= \frac{1}{6}2^{-6}$$

$\therefore$ our integral is convergent and $= \dfrac{1}{(6)2^6}$

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Examples

#2 Evaluate $\int_0^1 x^{-2}dx = \lim_{a \to 0^+} \int_a^1 x^{-2}dx$

$$\lim_{a \to 0} \int_a^1 x^{-2}\,dx = \lim_{a \to 0}\left[-x^{-1}\Big|_a^1 \right] = \lim_{a \to 0}\left[-1 + \frac{1}{a} \right]$$

$$\lim_{a \to 0}\left(\frac{1}{a} - 1 \right) = \infty$$

Hence our integral is divergent and has no value.

#3 Compute #2 for $\int_{-1}^{+1} x^{-2}dx$

$$\int_{-1}^{+1} x^{-2}dx = \int_{-1}^{0} x^{-2}dx + \int_0^1 x^{-2}dx\ ,$$

but we found above that

$$\lim_{a \to 0} \int_a^1 x^{-2}dx = \lim_{a \to 0}\left(\frac{1}{a} - 1 \right) = \infty$$

$\therefore$ it diverges.

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Differentiation of an integral

$$\frac{\partial}{\partial y}\int_{p(y)}^{q(y)} f(x,y)dx = \int_p^q f_y(x,y)dx + f(q,y)q'(y) - f(p,y)p'(y).$$

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Example: Consumer and Producer Surplus

- Define TV
- Define CS
- Consider perfectly discriminating monopoly
- Define PS
- Determine market optimum

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Some Notes on Multiple Integrals

- In this section, we will consider the integration of functions of more than one independent variable. The technique is analogous to that of partial differentiation.
- When performing integration with respect to one variable, other variables are treated as constants.

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Multiple Integrals

- Consider the following example:

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

- We read the integral operators from the inside out. The bounds a,b refer to those on x, while the bounds c,d refer to y. Likewise, dx appears first and dy appears second.

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Multiple Integrals

- The integral is computed in two steps:

#1. Compute $\int_a^b f(x,y)dx = g(y)$.

#2. Compute $\int_c^d g(y)dy = \int_c^d \int_a^b f(x,y)dxdy$.

- If there were n variables, you would follow the same recursive steps n times. Each successive integration eliminates a single independent variable.

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Examples

Example 1: Suppose that z = f(x,y). We wish to compute integrals of the form

$$\int_c^d \int_a^b f(x,y)dxdy.$$

Consider the example $f = x^2y$, where c = a = 0 and d = 2, b = 1. We have

$$\int_0^2 \int_0^1 x^2 ydxdy.$$

Begin by integrating with respect to x, treating y as a constant

$$\int_0^1 yx^2dx = \frac{1}{3}yx^3 |_0^1 = \frac{1}{3}y.$$

Next, we integrate the latter expression with respect to y.

$$\int_0^2 \frac{1}{3}ydy = \frac{1}{3}\frac{1}{2}y^2 |_0^2 = \frac{1}{6}4 = \frac{2}{3}.$$

Examples

Example 2: Compute

$$\iiint (3xy + 2z)dxdydz,$$

where the limits of integration are 0 and 1, in each case. Begin with x.

$$(\frac{3x^2y}{2} + 2zx) |_0^1 = \frac{3y}{2} + 2z.$$

Next, y

$$(\frac{3y^2}{4} + 2zy) |_0^1 = \frac{3}{4} + 2z.$$

Finally, z

$$(\frac{3z}{4} + z^2) |_0^1 = \frac{7}{4}.$$