Mathematics Review for MS Finance Students

Lecture Notes

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Lecture 1.1: Basics

I. Sets

1. A *set* is a list or collection of objects. The objects which compose a set are termed the *elements* or *members* of the set.

Remark 1: A set may be written in *tabular form*, that is, by listing all of its elements. As an example, consider the set A which consists of the positive integers 1, 2 and 3. We would write this set as

$$A = \{1, 2, 3\}.$$

Note that when we define a set, the order of listing is not important. That is, in our above example,

$$A = \{1, 2, 3\} = \{3, 1, 2\}.$$

Another notation for writing sets is the *set-builder* form. In the above example,

$$A = \{x \mid x \text{ is a positive integer, } 1 \le x \le 3\}.$$

or

$$A = \{x : x \text{ is a positive integer, } 1 \le x \le 3\}$$

The notation ":" or "|" reads "such that". The entire notation reads "A is the set of all numbers such that x is a positive integer and $1 \le x \le 3$."

Remark 2: The symbol "$\in$" reads "is an element of". Thus, in the same example

$$1 \in A \text{ and } 1, 2, 3, \in A.$$

2. If every element of a set S_1 is also an element of a set S_2, then S_1 is a *subset* of S_2 and we write

$$S_1 \subset S_2 \text{ or } S_2 \supset S_1.$$

The symbol "$\subset$" reads "is contained in" and "$\supset$" reads "contains".

<u>Examples</u>:

#1. If $S_1 = \{1, 2\}$, $S_2 = \{1, 2, 3\}$, then $S_1 \subset S_2$ or $S_2 \supset S_1$.

#2 The set of all positive integers is a subset of the set of real numbers.

Def 1: Two sets S_1 and S_2 are said to be *equal* if and only if $S_1 \subset S_2$ and $S_2 \subset S_1$.

Remark 3: The *largest subset* of a set S is the set S itself. The *smallest subset* of a set S is the set which contains no elements. The set which contains no elements is the *null set*, denoted $\varnothing$.

Remark 4: In Remark 3 we noted that the null set is the smallest subset of any set S. To see that the null set is contained in any set S, consider the following proof by contradiction.

Proof: Assume $\varnothing \not\subset S$, then there exists at least one $x \in \varnothing$ such that $x \notin S$. However, $\varnothing$ has no elements. Thus, we have a contradiction and it must be that $\varnothing \subset S$. ||

Remark 5: The set S which contains zero as its only element, i.e. $S = \{0\}$, is *not* the null set. That is, if $S = \{0\}$, then $S \neq \varnothing$ since $0 \in S$. As an example of the null set, consider the set A where

$$A = \{x : x \text{ is a living person 1000 years of age}\}$$

Clearly, $A = \varnothing$.

Def 2: Two sets S_1 and S_2 are *disjoint* if and only if there does not exist an x such that $x \in S_1$ and $x \in S_2$.

Example: If $S_1 = \{0\}$ and $S_2 = \{1, 2, 4\}$, then S_1 and S_2 are disjoint.

Def 3: The operations of *union*, *intersection*, *difference* (relative complement), and *complement* are defined for two sets A and B as follows:

(i) $A \cup B \equiv \{x : x \in A \text{ or } x \in B\}$,

(ii) $A \cap B \equiv \{x : x \in A \text{ and } x \in B\}$,

(iii) $A - B \equiv \{x : x \in A, x \notin B\}$,

(iv) $A' \equiv \{x : x \notin A\}$.

Remark 6: In the application of set theory all sets under investigation are likely to be subsets of a fixed set. We call this set the *universal set* and denote it as U. For example, in human population studies U would be the set of all people in the world.

Remark 7: We have the following laws of the algebra of sets:

1. *Idempotent laws*

   1. a. $A \cup A = A$ 1. b. $A \cap A = A$

2. *Associative laws*

   2. a. $(A \cup B) \cup C = A \cup (B \cup C)$

   2. b. $(A \cap B) \cap C = A \cap (B \cap C)$

3. *Commutative laws*

   3. a. $A \cup B = B \cup A$

   3. b. $A \cap B = B \cap A$

4. *Distributive laws*

   4. a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   4. b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

5. *Identity laws*

   5. a. $A \cup \varnothing = A$

   5. b. $A \cup U = U$

   5. c. $A \cap U = A$

   5. d. $A \cap \varnothing = \varnothing$

6. *Complement laws*

   6. a. $A \cup A' = U$

   6. b. $(A')' = A$

   6. c. $A \cap A' = \varnothing$

   6. d. $U' = \varnothing$, $\varnothing' = U$

Practice Problems I

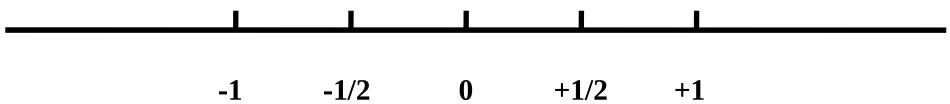
1. Given the sets $S_1 = \{2,3,5\}$, $S_2 = \{3,5,6\}$ and $S_3 = \{7\}$, find:

a. $S_1 \cap S_3$ b. $S_1 \cup S_2$ c. $S_2 \cap S_2$ d. $S_2 \cap U$ e. $S_2 \cap \varnothing$

2. Given A = {4,5,6}, B = {3,4,6,7}, and C = {2,3,6}, verify the distributive law.

II. The Real Number System

1. The real numbers can be geometrically represented by points on a straight line. We would choose a point, called the origin, to represent 0 and another point, to the right, to represent 1. Then it is possible to pair off the points on the line and the real numbers such that each point will represent a unique real number and each real number will be represented by a point. We would call this line the *real line*. Numbers to the right of zero are the *positive numbers* and those to the left of zero are the *negative numbers*. Zero is neither positive nor negative.



We shall denote the set of real numbers and the real line by R.

2. The *integers* are the "whole" real numbers. Let I be the set of integers such that

$$I = \{..., -2, -1, 0, 1, 2, ...\}.$$

3. The *rational numbers*, Q, are those real numbers which can be expressed as the *ratio* of two *integers*. Hence,

Def: $Q = \{x \mid x = p/q, p \in I, q \in I, q \neq 0\}$.

Clearly, *each integer is also a rational*, because any $x \in I$ may be expressed as $(x/1) \in Q$. So we have that the set of integers is a subset of the set of rational numbers, i.e.,

$$I \subset Q.$$

4. The *irrational numbers*, Q', are those real numbers which cannot be expressed as the ratio of two integers. Hence, the irrationals are those real numbers which are not rational. They are the non-repeating infinite decimals. The set of irrationals Q' is just the *complement of the set of*

rationales Q in the set of reals (Here we speak of *R as our universal set*). Some examples of irrational numbers are $\sqrt{5}$, $\sqrt{3}$ and $\sqrt{2}$.

5. The *natural numbers* N are the positive integers:

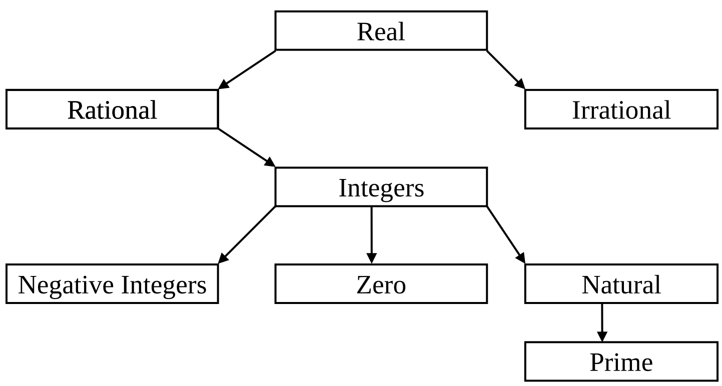
$N = \{x \mid x > 0, x \in I\}$, or

$N = \{1, 2, 3, ...\}$.

