Lecture 2. Marginal Functions, Average Functions, Elasticity, the Marginal Principle, and Constrained Optimization

2.1. Introduction

Suppose that an economic relationship can be described by a real-valued function

$\pi = \pi(x_1,x_2,...,x_n)$.

For the sake of example, π might be thought of as the profit of the firm and the x_i as the firm's n discretionary strategy variables determining profit. Such variables might include the levels of output, the prices of the firm's products, or variables representing the levels of the firm's advertising efforts. In order to keep the analysis simple, suppose for now that n - 1 of these variables have been set at particular levels and that the firm is considering a change in one of the strategy variables. Take x_2 through x_n as fixed, and let x_1 be the only strategy variable subject to change. To keep notation simple redefine π as

$\pi \equiv \pi(x)$,

where x now denotes the single discretionary variable x_1.

2.2. The Marginal Function

Relative to some given level of x, we might be interested in the effect on π of changing x by some amount Δx (Δx denotes a change in x). For example, if x takes on the two values x' and x", then $\Delta x = (x'' - x')$. We could form the difference quotient

(1) $$\frac{\Delta\pi}{\Delta x} = \frac{\pi(x'+\Delta x) - \pi(x')}{\Delta x}.$$

The ratio in (1) measures the rate of change of π with respect to the change in the independent variable x. In the following figure, we illustrate $\Delta\pi/\Delta x$ by the slope of the line segment $\overline{AB}$.

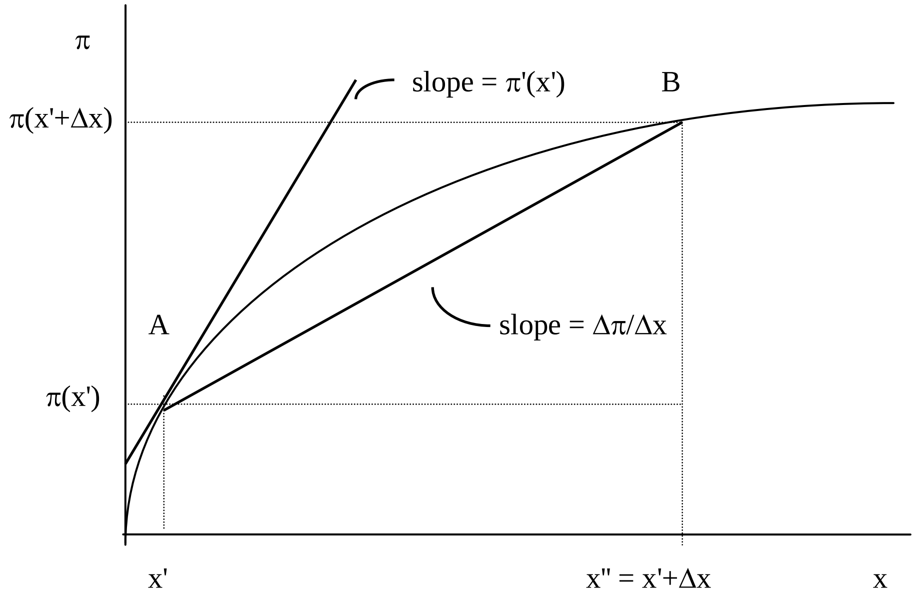


Figure 1

If we take the limit of $\Delta\pi/\Delta x$ as $\Delta x \rightarrow 0$, that is,

$$\lim_{\Delta x \rightarrow 0} \Delta\pi/\Delta x = \pi'(x'),$$

then we obtain the *marginal or derivative function of π.* Geometrically, the value of the derivative or marginal function is given by the slope of the tangent to the graph of π at the point A (i.e., the point (x', π(x'))).

