Lecture 2: Marginal Functions, Average Functions, Elasticity, the Marginal Principle, and Constrained Optimization

- The marginal or derivative function and optimization-basic principles
- The average function
- Elasticity
- Basic principles of constrained optimization

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Introduction

- Suppose that an economic relationship can be described by a real-valued function
  $\pi = \pi(x_1,x_2,...,x_n)$.
- π might be thought of as the profit of the firm and the x_i as the firm's n discretionary strategy variables determining profit.
- Suppose that, other variables constant, the firm is proposing a change in x_i

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Introduction

- Redefine x_i as x and write

  $\pi \equiv \pi(x)$,

where x now denotes the single discretionary variable x_i.

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The marginal or derivative function

- Relative to some given level of x, we might be interested in the effect on π of changing x by some amount Δx (Δx denotes a change in x).
- If x takes on the two values x' and x'', then $\Delta x = (x'' - x')$. We could form the difference quotient

$$(1) \qquad \frac{\Delta\pi}{\Delta x} = \frac{\pi(x'+\Delta x) - \pi(x')}{\Delta x}.$$

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Marginal function

- In the following figure, we illustrate $\Delta\pi/\Delta x$ by the slope of the line segment AB.

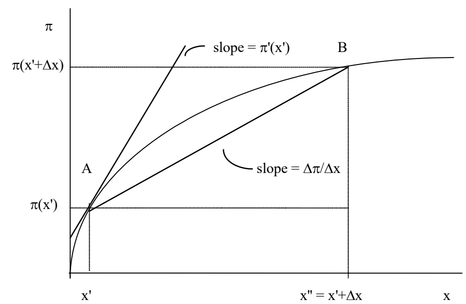

Figure 1

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Marginal function

- If we take the limit of $\Delta\pi/\Delta x$ as $\Delta x \to 0$, that is,

$$\lim_{\Delta x \to 0} \Delta\pi/\Delta x = \pi'(x'),$$

then we obtain the *marginal or derivative function of π.*

- Geometrically, the value of the derivative function is given by the slope of the tangent to the graph of π at the point A (i.e., the point $(x', \pi(x'))$).

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Illustrations: Total and Marginal

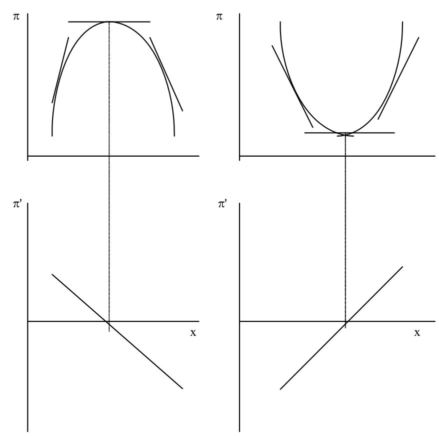

Figure 2 Figure 3

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Discussion of Figures 2,3

- In Figures 2,3 we show two total functions and their respective marginal functions.
- Figure 2 depicts a total function having a maximum and Figure 3 depicts a total function having a minimum.
- Note that at a maximum or a minimum point, the total function flattens out, or its marginal function goes to zero. In economics, we refer to this as the marginal principle.

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Marginal Principle

- The marginal principal states that the value of the marginal function is zero at any extremum (maximum or minimum) of the total function.
- This principal can be extended to state that if at a point x we have that $\pi'(x) > 0$, then in a neighborhood of x, we should raise x if we are interested in maximizing π and lower x if we are interested in minimizing π.

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Marginal Principal

- This principle assumes that the total function is hill shaped in the case of a maximum and valley shaped in the case of a minimum.
- There are second order conditions which suffice to validate a zero marginal point as a maximum or a minimum.

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Second order conditions

- For a maximum, it would be true that in a neighborhood of the extremum, we have that the marginal function is decreasing or downward sloping.
- For a minimum, the opposite would be true.

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Second order conditions

- For a maximum, it would be true that in a neighborhood of the extremum, we have that the marginal function is decreasing or downward sloping.
- For a minimum, the opposite would be true.