6. The *prime numbers* are those natural numbers say p, excluding 1, which are divisible only by 1 and p itself. A few examples are

2, 3, 5, 7, 11, 13, 17, 19, and 23.

7. We may illustrate the real number system with the following line diagram.



8. The *extended real number system.*

The set of real numbers R may be extended to include $-\infty$ and $+\infty$. Accordingly we would add to the real line the points $-\infty$ and $+\infty$. The result would be *the extended real number system* or the *augmented real line*. We denote these by $\hat{R}$. The following operational rules usually apply.

(i) If "a" is a real number, then $-\infty < a < +\infty$

(ii) $a + \infty = \infty + a = \infty$, if $a \neq -\infty$

(iii) $a + (-\infty) = (-\infty) + a = -\infty$, if $a \neq +\infty$

(iv) If $0 < a \leq +\infty$, then

$a \bullet \infty = \infty \bullet a = \infty$

$a \bullet (-\infty) = (-\infty) \bullet a = -\infty$

(v) If $-\infty \leq a < 0$, then

$$a \bullet \infty = \infty \bullet a = -\infty$$

$$a \bullet (-\infty) = (-\infty) \bullet a = +\infty$$

(vi) If "a" is a real number, then $\frac{a}{\infty} = \frac{a}{-\infty} = 0$

9. Absolute Value

Def 1: The *absolute value* of any real number x, denoted |x|, is defined as follows:

$$|x| \equiv \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Remark 1: If x is a real number, its absolute value |x| geometrically represents the distance between the point x and the point 0 on the real line. If a, b are real numbers, then |a-b| = |b-a| would represent the distance between a and b on the real line.

Remark 2: The following properties characterize absolute values:

(i) $|a| \geq 0,$

(ii) $|a| + |b| \geq |a + b|,$

(iii) $|a| \cdot |b| = |a \cdot b|,$

(iv) $\frac{|a|}{|b|} = \left|\frac{a}{b}\right|, \quad b \neq 0,$

where a, b $\in$ R.

10. *Intervals* on the real line.

Let a, b $\in$ R where a < b, then we have the following terminology:

(i) The set A = {x | a ≤ x ≤ b}, denoted A = [a, b], is termed a *closed interval* on R. (note that a, b $\in$ A)

(ii) The set B = {x | a < x ≤ b}, denoted B = (a, b], is termed an *open-closed interval* on R. (note a $\notin$ b, b $\in$ B)

(iii) The set C = $\{x \mid a \le x < b\}$, denoted C = $[a, b)$, is termed a *closed-open interval* on R. (note $a \in C$, $b \notin c$)

(iv) The set D = $\{x \mid a < x < b\}$, denoted D = (a, b), is termed an *open-interval* on the real line. (note $a, b \notin D$)

Practice Problems II

1. Write the following in set notation and in interval notation:

a. The set of real numbers between 2 and 10, inclusive.

b. The set of real numbers less than 15.

2. Determine the following:

a. $|5\text{-}6|$ b. $6/\infty$ c. $10\times(-\infty)$ d. $12 + (-\infty)$ e. $12/-\infty$

III. Functions, Ordered Tuples and Product Sets

Ordered Pairs and Ordered Tuples

1. If we are given a set {a, b}, we know that {a, b} = {b, a}. As we noted above, order does not matter. We could call {a, b} an *unordered pair*.

2. However, if we were to designate the element "a" as the first listing of the set and the element "b" as the second listing of the set, then we would have an *ordered pair*. We denote this ordering by

$$(a, b).$$

Similarly it would be possible to define an *ordered triple* say

$$(x_1, x_2, x_3),$$

where the ordered triple is a set with three elements such that x_1, is the first, x_2 the second, and x_3 the third element of the set.

Likewise, an *ordered N-tuple*

$(x_1, x_2, ..., x_n)$

could be defined analogously. An ordered N-tuple of numbers may be given various interpretations. For example, it could represent a vector whose components are the n numbers, or it could represent a point in n-space, the n numbers being the coordinates of the point. (Note (a, b) = (b, a) iff b = a, (a, b) = (c, d) iff a = c, d = b)

The product set

1. Let X and Y be two sets. The *product set* of X and Y or the *Cartesian product* of X and Y consists of all of the possible ordered pairs (x, y), where $x \in X$ and $y \in Y$. That is, we construct every possible ordered pair (x, y) such that the first element comes from X and the second from Y. The product set of X and Y is denoted

$$X \times Y,$$

which reads "X cross Y." More formally we have

Def 1: The *product set* of two sets X and Y is defined as follows:

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

Examples:

#1. If A = {a, b}, B = {c, d}, then $A \times B$ = {(a, c), (a, d), (b, c), (b, d)}

#2. If A = {a, b}, B = {c, d e}, then $A \times B$ = {(a, c), (a, d) ,(a, e), (b, c), (b, d), (b, e)}.

#3 The Cartesian plane or Euclidean two-space, R^2, is formed by

$$R \times R = R^2.$$

Each point x in the plane is an ordered pair $x = (x_1, x_2)$, where x_1 represents the coordinate on the axis of abscissas and x_2 represents the coordinate on the axis of ordinates.

The n-fold Cartesian product of R is Euclidean n-space.

Functions

Def 2: A *function* from a set X into a set Y is a rule f which assigns to every member x of the set X is single member y = f(x) of the set Y. The set X is said to be the *domain* of the function f and the set Y will be referred to as the *codomain* of the function f.

Remark 1: If f is a function from X into Y, we write

$$f : X \to Y,$$

or

$$X \xrightarrow{f} Y.$$

This notation reads "f is a function from X into Y" or "f maps from X into Y".

Def. 2. a: The element in Y assigned by f to an $x \in X$ is the *value of f at x* or the *image of x under f*.

Remark 2: We denote the value of f at x or the image of x under f by f(x). Hence, the function may be written

$$y = f(x),$$

which reads "y is a function of x", where $y \in Y$, $x \in X$.

Def. 2. b: The *graph* Gr(f) of the function $f : X \to Y$ is defined as follows:

$$Gr(f) \equiv \{(x, f(x)) : x \in X\},$$

where $Gr(f) \subset X \times Y$.

Def 2. c: The *range* f [X] of the function $f : X \to Y$ is the set of images of $x \in X$ under f or

$$f[X] \equiv \{ f(x): x \in X\}.$$

Remark 3: Note that the range of a function f is a subset of the codomain of f, that is

$$f[X] \subset Y.$$

Def 3: A function $f: X \to Y$ is said to be *onto* if and only if

$$f[X] = Y.$$

Def 4: A function f: X $\rightarrow$ Y is said to be *one-to-one* if and only if images of distinct members of the domain of f are always distinct; in other words, if and only if, for any two members x, $x' \in X$, $f(x) = f(x')$ implies $x = x'$.

One-to-one functions have a useful property in that they are capable of being "inverted". That is, it is possible to reverse the mapping and write x as a function of y. The point is that if y = f(x) is one-to-one, then each y has an associated unique x in the original mapping. Thus, there is a reverse functional relationship.

Result: If y = f(x) is a one-to-one function, then an inverse function $x = f^{-1}(y)$ exists and f^{-1}: f[X] $\rightarrow$ X. That is the inverse function assigns to each image value f(x') the value x', for all x'.

Examples: Given the function y = 2x, the inverse function exists and is given by x = y/2. Given the function y = 3 + 4x, an inverse function exists and is given by x = -3/4 + y/4.

Practice Problems III

1. Given $S_1 = \{1,3,4\}$, $S_2 = \{a,b\}$ and $S_3 = \{m,n\}$, find the Cartesian products:

a. $S_1 \times S_2$ b. $S_2 \times S_3$ c. $S_1 \times S_3$ d. $S_1 \times S_2 \times S_3$.

2. Is it true that $S_1 \times S_2 = S_2 \times S_1$? Explain.

3. If the domain of the function y = 5 + 3x is the set $\{x| 1 \leq x \leq 4\}$, find the range of this function and express it as a set.