In many business problems it is important to be able to visualize the marginal function of a given total function, by looking at that total function. In Figures 2 and 3 below we show two total functions and their respective marginal functions. Figure 2 depicts a total function having a maximum and Figure 3 depicts a total function having a minimum. Note that at maximum or a minimum point, the total function flattens out, or its marginal function goes to zero. In economics, we refer to this as the marginal principle. It simply states that the value of the marginal function is zero at any extremum (maximum or minimum) of the total function. This principle can be extended to state that if at a point x we have that $\pi'(x) > 0$, then in a neighborhood of x, we should raise x if we are interested in maximizing π and lower x if we are interested in minimizing π. This principle, of course assumes that the total function is hill shaped in the case of a maximum and valley shaped in the case of a minimum. There are second order conditions which suffice to validate a zero marginal point as a maximum or a minimum. For a maximum, it would be true that in a neighborhood of the extremum, we have that the marginal function is decreasing or downward sloping. For a minimum, the opposite would be true.

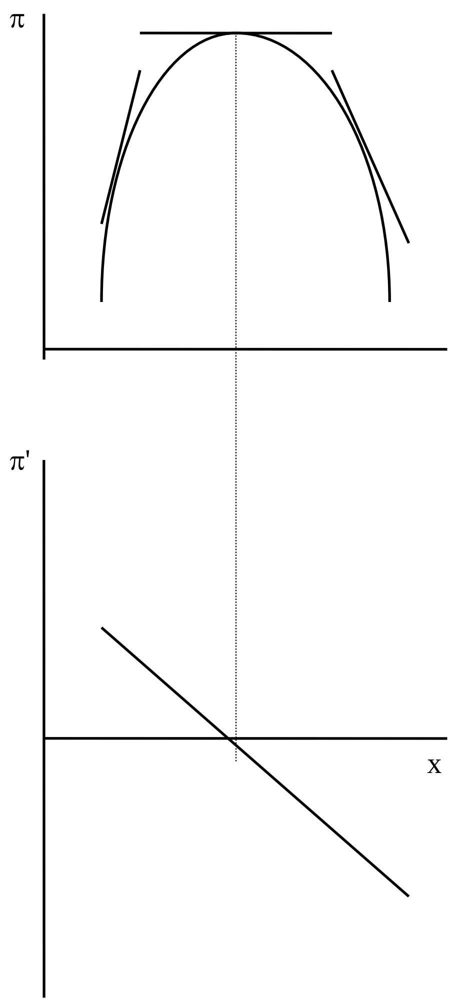


Figure 2

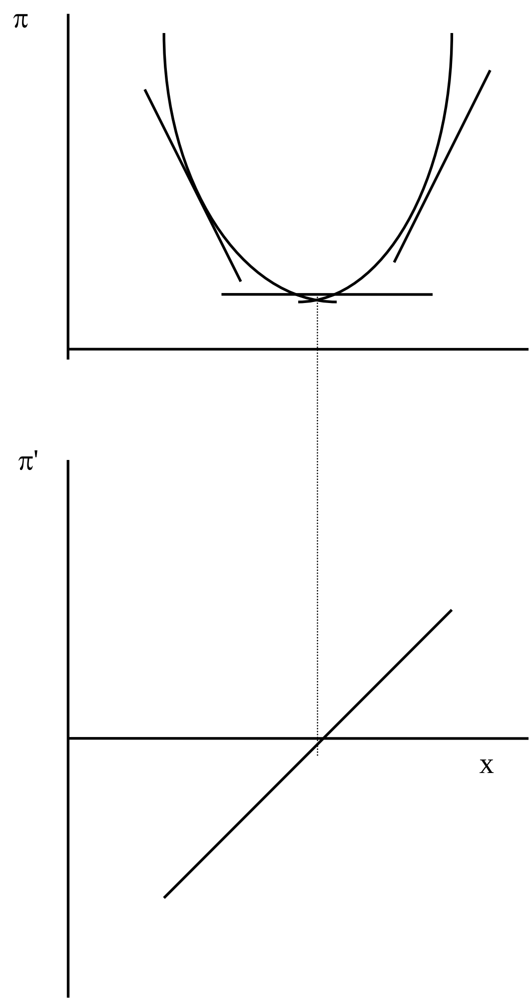


Figure 3

Example. Let $\pi(x) = R(x) - C(x)$, where R is a revenue function and C is a cost function. The variable x might be thought of as the level of the firm's output. Suppose that a maximum of the firm's profit occurs at the output level x^o. Then we have that $\pi'(x^o) = R'(x^o) - C'(x^o) = 0$, or that

$$R'(x^o) = C'(x^o).$$

At a profit maximum, marginal revenue is equal to marginal cost. Using the marginal principle, the firm should raise output when marginal revenue is greater than marginal cost, and it should lower output when marginal revenue is less than marginal cost. This makes sense, because, on the margin, profit goes up with x, if the extra revenue is greater than the extra cost and vice versa.