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Example

- Let $\pi(x) = R(x) - C(x)$, where R is a revenue function and C is a cost function. The variable x might be thought of as the level of the firm's output. Suppose that a maximum of the firm's profit occurs at the output level x^o. Then we have that $\pi'(x^o) = R'(x^o) - C'(x^o) = 0$, or that
- $R'(x^o) = C'(x^o)$.

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Example

- At a profit maximum, marginal revenue is equal to marginal cost.
- Using the marginal principle, the firm should raise output when marginal revenue is greater than marginal cost, and it should lower output when marginal revenue is less than marginal cost.

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Many choice variables

- If the firm's objective function has n strategy variables $x = (x_1,...,x_n)$, then the marginal function of the i^{th} strategy variable is denoted as π_i
- We define π_i in the same way that π' was defined above with the stipulation that all other choice variables are held constant when we consider the marginal function of the i^{th}.

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Many choice variables

- For example if we were interested in π_1

$$(2) \qquad \Delta\pi/\Delta x_1 = \frac{\pi(x_1' + \Delta x_1, x_2', \ldots, x_n') - \pi(x_1', \ldots, x_n')}{\Delta x_1}.$$

- Taking the limit of this quotient as Δx_1 tends to zero we obtain the marginal function π_1. The other π_i are defined in an analogous fashion.

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Marginal principle: many choice variables

- At a maximum or at a minimum of the total function, all of the values of the marginal functions go to zero.
- If $x^o = (x_1^o,..., x_n^o)$ is the extremum, then we would have that $\pi_i(x_1^o,..., x_n^o) = 0$. for all i.

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Marginal principle: many choice variables

- If we were searching for a maximum, then we would raise any strategy variable whose marginal function has a positive value at a point, and we would lower a strategy variable whose marginal function has a negative value at a point.
- The reverse recommendations would be made if we were interested in finding a minimum.

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The average function

- Given the total function $\pi(x)$, the corresponding *average function* is defined by

$$(3) \qquad \frac{\pi(x)}{x}, \text{ for } x \neq 0.$$

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Illustration of average function

- Geometrically, at any x^o, the average function at x^o is given by the slope of the line segment joining zero and the point $(x^o, \pi(x^o))$.

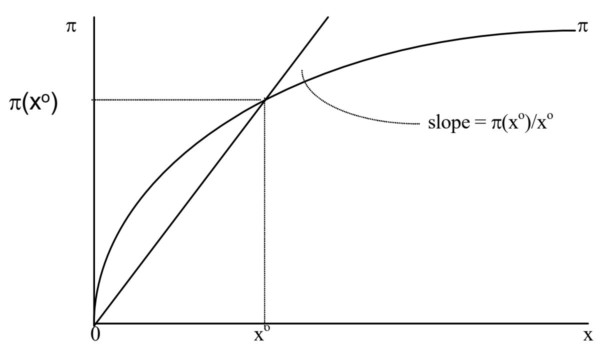

Figure 4

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Example

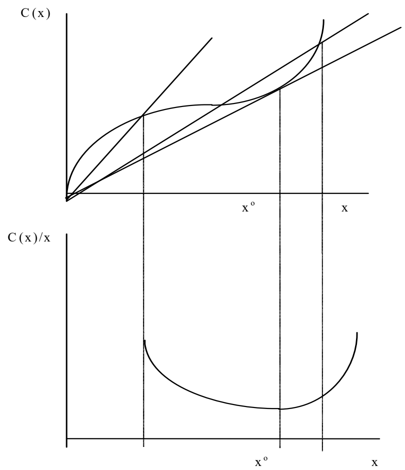

Figure 5

Discussion of example

- Let C = C(x) denote a firm's cost function and let x be the firm's level of output.
- In the lower diagram, we show C(x)/x, termed average cost.
- Average cost has two regions. In the initial region, average cost is declining as output is increased, and, in the second region, the opposite is true.

Elasticity

- The notion of elasticity is used in many business applications where the objective is to gauge the responsiveness of one variable to a change in another variable.
- A firm to wants to know how a rival's price change might impact the quantity demanded of their product.
- Alternatively, the same firm might want to quantify the impact on quantity demanded of their product of a change in the price of their product or a change in advertising outlay for that product.