4. Does a circle drawn in a rectangular coordinate plane represent a function? Explain.

5. Is a hill shaped function one-to-one? Explain.

6. Does y = 2 + 4x have an inverse? If so what is it?

IV. Common Functions

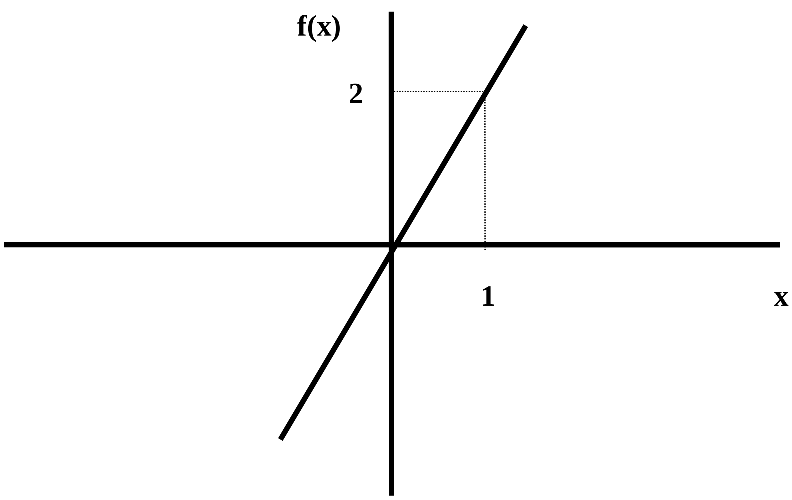
1. When we write $y = f(x)$, we mean that a functional relationship between y and x exits (each x maps into one y), however, we have not made the rule of the mapping explicit. In this section, we consider several specific functional types. Each is used in business applications.

2. The first example is a specific example of a linear function. Let the function f assign to every real number its double. Hence, for every real number x,

$$f(x) = 2x, \text{ or}$$

$$y = 2x.$$

Both the domain and the codomain of f are the set of real numbers, R. Hence, $f: R \to R$. The image of the real number 2 is $f(2) = 4$. The range of f is given by $f[R] = \{2x : x \in R\}$. The Gr(f) is given by $Gr(f) = \{(x, 2x): x \in R\}$. We illustrate a portion of $Gr(f) \subset R \times R = R^2$.



Clearly, the function $f(x) = 2x$ is one-to-one. Moreover, $f(x) = 2x$ is onto, that is, $f[R] = R$.

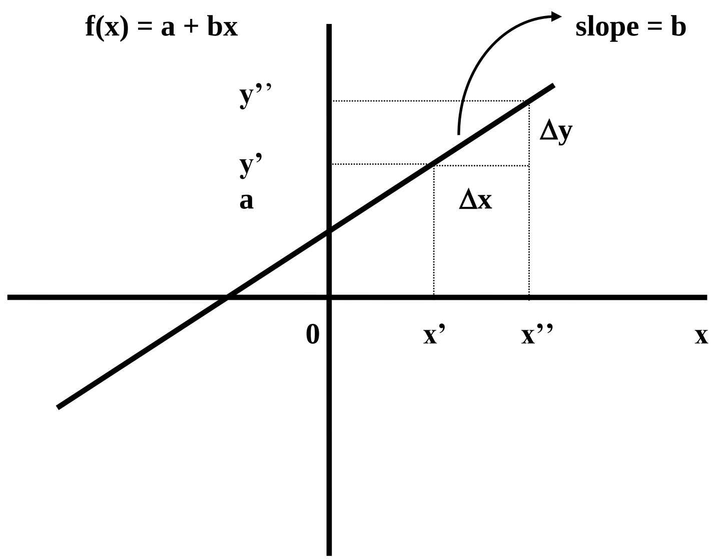
3. Example #1 is a specific example of a general class of linear functions. Let a and b be two real numbers. The function $y = f(x) = a + bx$ is a linear function of x. The graph of this function is shown below for the case where both a and b are positive. The constant a is called the intercept of the function, because, for $x = 0$, $y = a$. That is the function crosses the y axis at the point a. The constant b is called the slope coefficient, because the slope of the graph of this function is b at each

point. By slope, we mean the rate of change of y per change in x. Let x change by some arbitrary amount $\Delta x = x'' - x'$. This change in x generates a change in y through the functional relationship. The induced change in y is given by

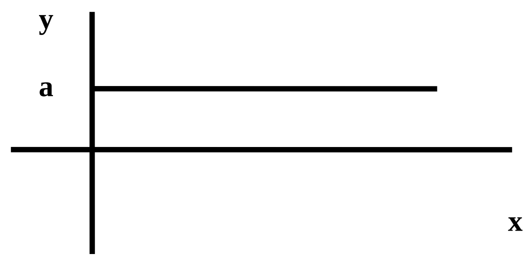
$$\Delta y = (a + bx'') - (a + bx') = b(x'' - x').$$

Thus, the rate of change of y per change in x is just

$$\frac{\Delta y}{\Delta x} = \frac{b(x''-x')}{(x''-x')} = b.$$



If in the above example, b = 0, then f is said to be a constant function. For every value of x , y is equal to the constant a. In this case, the graph of f is a flat line (i.e., the slope is zero).



3. Next, we consider a general class of functions termed polynomial. The term polynomial means multi-term. A polynomial function has the general form

$$y = a_o + a_1x + a_2x^2 + \bullet\bullet\bullet + a_nx^n.$$

If n = 0, then $y = a_o$ and we have a *constant function*.

If n = 1, then $y = a_o + a_1x$ and we have a *linear function.*

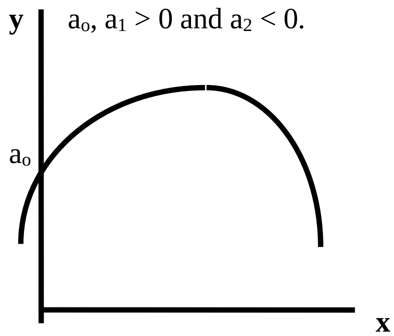
If n = 2, then $y = a_o + a_1x + a_2x^2$ and we have a *quadratic function*.

If n = 3, then $y = a_o + a_1x + a_2x^2 + a_3x^3$ and we have a *cubic function.*

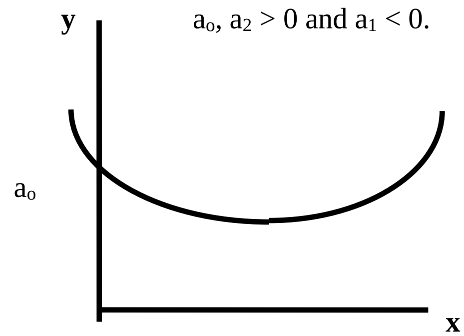
The superscript indicators of the powers of x are called exponents and the highest power involved in the function is called the degree of the polynomial function. A cubic function is therefore called a third-degree polynomial.

We have already discussed the case of a linear function.

When a quadratic function is plotted, it appears as a parabola. This is a curve with a single "bump". An example is given in the diagram below.

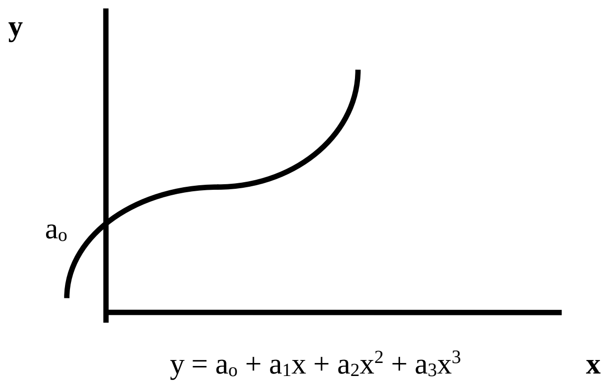


$y = a_o + a_1x + a_2x^2$



$y = a_o + a_1x + a_2x^2$

When a cubic function is plotted, it exhibits two bumps as shown in the example below.