The above marginal analysis can be extended to the case of finitely many choice variables. If the firm's objective function has n strategy variables $x = (x_1,...,x_n)$, then the marginal function of the i^{th} strategy variable is denoted as π_i. We define π_i in the same way that π' was defined above with the stipulation that all other choice variables are held constant when we consider the marginal function of the i^{th}. That is, for example, if we were interested in the marginal function of the first choice variable, we would define this as

(2) $$\Delta\pi/\Delta x_1 = \frac{\pi(x_1' + \Delta x_1, x_2', \ldots, x_n') - \pi(x_1', \ldots, x_n')}{\Delta x_1}.$$

Taking the limit of this quotient as Δx_1 tends to zero we obtain the marginal function π_1. The other π_i are defined in an analogous fashion.

At a maximum or at a minimum of the total function, all of the values of the marginal functions go to zero. If $x^o = (x_1^o,..., x_n^o)$ is the extremum, then we would have that $\pi_i(x_1^o,..., x_n^o) = 0$. for all i. If we were searching for a maximum, then we would raise any strategy variable whose marginal function has a positive value at a point, and we would lower a strategy variable whose marginal function has a negative value at a point. The reverse recommendations would be made if we were interested in finding a minimum.

2.3. The Average Function

Given the total function $\pi(x)$, the corresponding *average function* is defined by

(3) $$\frac{\pi(x)}{x}, \text{ for } x \neq 0.$$

Geometrically, at any x^o, the average function at x^o is given by the slope of the line segment joining zero and the point $(x^o, \pi(x^o))$. In Figure 4, we show a function π and the value of the average function at a point x^o.

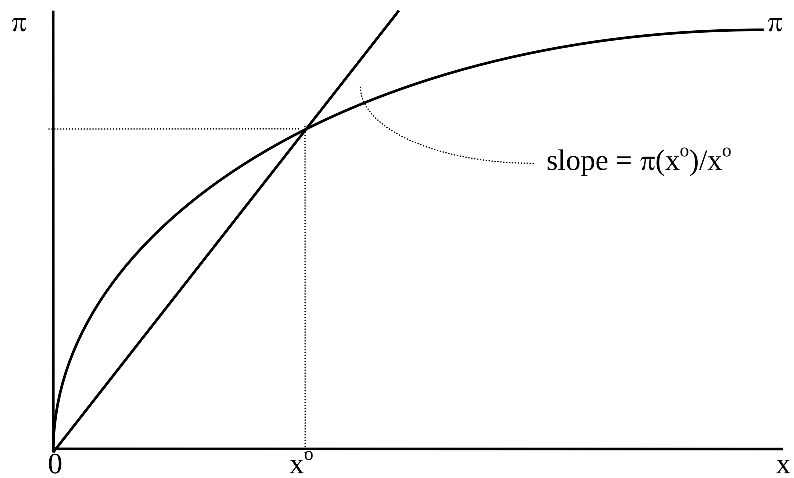


Figure 4

Example. Let $C = C(x)$ denote a firm's cost function and let x be the firm's level of output. In many applications, we expect the cost function to have the shape depicted in Figure 5. In the lower diagram, we show the average function $C(x)/x$, termed average cost. Average cost has two regions. In the initial region, average cost is declining as output is increased, and, in the second region, the opposite is true. The initial region is said to exhibit economies of scale (falling average cost), and the second exhibits diseconomies of scale (rising average cost). As you can see, average cost has a unique minimum point at the level of output x^o. This point separates the two regions. As an exercise, you might draw a diagram of the marginal cost function $C'(x)$ and compare this function to the average function.