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Elasticity

- Elasticity measures the impact of such changes by taking the percentage change in a dependent variable induced by some percentage change in an independent variable.

$$(4) \qquad \frac{\Delta\pi / \pi}{\Delta x / x} = \frac{\Delta\pi / \Delta x}{\pi / x} = (\text{marginal function}) / (\text{average function}).$$

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Measurement

- If one knows π through empirical estimation, then the marginal function can be used for the difference quotient $\Delta\pi/\Delta x$.
- In this case, the elasticity is called a *point elasticity*.
- In some cases, the function π is not known and only observations of x and π are available.

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Measurement

- If we have at least two observations of (x,π), then we can compute a different notion of elasticity called the *arc elasticity*.

$$(5) \qquad \frac{(\pi'' - \pi')}{(x'' - x')} \frac{x^a}{\pi^a}.$$

average observations are denoted x^a and π^a

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Example #1

- Let the function Q = 12 - 4p = Q(p) describe a demand relationship, where p is price and Q is quantity. Given that this function is linear, we have that Q'(p) = $\Delta Q/\Delta p$ = - 4. The point elasticity at the price level p = \$1 is given by

$$\frac{Q'(p)p}{Q} = \frac{-4(1)}{12-4(1)} = \frac{-4}{8} = \frac{-1}{2}.$$

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Example #2

- We observe that when p = \$2, Q = 10. Further when p = \$4, we have that Q = 6. Compute the arc elasticity of demand for these observations.

$$\frac{\Delta Q / \Delta p}{Q^a / p^a} = \frac{[(6-10)/(4-2)]}{8/3} = \frac{-2(3)}{8} = \frac{-3}{4}.$$

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Constrained Optimization: Basic Principles

- In many business problems, we are confronted with a feasibility constraint which limits our ability to choose values of our strategy variables.
- As an example, consider the problem of maximizing output flow given a limited budget to purchase the inputs used to produce output.
- Alternatively, a manager may be asked to achieve an output target with a cost minimal choice of inputs.

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Constrained Optimization

- A firm has just two strategy variables, x_1, x_2.
- We assume that the two strategy variables generate an output variable q which the firm sells. The relationship between the two variables and output is given by the multi-variable function

(6) $q = f(x_1,x_2)$.

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Constrained Optimization

- *The set of pairs (x_1,x_2) capable of generating a given level of q is called an* *iso-quant.* An iso-quant is then given by
- {(x_1,x_2) | q = f(x_1,x_2) and q fixed}.
- This is also called a *level surface* of the function f(x_1,x_2).
- In the economics of production, the x_i's might represent material and labor inputs and q would be output flow.

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Constrained Optimization

Figure 6 gives an example of such an iso-quant. The iso-quant map, i.e., the set of all iso-quants, can be used to describe the entire 3-dimensional function.

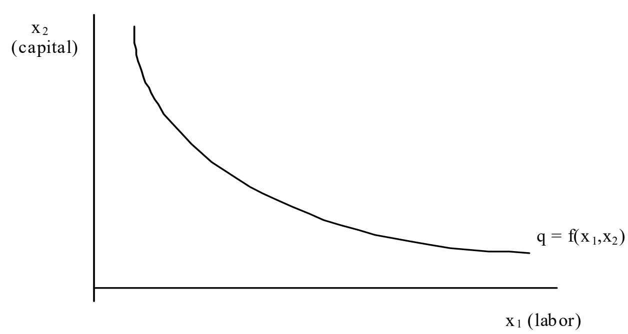

Figure 6

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Constrained Optimization

- Along the iso-quant, the firm is able to substitute one input for another and still achieve the same output target.
- The rate at which one input can be substituted for another along an iso-quant is called *the marginal rate of substitution between x_1 and x_2*. (This notion can be intuitively thought of as the number of units of x_2 that the firm can eliminate from the production process if it adds one more unit of x_1, holding q constant.)

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MRS

- Formally, we define the marginal rate of substitution, MRS, as

$$MRS \equiv \lim_{\Delta x_1 \to 0} -(\Delta x_2/\Delta x_1)|_{q=constant}.$$

- The MRS represents the absolute value of the slope of the iso-quant at a point.
- This is illustrated in Figure 7 below.