$y = a_o + a_1x + a_2x^2 + a_3x^3$
($a_o, a_1, a_2 > 0$ and $a_3 < 0$)

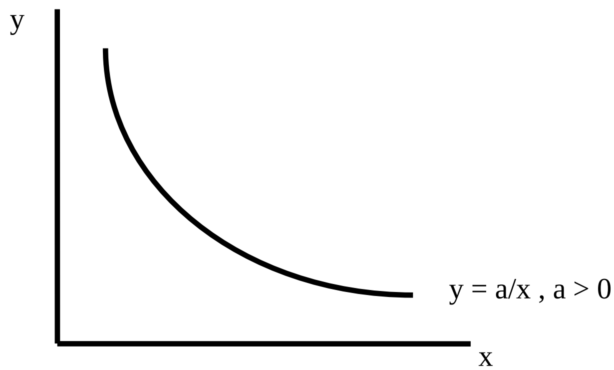
4. *Rational functions* are functions in which y is expressed as the ratio of two polynomial functions in x. An example is

$$y = \frac{x+1}{x^2 + 4x}.$$

Under this definition, every polynomial function is also a rational function. One common rational function encountered in business applications is the rectangular hyperbola. This function has the form

$$y = a/x.$$

Given $a > 0$ and $x \geq 0$, the graph of this function is as in the diagram below.



5. All of the functions discussed thus far are termed algebraic. Generally, such functions are expressed in terms of polynomials and/or roots of polynomials (e.g., square root of a polynomial.). For example, y = square root $\{(x^3 + x^2)\}$ is not rational, but it is algebraic. Nonalgebraic functions include two types that are used in business applications. The first is the *exponential function*, $y = b^x$, where the independent variable appears in the exponent. The second is the closely related *logarithmic function*, $y = \log_b x$. We will discuss both of these functions, but first let us digress on the topic of exponents.

A Digression on Exponents

Above, in the introduction to polynomial functions, we considered the exponent as the indicator of the power to which a variable or number is to be raised. For example, 3^2 means that the number 3 is to be raised to the second power or that 3 is to be multiplied by itself. We have that $3^2 = 3 \times 3 = 9$. Generally,

$$x \times x \times x \times \bullet\bullet\bullet \times x \text{ (n-terms)} = x^n.$$

As a special case, $x^1 = x$.

Exponents obey the following rules.

Rule 1. $x^n \times x^m = x^{n+m}$.

Rule 2. $x^n/x^m = x^{n-m}$.

The proofs of Rules 1 and 2 are obvious. However, if $n < m$, then the power of x becomes negative from Rule 2. What does this mean? Actually, Rule 2 tells us the answer. If $n = 4$ and $m = 6$, then

$$\frac{x^4}{x^6} = \frac{xxxx}{xxxxxx} = \frac{1}{xx} = \frac{1}{x^2}.$$

Thus $x^{-2} = 1/x^2$ and this can be generalized into another rule:

Rule 3. $x^{-n} = 1/x^n$.

Another special case of Rule 2 is where $n = m$, in which case $x^n/x^n = x^0$. This must be one. Thus, in accordance with Rule 2, any nonzero number raised to the zero power is one.

Rule 4. $x^0 = 1$. ($x \neq 0$)

So far we have thought of exponents as integers. How do we interpret fractional exponents? Using Rule 1, we can interpret a number such as $x^{1/2}$:

$x^{1/2} \times x^{1/2} = x^1 = x$.

That is, because $x^{1/2}$ multiplied by itself is x, $x^{1/2}$ is the square root of x. Likewise, $x^{1/3}$ is the cube root of x. Generally,

Rule 5. $x^{1/n} = \sqrt[n]{x}$.

Two other rules obeyed by exponents are as follows.

Rule 6. $(x^m)^n = x^{nm}$.

Rule 7. $x^m \times y^m = (xy)^m$.

Exponential and Logarithmic Functions

1. An exponential function is a function in which the independent variable appears as an exponent:

$y = b^x$, where $b > 1$.[1]

A logarithmic function is the inverse function of b^x. That is,

$x = \log_b y$.

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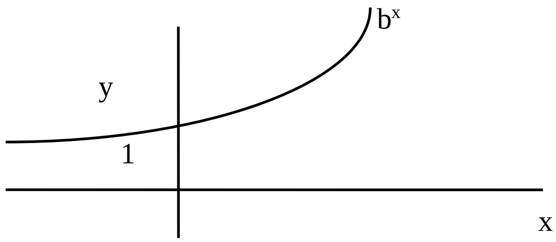
[1] We take $b > 0$ to avoid complex numbers. $b > 1$ is not restrictive because we can take $(b^{-1})^x = b^{-x}$ for this case.

The exponential function is graphed below.



2. Rules of logarithms.

Rule 1. If x, y > 0, then log (yx) = log y + log x.

Rule 2. If x, y > 0, then log (y/x) = log y - log x.

Rule 3. If x > 0, then log x^a = a log x.

3. A preferred base is the number e ≅ 2.72. (More formally, $e = \lim_{n\to\infty} [1 + 1/n]^n$.) e is called the natural logarithmic base. The corresponding log function is written x = ln y, meaning $\log_e y$.

4. Conversion and inversion of bases.

a. conversion

$\log_b u = (\log_b c)(\log_c u)$ ($\log_c u$ is known)

Proof: Let $u = c^p$. Then $\log_c u = p$. We know that $\log_b u = \log_b c^p = p\log_b c$. By definition, $p = \log_c u$, so that $\log_b u = (\log_c u)(\log_b c)$. ||

b. inversion

$\log_b c = 1/(\log_c b)$.

Proof: Using the conversion rule, $1 = \log_b b = (\log_b c)(\log_c b)$. Thus, $\log_b c = 1/(\log_c b)$. ||

Practice Problems IV

1. Graph the functions

a. y = 2 + 3x b. y = 2 - 3x.

In each case, let x ≥ 0.

2. Condense the following expressions:

a. $x^6 \times x^4$ b. x^3/x^{-2} c. $(x^{1/2} \times x^{1/3})/x^{2/3}$

3. Show that $x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$.

4. Simplify the following:

a. $\ln(x^6 y^\alpha z^5)$ b. $\log_{10} 1$ c. $\log_{10} 2 = \log_2$? d. $\ln(x/yz)$

5. Given $\log_e 2$, how do we convert this to $\log_{10}$? Explain.

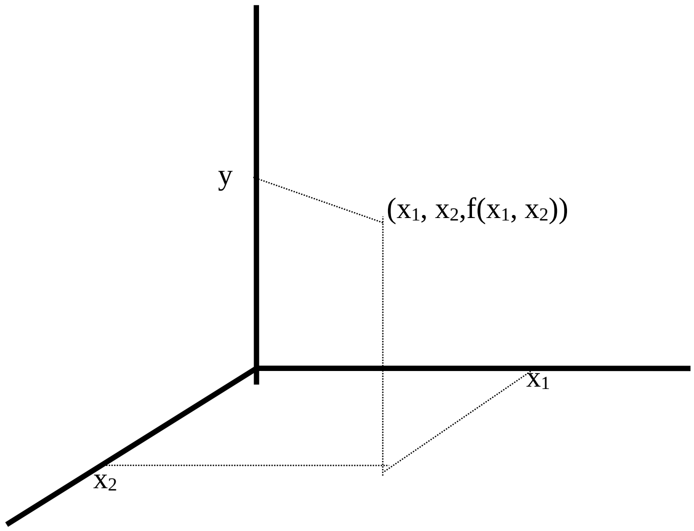
V. Functions of Two or More Independent Variables

1. So far we have only considered functions of a single independent variable. However, the concept can be extended to the case of two or more independent variables. If y is the dependent variable and x_1 and x_2 are two independent variables, then a function of the form $y = f(x_1,x_2)$ expresses a relationship where each ordered pair (x_1, x_2) is assigned one image value y. For example,

$$y = 6x_1 + 4x_2 \text{ and } y = x_1^2 x_2^3$$

exemplify such functions. In this case the domain of the function is a set of ordered pairs (x_1, x_2) and the co-domain is the real line. The function maps from a point in a two dimensional space to a point in one dimensional space.