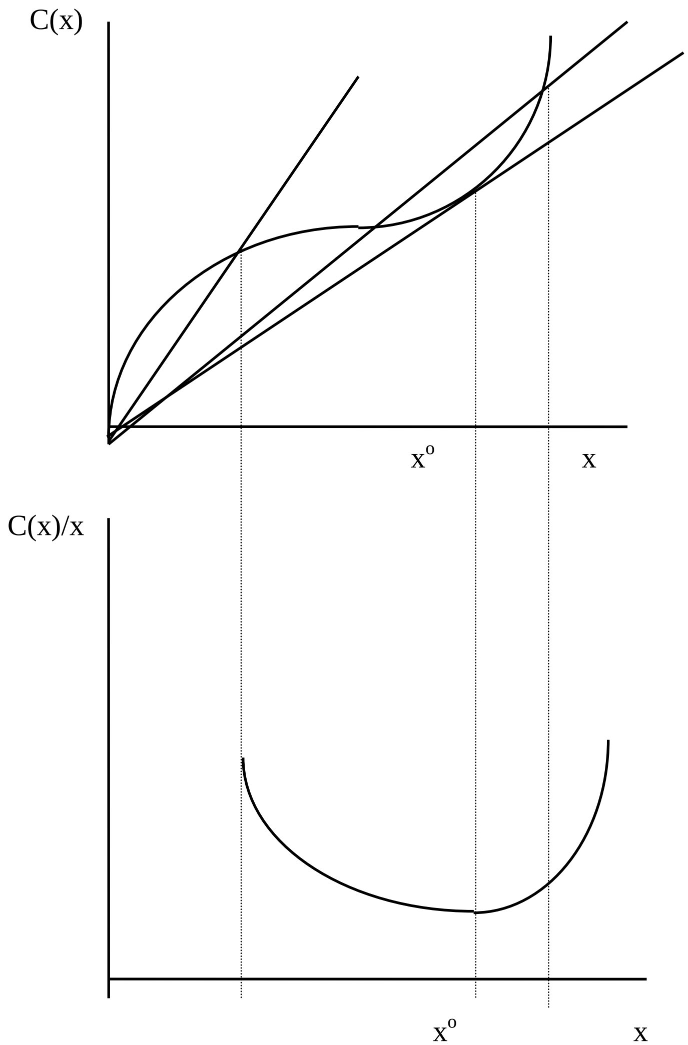


Figure 5

The average function can be defined for objective functions of many choice variables by focusing on one variable at a time and employing the same definition as above. If the objective function is given by $\pi(x_1,...,x_n)$, then the average function for the i^{th} variable is π/x_i.

2.4. Elasticity

The notion of elasticity is used in many business applications where the objective is to gauge the responsiveness of one variable to a change in another variable. For example it is of interest for a firm to know how a rival's price change might impact the quantity demanded of their product. Alternatively, the same firm might want to quantify the impact on quantity demanded of their product of a change in the price of their product or a change in advertising outlay for that product. Elasticity measures the impact of such changes by taking the percentage change in a dependent variable induced by some percentage change in an independent variable. We measure a percentage change by computing the change in a variable and dividing by the level of that variable.

Given a function $\pi(x)$, the *elasticity of π with respect to x* is given by

$$(4) \qquad \frac{\Delta\pi / \pi}{\Delta x / x} = \frac{\Delta\pi / \Delta x}{\pi / x} = \text{(marginal function) / (average function)}.$$

If one knows the function π through say empirical estimation, then the marginal function can be used for the difference quotient $\Delta\pi/\Delta x$. In this case, the elasticity is called a *point elasticity*. In some cases, the function π is not known and only observations of x and π are available.