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MRS Illustration

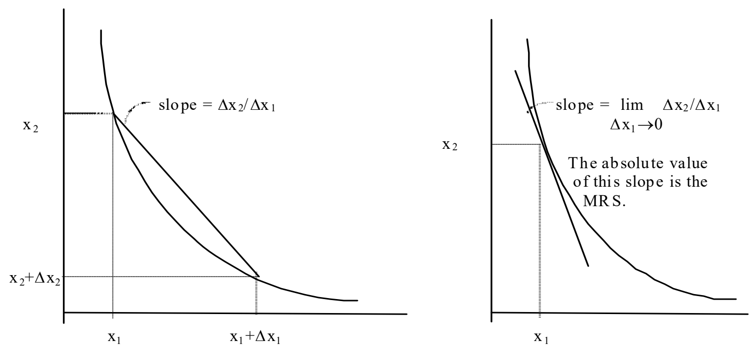

Figure 7

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MRS Computation

- Note that along an isoquant, $q = f(x_1,x_2)$ and q is constant. Thus, along an isoquant,

$$\Delta q = 0 = \frac{\Delta q}{\Delta x_1}\Big|_{x_2 = \text{constant}} \Delta x_1 + \frac{\Delta q}{\Delta x_2}\Big|_{x_1 = \text{constant}} \Delta x_2,$$

- and

$-(\Delta x_2/\Delta x_1)|_{q=\text{constant}} = (\Delta q/\Delta x_1)/(\Delta q/\Delta x_2) = MP_1/MP_2$.

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Constrained Optimization

- Suppose that a manager is asked to provide for the firm a given output target at a minimum expenditure level.
- The total expenditure on all inputs is given by the simple linear function

$$C = p_1x_1 + p_2x_2,$$

- The firm's output target is given by q^t . The constraint is then that

$$q^t = f(x_1,x_2).$$

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Constrained Optimization

- We would write this problem as

  $\min_{\{x_1,x_2\}} (p_1x_1 + p_2x_2)$ subject to $q^t = f(x_1,x_2)$.

- The firm's expenditure function can be rewritten as the linear function

$$x_2 = \frac{C}{p_2} - \frac{p_1}{p_2}x_1 .$$

- p_1/p_2 Measures amt of x_2 the firm *must give up* for another unit of x_1 purchased

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Illustration of Expenditure function

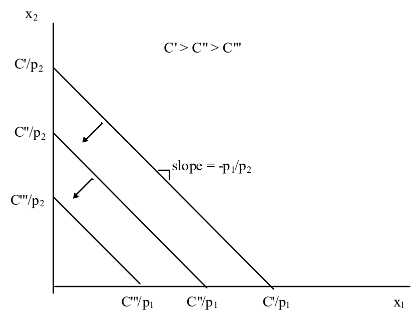

Figure 8

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Solution

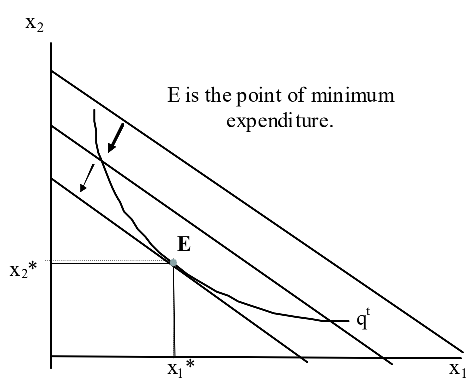

Figure 9

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Solution

- At a minimum of expenditure where both inputs would be utilized, we have that the slope of the iso-quant is equal to the slope of the expenditure line.
- At a minimum, the rate at which x_1 can be substituted for x_2 in production (the MRS) is equated to the rate at which x_1 must be substituted for x_2 in the market place, (p_1/p_2).

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Solution

- Thus, in the expenditure minimizing equilibrium, we have that the following condition is met

$$\text{MRS} = \frac{p_1}{p_2} \text{ or that } \frac{MP_1}{MP_2} = \frac{p_1}{p_2} \text{ and } \frac{MP_1}{p_1} = \frac{MP_2}{p_2}.$$

- In a later lecture, we will discuss computational methods for constrained and unconstrained optimization

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