2. When graphing a function of two independent variables, we plot (x_1, x_2) in a horizontal plane and let values of y be shown as verticals perpendicular to that plane. The graph of the function consists of a set of ordered triples $(x_1, x_2, f(x_1, x_2))$ and thus is a surface. In the diagram below, we show one point in the graph of such a function.



3. It is obvious that the concept of a multivariable function can be extended easily to the case of n independent variables. Such a function is a mapping from points in n-dimensional space to the real line. The domain consists of ordered n-tuples $(x_1,\ldots,x_n)$ and the function is written as

$$y = f(x_1,\ldots,x_n).$$

The graph of the function is a surface in (n+1)-dimensional space. For n greater than 2, it is of course impossible to graph the function.

Appendix to Lecture 1.1

Some Common Notation

1. Summation Notation: We have $\sum_{i=1}^{n} x_i \equiv x_1 + \cdots + x_n$.

2. Product Notation: We have $\prod_{i=1}^{n} x_i \equiv x_1 \cdots x_n$.

Notes on Logical Reasoning

1. Notation for logical reasoning:

a. $\forall$ means "for all"

b. ~ means "not"

c. $\exists$ means "there exists"

2. A Conditional

Let A and B be two statements.

$A \Rightarrow B$ means all of the following:

- If A, then B
- A implies B
- A is sufficient for B
- B is necessary for A

3. Proving a Conditional: Methods of Proof

a. Direct: Show that B follows from A.

b. Indirect: Find a statement C where $C \Rightarrow B$. Show that $A \Rightarrow C$.

c. Contrapositive: Show that $(\sim B) \Rightarrow (\sim A)$.

d. Contradiction: Show that (~ B and A) $\Rightarrow$ (false statement).

3. A Biconditional

Let A and B be two statements.

$A \Leftrightarrow B$ means all of the following:

- A if and only if B (A iff B)
- A is necessary and sufficient for B
- A and B are equivalent
- A implies B and B implies A

4. Proving a Biconditional

Use any of the above methods for proving a conditional and show that $A \Rightarrow B$ and that $B \Rightarrow A$.

Practice Problems I

1. Given the sets $S_1 = \{2,3,5\}$, $S_2 = \{3,5,6\}$ and $S_3 = \{7\}$, find:

a. $S_1 \cap S_3$ b. $S_1 \cup S_2$ c. $S_2 \cap S_2$ d. $S_2 \cap U$ e. $S_2 \cap \varnothing$

2. Given A = {4,5,6}, B = {3,4,6,7}, and C = {2,3,6}, verify the distributive law.

Answers:

1. a. $S_1 \cap S_3 = \varnothing$, b. {2,3,5,6}, c. {3,5,6}, d. S_2, e. $\varnothing$.

2. $A \cup (B \cap C) = \{4,5,6\} \cup \{3,6\} = \{3,4,5,6\}$
$(A \cup B) \cap (A \cup C) = \{3,4,5,6,7\} \cap \{2,3,4,5,6\} = \{3,4,5,6\}$.

Practice Problems II

1. Write the following in set notation and in interval notation:

a. The set of real numbers between 2 and 10, inclusive.

b. The set of real numbers less than 15.

2. Determine the following:

a. |5-6| b. 6/∞ c. 10×(-∞) d. 12 + (-∞) e. 12/-∞

Answers:

1a. [2,10] = {x | x is real, $2 \leq x \leq 10$}.

1b. $(-\infty, 15) = \{x | x\ is\ real, 15 > x\}$.

2. a. 1, b. 0, c. -∞, d. -∞, e. 0.

Practice Problems III

1. Given $S_1 = \{1,3,4\}$, $S_2 = \{a,b\}$ and $S_3 = \{m,n\}$, find the Cartesian products:

a. $S_1 \times S_2$ b. $S_2 \times S_3$ c. $S_1 \times S_3$ d. $S_1 \times S_2 \times S_3$.

2. Is it true that $S_1 \times S_2 = S_2 \times S_1$? Explain.

3. If the domain of the function y = 5 + 3x is the set $\{x | 1 \leq x \leq 4\}$, find the range of this function and express it as a set.

4. Does a circle drawn in a rectangular coordinate plane represent a function? Explain.

5. Is a hill shaped function one-to-one? Explain.

6. Does y = 2 + 4x have an inverse? If so what is it?

Answers

1. a. $S_1 \times S_2 = \{1,3,4\} \times \{a,b\} = \{(1,a),(1,b),(3,a),(3,b),(4,a),(4,b)\}$.
b. $S_2 \times S_3 = \{a,b\} \times \{m,n\} = \{(a,m),(a,n),(b,m),(b,n)\}$.
c. $S_1 \times S_3 = \{(1,m),(1,n),(3,m),(3,n),(4,m),(4,n)\}$.
d. $S_1 \times S_2 \times S_3 = \{1,3,4\} \times \{a,b\} \times \{m,n\} = \{(1,a,m), (1,a,n), (1,b,m), (1,b,n), (3,a,m), (3,a,n), (3,b,m), (3,b,n), (4,a,m), (4,a,n), (4,b,m), (4,b,n)\}$.

2. This is true if the two sets are equal. In discrete case, if S_i have unequal # elements but all the same element, this also is sufficient.

3. Range is [8,17].

4. No, an x can be assigned two y's.

5. No, there are y's which are common to two x's.

6. Yes, x = -0.5 + .25y.

Practice Problems IV

1. Graph the functions

a. y = 2 + 3x   b. y = 2 - 3x.

In each case, let $x \geq 0$.

2. Condense the following expressions:

a. $x^6 \times x^4$ b. x^3/x^{-2} c. $(x^{1/2} \times x^{1/3})/x^{2/3}$

3. Show that $x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$.

4. Simplify the following:

a. $\ln(x^6 y^\alpha z^5)$ b. $\log_{10} 1$ c. $\log_{10} 2 = \log_2$?   d. ln(x/yz)

5. Given $\log_e 2$, how do we convert this to $\log_{10}$? Explain.

Answers

1. This can be done on your own.

2. a. x^{10}, b. x^5, c. $x^{1/6}$

3. By definition true. $x^{m/n} = (x^m)^{1/n} = (x^{1/n})^m$.
4. a. $\ln(x^6 y^\alpha z^5) = 6\ln x + \alpha \ln y + 5\ln z$, b. 0, c. $\log_{10} 2 = 1/\log_2 10$ ($\log_b c = 1/(\log_c b)$.), d. $\ln(x/yz) = \ln x - (\ln y + \ln z)$.

5. $\log_{10} 2 = (\log_{10} e)(\log_e 2)$. Use $\log_b u = (\log_b c)(\log_c u)$ ($\log_c u$ is known).

Lecture 1.2: Introduction to Matrix Algebra

General

1. A matrix, for our purpose, is a *rectangular array* of *objects* or elements. We will take these elements as being real numbers and indicate an element by its row and column position. A matrix is then an ordered set.

2. Let $a_{ij} \in R$ denote the *element* of a matrix which occupies the position of the i^{th} row and j^{th} column. The dimension of the matrix is defined or stated by indicating first the number of rows and then the number of columns. We will adopt the convention of indicating a *matrix by a capital letter and its elements by the corresponding lower case letter.*

Example 1. $\underset{2\times 2}{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Example 2. $\underset{1\times 1}{A} = \begin{bmatrix} a_{11} \end{bmatrix}$

Example 3.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & . \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Example 4. $\underset{2\times 3}{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

3. A matrix is said to be (i) *square* if # rows = # columns and a square matrix is said to be (ii) *symmetric* if $a_{ij} = a_{ji}$ $\forall i, j, i \neq j$.