If we have at least two observations of (x,π), then we can compute a different notion of elasticity called the *arc elasticity*. Let (x',π') and (x'',π'') represent the two observations. For this case, we compute $\Delta\pi/\Delta x$ as $(\pi''-\pi')/(x''-x')$ and the basis levels for the percentage computation are taken as the average observations: $x^a = (x'+x'')/2$ and $(\pi'+\pi'')/2$. The arc elasticity formula becomes

$$(5) \qquad \frac{(\pi'' - \pi')}{(x''-x')}\frac{x^a}{\pi^a}.$$

Both of the above notions of elasticity are extended to the case of n strategy variables, by holding the other variables fixed, when a single variable's elasticity is computed.

Example 1. Let the function Q = 12 - 4p = Q(p) describe a demand relationship, where p is price and Q is quantity. Given that this function is linear, we have that $Q'(p) = \Delta Q/\Delta p = -4$. The point elasticity at the price level p = \$1 is given by

$$\frac{Q'(p)p}{Q} = \frac{-4(1)}{12-4(1)} = \frac{-4}{8} = \frac{-1}{2}.$$

This is interpreted as follows: If price were to rise by one percent in a neighborhood of p = \$1, then quantity demanded would drop by one half of one percent. It is interesting to note that the elasticity of demand for gasoline in the United States is equal to -1/2.

Example 2. We observe that when p = \$2, Q = 10. Further when p = \$4, we have that Q = 6. Compute the arc elasticity of demand for these observations.

$$\frac{\Delta Q / \Delta p}{Q^a / p^a} = \frac{[(6-10)/(4-2)]}{8/3} = \frac{-2(3)}{8} = \frac{-3}{4}.$$

2.5. Iso-Surfaces and Constrained Optimization

In many business problems, we are confronted with a feasibility constraint which limits our ability to choose values of our strategy variables. As an example, consider the problem of maximizing output flow given a limited budget to purchase the inputs used to produce output. Alternatively, a manager may be asked to achieve an output target with a cost minimal choice of inputs. We would like to develop a simple methodology for studying this type of problem. To keep the analysis simple, we will assume that the firm has just two strategy variables, x_1, x_2. We assume that the two strategy variables generate an output variable q which the firm sells. The relationship between the two variables and output is given by the multi-variable function

(6) $q = f(x_1,x_2)$.

For a fixed or target output level of q there are generally many combinations of x_1 and x_2 capable of generating that target. *The set of pairs (x_1,x_2) capable of generating a given level of q is called an iso-quant.*[1] An iso-quant is then given by

$$\{(x_1,x_2) \mid q = f(x_1,x_2) \text{ and } q \text{ fixed}\}.$$

In the economics of production, the x_i's might represent material and labor inputs and q would be output flow. The iso-quant would be the various input mixes of material and labor inputs which could generate a given output flow. Figure 6 gives an example of such an iso-quant. Here, x_1 is interpreted as labor and x_2 is material or capital.

[1] This is also called a *level surface* of the function $f(x_1,x_2)$.

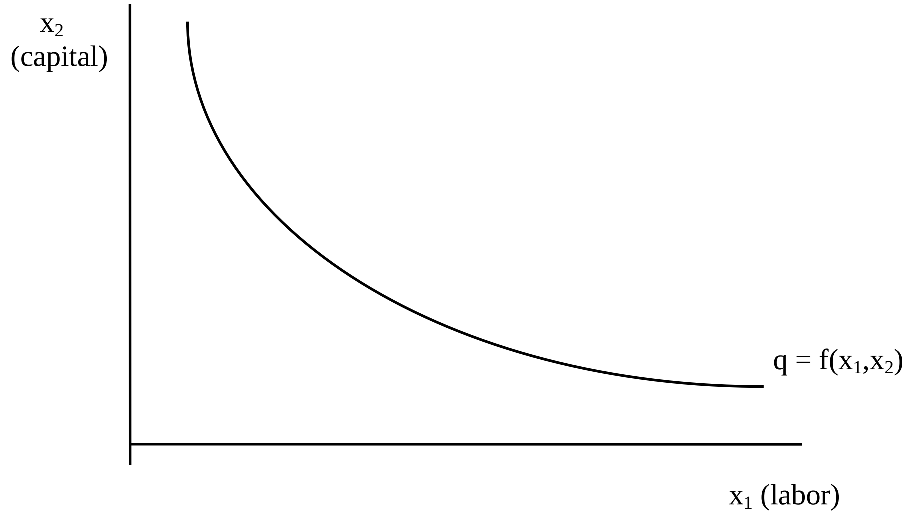


Figure 6

The iso-quant map, i.e., the set of all iso-quants, can be used to describe the entire 3-dimensional function.