Example. The matrix $\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$ is square but not symmetric, since $a_{21} = 2 \neq 3 = a_{12}$. The square matrix $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 4 & 3 & 1 \end{bmatrix}$ is symmetric since $a_{12} = a_{21} = 2$, $a_{31} = a_{13} = 4$, and $a_{32} = a_{23} = 3$.

4. The *principle diagonal elements* of a square matrix A are given by the elements a_{ij}, i = j. The *principle diagonal* is the ordered n-tuple (a_{11}, ..., a_{nn}). The *trace* of a square matrix is defined as the sum of the principal diagonal elements. It is denoted tr(A) = $\sum_i a_{ii}$.

5. A *diagonal matrix* is a square matrix whose only nonzero elements appear on the principal diagonal.

6. A *scalar matrix* is a diagonal matrix with the same value in all of the diagonal elements.

7. An *identity matrix* is a scalar matrix with ones on the diagonal.

8. A *triangular matrix* is a square matrix that has only zeros either above or below the principal diagonal. If the zeros are above the diagonal, then the matrix is lower triangular and conversely for upper triangular.

Remark: The following notations for indicating an n × m matrix A are equivalent

$$[a_{ij}]_{\substack{i=1,\ldots,n \\ j=1,\ldots m}}, \begin{pmatrix} & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \end{pmatrix}, \begin{bmatrix} & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \end{bmatrix}, \text{or} \left\| \qquad \right\|.$$

9. If a matrix A is of dimension 1 × n, then it is termed a *row vector*, A = [a_{11},…,a_{1n}]. Since there is only one row, the row index is sometimes dropped and A is written [a_1,…,a_n] = a'. A matrix A of dimension n × 1 is termed a *column vector*, $A = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}$. Likewise, since there is only one column, this is sometimes written as $a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$.

Algebraic Operations on Matrices

1. *Equality*. Two matrices say A and B, $[a_{ij}]_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}$, $[b_{ij}]_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}$ are said to be equal iff $a_{ij} = b_{ij}$ $\forall$i, j.

2. *Addition and Subtraction*. Take A and B as above with the same dimensions we have

$$A_{nxm} \pm B_{nxm} = \begin{bmatrix} a_{11} \pm b_{11} & \cdots & a_{1m} \pm b_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} \pm b_{n1} & \cdots & a_{nn} \pm b_{nn} \end{bmatrix}$$

3. *Scalar Multiplication*. Let k ∈ R. $k \underset{n \times m}{A} = \left[ka_{ij}\right]$.

4. *Multiplication*. Two matrices A and B can be multiplied to form AB, only if the column dimension of A = row dimension of B. If this *conformability requirement* is met, then it is possible to define the product AB. In words, the column dimension of the lead matrix must equal the row dimension of the lag matrix, for conformability.

Example If $\underset{2\times3}{A}$ and $\underset{4\times2}{B}$, then AB cannot be defined, but $\underset{4\times2}{B}\underset{2\times4}{A}$ can be defined.

In order to precisely present the mechanics of matrix multiplication, let us introduce the idea of an inner (dot) product of two n-tuples of real numbers. Suppose x, y ∈ R^n. Then the *inner product* of x and y is defined by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

Note that $x \cdot x = \sum_i x_i^2$ and $x \cdot y = y \cdot x$. That is, the dot product is commutative. Given an n × m matrix A, let us associate the k^{th} *column of A* with the ordered n-tuple $a_{ok} = (a_{1k},\ldots,a_{nk})$. Moreover associate the j^{th} *row of A* with the ordered m-tuple $a_{jo} = (a_{j1},\ldots,a_{jm})$.

Example. $\underset{2\times3}{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$

$a_{02} = (2, 4)$

$a_{20} = (0, 4, 5)$

With this notation in hand, consider two matrices $\underset{n\times m}{A}\underset{m\times k}{B}$ which are conformable for multiplication in the order AB. The *product AB* is then given by

$$A_{nxm}B_{mxk} = \begin{bmatrix} a_{10}\cdot b_{01} & \cdots & a_{10}\cdot b_{0k} \\ \vdots & \ddots & \vdots \\ a_{n0}\cdot b_{01} & \cdots & a_{n0}\cdot b_{0k} \end{bmatrix}$$

That is if AB = C, then

$$c_{jl} = \sum_{i=1}^{m} a_{ji} b_{il}.$$

Note it must be that $\underset{n\times k}{AB}$.

Example 1

$$\underset{2\times 2}{A}\,\underset{2\times 2}{B} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{2} a_{1i}b_{i1} & \sum_{i=1}^{2} a_{1i}b_{i2} \\ \sum_{i=1}^{2} a_{2i}b_{i1} & \sum_{i=1}^{2} a_{2i}b_{i2} \end{bmatrix} = \begin{bmatrix} a_{10}\cdot b_{01} & a_{10}\cdot b_{02} \\ a_{20}\cdot b_{01} & a_{20}\cdot b_{02} \end{bmatrix}$$

Example 2

$$A = \underset{2\times 2}{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}} \quad B = \underset{2\times 1}{\begin{bmatrix} 1 \\ 4 \end{bmatrix}}$$

$$\underset{2\times 1}{AB} = \underset{2\times 2}{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}}\underset{2\times 1}{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 6 \\ 8 \end{bmatrix}}$$

Example 3. Suppose that A is a l × n row vector A = a′ = (a_{11} a_{12} … a_{1n}) and B an n × 1 col vector

$B = b = \begin{bmatrix} b_{11} \\ \vdots \\ b_{n1} \end{bmatrix}$. Hence, we have

$$a'b = \sum_{i=1}^{n} a_{1i}\, b_{i1}.$$

This is a scalar and the operation is termed a *scalar product*. Note that $a'b = a' \bullet b'$. (The scalar product is same as the inner product of 2 row vectors.)

Moreover suppose that $a' = (a_{11},\ldots,a_{1n})$ and $b = \begin{bmatrix} b_{11} \\ \vdots \\ b_{m1} \end{bmatrix}$. Then

$$ba' = \begin{bmatrix} b_{11}a_{11} & \cdots & b_{11}a_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1}a_{11} & \cdots & b_{m1}a_{1n} \end{bmatrix}.$$

5. The operation of addition is both commutative and associative. We have

(Com. Law) $A + B = B + A$

(Associative) $(A + B) + C = A + (B + C)$

The operation of multiplication is not commutative but it does satisfy the associative and distributive laws.

(Associative) $(AB)C = A(BC)$

(Distributive) $A(B + C) = AB + AC$

$(B + C)A = BA + CA$

To see that $AB \neq BA$ consider the example $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

6. Generally, when we take the product of a matrix and a vector, we can write the result as

$$c = Ab.$$

In this example, the matrix A is n by n and the column vectors c and b are n by 1. This product can be interpreted in two different ways. Taking the case of a 2×2 matrix A, we have

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

First, this can be a compact way of writing the two equations

$$1 = a + 3b$$

$$4 = 3a + 2b.$$

Alternatively, we can write the relationship as a linear combination of the columns of A

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

In the general case where A is n×n, we have

$$c = Ab$$

$$= b_1\mathbf{a_1} + \cdots + b_n\mathbf{a_n},$$

where $\mathbf{a_i}$ is the ith column of A. Further, in the product C = AB, each column of the matrix C is a linear combination of the columns of A where the coefficients are the elements in the corresponding columns of B. That is,

$$C = AB \text{ if and only if } \mathbf{c_i} = A\mathbf{b_i}.$$

7. Transpose of a Matrix. The *transpose* of a matrix A, denoted A′, is the matrix formed by interchanging the rows and columns of the original matrix A.