Along the iso-quant, the firm is able to substitute one input for another and still achieve the same output target. That is, there is a trade-off in the sense that if we add a unit of x_1 to the production process, then we can subtract a certain number of units of x_2 from production and still achieve the given target output flow. The rate at which one input can be substituted for another along an iso-quant is called *the marginal rate of substitution between x_1 and x_2*. This notion can be intuitively thought of as the number of units of x_2 that the firm can eliminate from the production process if it adds one more unit of x_1, holding q constant. Formally, we define the marginal rate of substitution, MRS, as

$$MRS \equiv \lim_{\Delta x_1 \to 0} -(\Delta x_2/\Delta x_1)|_{q=\text{constant}}.$$

This is illustrated in Figure 7 below. The MRS represents the absolute value of the slope of the iso-quant at a point.

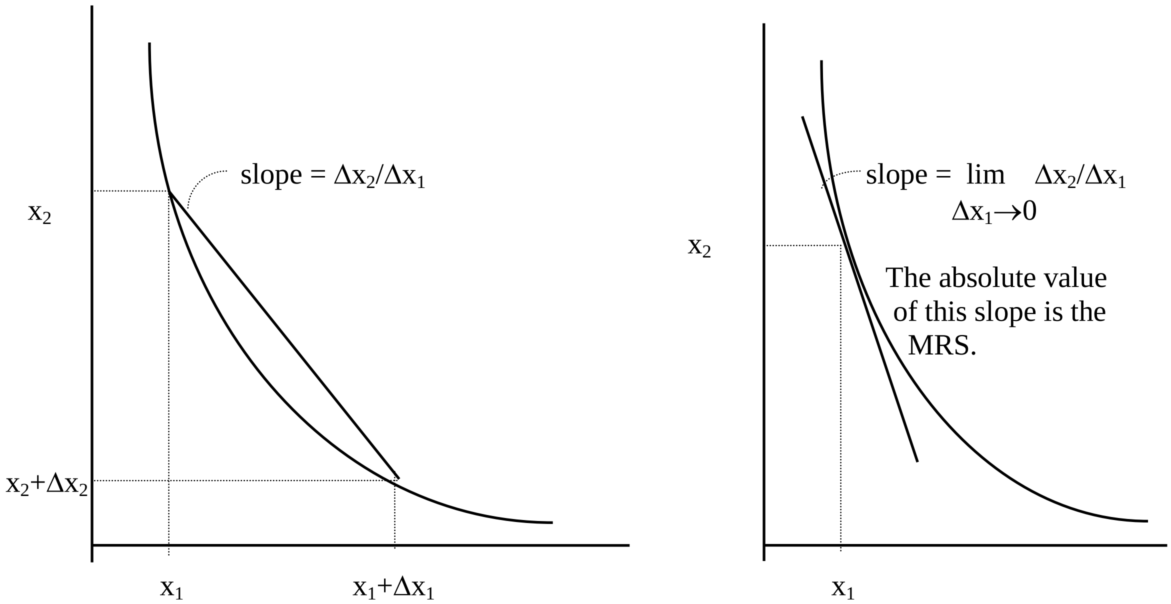


Figure 7

Note that along an isoquant, $q = f(x_1,x_2)$ and q is constant. Thus, along an isoquant,

$$\Delta q = 0 = \frac{\Delta q}{\Delta x_1}\Big|_{x_2 = \text{constant}} \Delta x_1 \quad + \quad \frac{\Delta q}{\Delta x_2}\Big|_{x_1 = \text{constant}} \Delta x_2,$$

and

$$-(\Delta x_2/\Delta x_1)|_{q=\text{constant}} \;=\; (\Delta q/\Delta x_1)/(\Delta q/\Delta x_2) \;=\; MP_1/MP_2.$$

That is, $MRS = MP_1/MP_2$.

Now suppose that a manager is asked to provide for the firm a given output target at a minimum expenditure level. Suppose that the firm must pay fixed unit prices for the two inputs, and let these prices be denoted as p_1 and p_2. The total expenditure on all inputs is given by the simple linear function

$$C = p_1x_1 + p_2x_2,$$

and the firm's output target is given by q^t. The firm's formal problem is to choose x_i so as to minimize C subject to the constraint that q^t be produced. The constraint is then that

$$q^t = f(x_1,x_2).$$

We would write this problem as

$$\min_{\{x_1,x_2\}} (p_1x_1 + p_2x_2) \text{ subject to } q^t = f(x_1,x_2).$$

The firm's expenditure function can be rewritten as the linear function

$$x_2 = \frac{C}{p_2} - \frac{p_1}{p_2} x_1.$$

This is a linear function with intercept C/p_2 and slope coefficient $-p_1/p_2$. The slope coefficient,

$-p_1/p_2$,

represents the MRS between x_1 and x_2 along a given expenditure line. It measures the number of units of x_2 the firm *must give up* for another unit of x_1 purchased, assuming that expenditure is at a given level. As can be seen in Figure 8, as the firm lowers its expenditure, the expenditure line shifts in towards the origin in a parallel fashion.

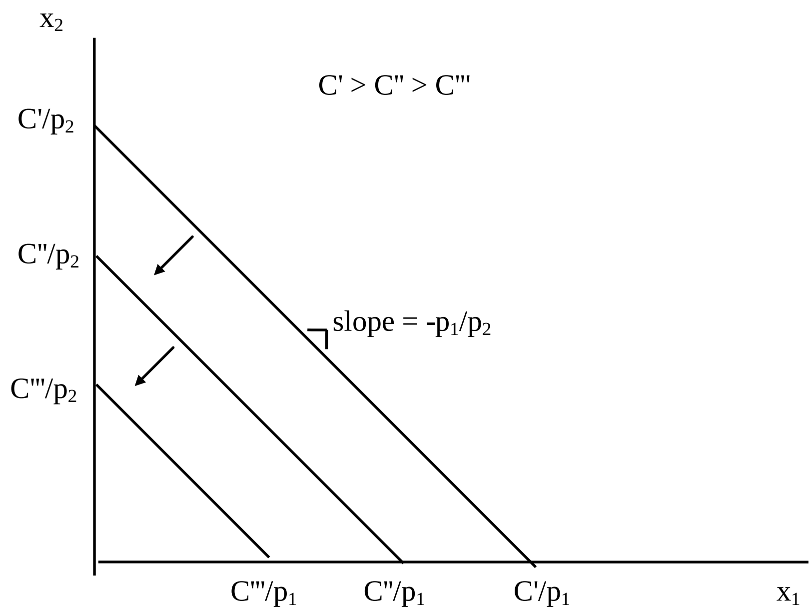


Figure 8

The solution to the firm's minimization problem can be shown graphically. The iso-quant in Figure 9 represents the feasible set of (x_1,x_2), and the set of parallel expenditure lines represents alternative levels of expenditure with lower levels of expenditure being represented by those lines closer to the origin. At a minimum of expenditure where both inputs would be utilized, we have that the slope of the iso-quant is equal to the slope of the expenditure line. At a minimum, the rate at which x_1 can be substituted for x_2 in production (the MRS) is equated to the rate at which x_1 must be substituted for x_2 in the market place, (p_1/p_2). Thus, in the expenditure minimizing equilibrium, we have that the following condition is met

$$MRS = \frac{p_1}{p_2} \text{ or that } \frac{MP_1}{MP_2} = \frac{p_1}{p_2} \text{ and } \frac{MP_1}{p_1} = \frac{MP_2}{p_2}.$$

The final version of the optimality condition says that the firm should equate the marginal benefit to marginal cost ratio of each input.