Example 1. Let A = (1 2) then $A' = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Example 2. Let $\underset{3\times 2}{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, then $\underset{2\times 3}{A'} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

Properties :

(i) (A′)′ = A (obvious)

(ii) (A + B)′ = A′ + B′

Proof: Let A + B = C, then $c_{ij} = a_{ij} + b_{ij}$. Let c'_{ij} denote an element of C′. Clearly, $c'_{ij} = c_{ji} = a_{ji} + b_{ji}$. Let a'_{ij}, b'_{ij} be elements of A′ and B′ respectively such that $a'_{ij} = a_{ji}$ and $b'_{ij} = b_{ji}$

$$c'_{ij} = a_{ji} + b_{ji} = a'_{ij} + b'_{ij}.$$

Thus, the elements of C′ and A′ + B′ are identical.

(iii) (AB)′ = B′A′

Proof: Let $\underset{n\times m}{A}, \underset{m\times k}{B}$ then $A_{nxm}B_{mxk} = \begin{bmatrix} a_{10}\cdot b_{01} & \cdots & a_{10}\cdot b_{0k} \\ \vdots & \ddots & \vdots \\ a_{n0}\cdot b_{01} & \cdots & a_{n0}\cdot b_{0k} \end{bmatrix}$. Then

(AB)’ = $\begin{bmatrix} a_{10}\cdot b_{01} & \cdots & a_{n0}\cdot b_{01} \\ \vdots & \ddots & \vdots \\ a_{10}\cdot b_{0k} & \cdots & a_{n0}\cdot b_{0k} \end{bmatrix}$. We have that $B'_{kxm} = \begin{bmatrix} b_{01}' \\ b_{0k}' \end{bmatrix}$ and $A'_{mxn} = [a_{10}' \quad \ldots \quad a_{n0}']$. Thus,

$B'A' = \begin{bmatrix} b_{01}'\cdot a_{10}' & \cdots & b_{01}'\cdot a_{n0}' \\ \vdots & \ddots & \vdots \\ b_{0k}'\cdot a_{10}' & \cdots & b_{0k}'\cdot a_{n0}' \end{bmatrix}$. ||

7. The Identity and Null Matrices.

a. An *identity matrix* is a square matrix with ones in its principle diagonal and zeros elsewhere. An n × n identity matrix is denoted I_n.

Properties:

(i) Let A be n × p. Then we have $I_nA = AI_P = A$.

Proof: Exercise

(ii) Let A be n × p and B be p × m. Then we have

$$\underset{n\times p}{A}\ \underset{p\times p}{Ip}\ \underset{p\times m}{B} = (AI_p)B = AB.$$

(iii) $I_n\cdot I_n\cdot I_n\cdots I_n = I_n$. The product of any number of identity matrices is the identity matrix. In general, a matrix is termed idempotent, when it satisfies the property AA = A.

b. The *null matrix,* denoted [0] is a matrix whose elements are all zero. Subject to dimensional conformability we have

Properties:

(i) A + [0] = [0] + A = A

(ii) [0]A = A[0] = [0].

Proofs: Exercise

Remark : If AB = [0], it need not be true that A = [0] or B [0].

Example.

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$$

It is easy to show that AB = [0].

8. Determinants and Related Concepts.

a. A determinant is defined only for square matrices. When taking the determinant of a matrix we attach a sign + or - to each element:

sign attached to a_{ij} = sign $(-1)^{i+j}$.

Thus, for example, sign a_{12} = - , sign of a_{43} = -, and sign a_{13} = +.

b. The determinant of a scalar x, $|x|$,is the matrix itself. The determinant of a 2 × 2 matrix A, denoted $|A|$ or det A, is defined as follows:

$$|A| = \left|\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right| = +1(a_{11})(a_{22}) - (1)(a_{21}) \cdot a_{12}.$$

Example $A = \begin{bmatrix} 3 & 6 \\ -5 & 6 \end{bmatrix}$. $|A| = 3 \cdot 6 - (-5)(6) = 18 + 30 = 48.$

c. The determinant of an arbitrary $n \times n$ $(n \geq 2)$ matrix A can be found via the *Laplace Expansion* process. In order to introduce this process, let us consider some preliminary definitions. Let $\underset{n \times n}{A}$.

Definition. The *minor* of the element a_{ij}, denoted $|M_{ij}|$, is the determinant of the submatrix formed by deleting the i^{th} row and j^{th} column.

Example 1. Let A be 2 × 2, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. $|M_{11}| = |a_{22}| = a_{22}$. Moreover $|M_{12}| = a_{21}, |M_{21}| = a_{12}$ and $|M_{22}| = a_{11}$.

Example 2. Let A be 3 × 3.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \quad |M_{13}| = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}.$$

Definition. The *cofactor* of the element a_{ij} denoted $|C_{ij}|$ is given by

$$(-1)^{i+j}|M_{ij}|.$$

Example Let A be 3 × 3. Then $|C_{21}| = -1\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = -1(a_{12}a_{33} - a_{32}a_{13})$.

Definition. The *principle minor* of the principle diagonal element a_{ii}, denoted $|PM_i|$ is the determinant of the submatrix formed by retaining only the first i rows and first i columns. The order of $|PM_i|$ is its row = col. dimension.

Example

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|PM_1| = a_{11} \qquad |PM_2| = a_{11}a_{22} - a_{21}a_{12} \qquad |PM_3| = |A|.$$

9. *Laplace Expansion*: Let A be n × n. Then

$|A| = \sum_{i=1}^{n} a_{ij}|C_{ij}|$ (expansion by j^{th} col) $\qquad |A| = \sum_{j=1}^{n} a_{ij}|C_{ij}|$ (expansion by i^{th} row)

Note that eventually cofactors degenerate to the 2 × 2 case.

Example 1 3 × 3. Expansion by 2nd col.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$$|A| = \sum_{i=1}^{3} a_{i2}C_{i2} = -a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{22}(a_{11}a_{33} - a_{31}a_{13}) - a_{32}(a_{11}a_{23} - a_{21}a_{13}).$$

Next consider expansion via the 3<u>rd</u> row.

$$|A| = \sum_{j=1}^{3} a_{3j}c_{3j} = a_{31}(a_{12}a_{23} - a_{22}a_{13}) - a_{32}(a_{11}a_{23} - a_{21}a_{13}) + a_{33}(a_{11}a_{22} - a_{21}a_{12}).$$

Let's check the two terms to see if they are equal. The middle term of the second expression is the same as the last term of the first expression. Checking the remaining two terms, we have the following. In the first case $-a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{22}a_{11}a_{33} - a_{22}a_{31}a_{13}$. In the second case $a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} + a_{33}a_{11}a_{22} - a_{33}a_{21}a_{12}$. Thus, they are the same.

Example 2. A is 3 × 3 and given by

$\begin{bmatrix} 0 & 4 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. In this case it is easiest to expand via the first col.

$$|A| = -2\begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} = -2(4-1) + 1(4-1) = -2(3) + 3 = \underline{\underline{-3}}.$$

10. *Properties of Determinants*.

(i) $|A| = |A'|$

(ii) The interchange of any two rows (or two col.) will change the sign of the determinant, but will not change its absolute value.

<u>Examples of Properties (i) and (i)</u>

#1 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$|A| = -2$ $|A'| = -2$

#2 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

$|A| = -2$ $|B| = +2$

(iii) The multiplication of any p rows (or col) of a matrix by a scalar k will change the value of the determinant to $k^p|A|$.

(iv) The addition (subtraction) of any multiple of any row to (from) another row will leave the value of the determinant unaltered, if the linear combination is placed in the initial (the transformed) row slot. The same holds true if we replace the word "row" by column.

<u>Examples of Properties (iii) and (iv)</u>

(iii) Take A, 2 x 2, and multiply by 2.

$|2A| = 2a_{11}\,2a_{22} - 2a_{21}\,2a_{12} = 4|A|$

(iv) Take A, 2 x 2, and add 2 times the second row to the first row

$$\tilde{A} = \begin{bmatrix} a_{11} + 2a_{21} & a_{12} + 2a_{22} \\ a_{21} & a_{22} \end{bmatrix}, |\tilde{A}| = a_{11}a_{22} - a_{12}a_{21}$$

(v) If one row (col) is a multiple of another row (col), the value of the determinant will be zero.

(vi) If A and B are square, then |AB| = |A||B|.

<u>Examples of Properties (v) and (vi)</u>

(v) Let

$$A = \begin{bmatrix} 3a & 3b \\ a & b \end{bmatrix}, |A| = 3ab - 3ab = 0$$

(vi) Let

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 1 \end{bmatrix}, |A| = -3, B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, |B| = -2, |A||B| = 6.$$

$$AB = \begin{bmatrix} 12 & 18 \\ 5 & 8 \end{bmatrix}, |AB| = 96 - 90 = 6.$$

Inverse Matrix

1. *Def*. Given an n × n square matrix A, the *inverse matrix* of A, denoted A^{-1}, is that matrix which satisfies

$$A^{-1}A = AA^{-1} = I_n.$$

When such a matrix exists, A is said to be *nonsingular*. If A^{-1} exists it is unique.

2. Computing the Inverse.

a. Let us begin by assuming that the matrix we wish to invert is an n × n matrix A with $|A| \neq 0$.

b. *Def*. The *cofactor matrix* of A is given by

$$C = \left[|C_{ij}| \right].$$

Def. The *adjoint matrix* of A is given by adj A = C′.

c. *Computation of Inverse*: $A^{-1} = \dfrac{\text{adj } A}{|A|}$.

Example: Let $A = \begin{bmatrix} 1 & 3 \\ 9 & 2 \end{bmatrix}$ $\quad |A| = 2 - 27 = -25,$ $\quad C = \left[|C_{ij}| \right] = \begin{bmatrix} 2 & -9 \\ -3 & 1 \end{bmatrix},$

$$\text{adj } A = \begin{bmatrix} 2 & -3 \\ -9 & 1 \end{bmatrix}.$$

$$A^{-1} = \frac{1}{-25}\begin{bmatrix} 2 & -3 \\ -9 & 1 \end{bmatrix} = \begin{bmatrix} -2/25 & 3/25 \\ 9/25 & -1/25 \end{bmatrix}.$$

$$AA^{-1} = \begin{bmatrix} 1 & 3 \\ 9 & 2 \end{bmatrix}\begin{bmatrix} -2/25 & 3/25 \\ 9/25 & -1/25 \end{bmatrix} = \begin{bmatrix} -\frac{2}{25} + \frac{27}{25} & \frac{3}{25} - \frac{3}{25} \\ -\frac{18}{25} + \frac{18}{25} & \frac{27}{25} - \frac{2}{25} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

3. Key Properties of the Inverse Operation

a. $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: $B^{-1}A^{-1}AB = B^{-1}IB = I$ and $ABB^{-1}A^{-1} = I$. It follows that $B^{-1}A^{-1}$ is the inverse of AB. ||

b. $(A^{-1})^{-1} = A$.

Proof: $AA^{-1} = I$ and $A^{-1}A = I$. Thus, the result holds. ||

c. $(A')^{-1} = (A^{-1})'$.

Proof: $AA^{-1} = A^{-1}A = I$. Transposing and noting that $I' = I$, we have $(A^{-1})'A' = I = A'(A^{-1})'$. ||

d. $I^{-1} = I$.

Proof: $II = I$.

4. a. Note that AB = 0 does not imply that A = 0 or that B = 0. If either A or B is nonsingular and AB = 0, then the other matrix is the null matrix. That is, the product of two non-singular matrices cannot be null.

Proof: Let $|A| \neq 0$ and $AB = 0$. Then $A^{-1}AB = B = 0$. ||

b. For sqaure matrices, it can be shown that $|AB| = |A||B|$, so that, in this case, $|AB| = 0$ if and only if $|A| = 0$, $|B| = 0$, or both.

Linear Equation Systems, the Inverse Matrix and Cramer's Rule.

1. Consider an equation system with n unknowns x_i, i = 1, …, n. In matrix notation this system can be written as

$$Ax = d,$$

where $A=[a_{ij}]_{\substack{i=1,\ldots,n\\ j=1,\ldots,n}}$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $d = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$. If $|A| \neq 0$, A^{-1} exists and we can write

$$A^{-1}Ax = A^{-1}d$$

$$I_n x = A^{-1}d$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A^{-1}d.$$

Thus, if there is no linear dependence in the rows or columns of the coefficient matrix we can obtain a solution to the equation system. Since A^{-1} is unique if it exists, this solution is unique. Hence, an easy way to test for existence and uniqueness of a solution to a set of linear equations is to determine whether the coefficient matrix has a nonvanishing determinant.

2. This solution gives us values of the solution variables, in terms of A^{-1}, in vector form. A formula known as *Cramer's Rule* gives explicit solutions for each x_i. If $|A| \neq 0$, we have

$$x = A^{-1}d = \frac{1}{|A|}[\text{adj}A]d = \frac{1}{|A|}\begin{bmatrix} |C_{11}| & \cdots & |C_{n1}| \\ \vdots & \ddots & \vdots \\ |C_{1n}| & \cdots & |C_{nn}| \end{bmatrix}\begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix},$$

$$x = \frac{1}{|A|}\begin{bmatrix} d_1|C_{11}| & \cdots & d_n|C_{n1}| \\ \vdots & \ddots & \vdots \\ d_1|C_{1n}| & \cdots & d_n|C_{nn}| \end{bmatrix},$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} \sum_{i=1}^{n} |C_{i1}| d_i \\ \vdots \\ \sum_{i=1}^{n} |C_{in}| d_i \end{bmatrix}$$

Thus,

$$x_j = \frac{1}{|A|}\sum_{i=1}^{n} |C_{ij}| di \, .$$

Consider the term $\sum_{i=1}^{n} |C_{ij}| di$. Recall $|A| = \sum_{i=1}^{n} a_{ij}|C_{ij}|$ (expansion by jth col.). Thus

$$\sum_{i=1}^{n} d_i|C_{ij}| = \left| \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} \cdots \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \ldots \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix} \right| \equiv |A_j|.$$

Using this notation, we obtain for j = 1, …, n, $x_j = \frac{|A_j|}{|A|}$. (Cramer's Rule)

Remark. This method does not involve computation of A^{-1}.

Example

$$3x_1 + 4x_2 = 10$$
$$6x_1 + 1x_2 = 20$$
$$\begin{bmatrix} 3 & 4 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$
$$x_1 = \frac{|A_1|}{|A|} = \frac{10-80}{3-24} = \frac{-70}{-21} = \frac{70}{21}$$
$$x_2 = \frac{|A_2|}{|A|} = \frac{60-60}{-21} = 0.$$

Let's check this by computing A^{-1}

$$C = \begin{bmatrix} 1 & -6 \\ -4 & 3 \end{bmatrix} \Rightarrow C' = adjA = \begin{bmatrix} 1 & -4 \\ -6 & 3 \end{bmatrix}$$
$$A^{-1} = -\frac{1}{21}\begin{bmatrix} 1 & -4 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} -1/21 & +4/21 \\ 6/21 & -3/21 \end{bmatrix}$$

Check

$$A^{-1}A = \begin{bmatrix} \frac{-1}{21} & \frac{4}{21} \\ \frac{6}{21} & -\frac{3}{21} \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{21}+\frac{24}{21} & -\frac{4}{21}+\frac{4}{21} \\ \frac{18}{21}-\frac{18}{21} & \frac{24}{21}-\frac{3}{21} \end{bmatrix} = I$$
$$x = \begin{bmatrix} -\frac{1}{21} & \frac{4}{21} \\ \frac{6}{21} & -\frac{3}{21} \end{bmatrix}\begin{bmatrix} 10 \\ 20 \end{bmatrix}$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{10}{21} & +\frac{80}{21} \\ \frac{60}{21} & -\frac{60}{21} \end{bmatrix} = \begin{bmatrix} \frac{70}{21} \\ 0 \end{bmatrix}.$$