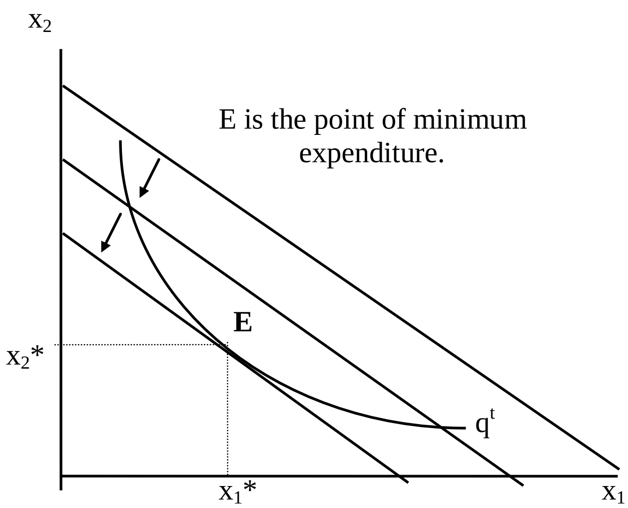


Figure 9

Practice Problems

1. Show that, for a linear function of the form y = a + bx, the marginal function and the difference quotient Δy/Δx are given by b, for all x.

2. Consider the linear function of 1, where x > 0 and a,b > 0. Write the equation for the average function. Graph the marginal and the average functions in the same diagram, with x on the horizontal axis. Write the equation for the point elasticity function.

3. Let the function R = R(Q) represent the amount of revenue a firm takes in as a function of the number of units it sells to consumers. Answer each of the following questions.

a. Suppose that the firm's forecasting group has estimated the above function to be given by

$$R = 100{,}000 + \frac{Q}{2}, \text{ for } 50{,}000 < Q < 100{,}000.$$

Determine the equations for the marginal revenue function and the average revenue function. Sketch these schedules in the same diagram with Q on the horizontal axis and marginal and average revenue on the vertical axis.

b. If the firm is currently selling 50,000 units, what is the arc revenue elasticity of increasing output to 55,000 units?

c. If the firm estimates that its marginal cost is

$$\frac{Q}{150{,}000}, \text{ for } 50{,}000 < Q < 100{,}000,$$

then is there an output in this interval that would provide maximum profit? Explain your answer and use the marginal principle to determine your results.

4. Consider the quadratic revenue function $R = 10q - q^2$, where q is output and R is revenue. For this revenue function, the marginal revenue function is given by 10 - 2q. What output level maximizes revenue? Explain you answer in detail, using a graph to motivate your discussion.

5. Suppose a firm has a production function given by $q = f(x_1,x_2)$. The division manager in change of this production process is asked to maximize output flow, but is given a budget constraint to purchase the two inputs x_1 and x_2. The manager is given a total amount C to spend and must pay prices r_i for each of the inputs.

a. Write the constraint for the basic optimization problem, with x_2 as the dependent variable. What is the slope of this constraint?

b. Use a graphical approach to characterize the constrained optimum. What marginal condition is satisfied at a maximum? How does this problem compare to the one presented in the text?

6. Suppose that the production function takes on the specific form

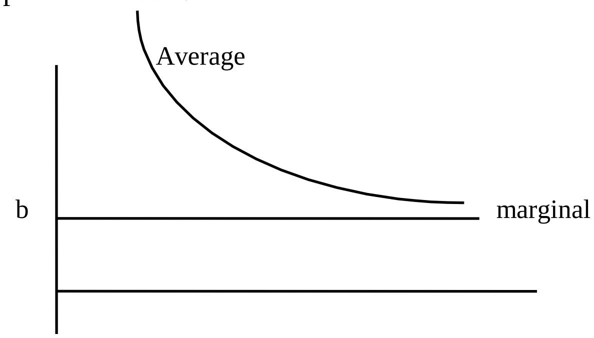
$q = 2LK + 3L.$

If the prices of K and L are each \$1, $MP_L = 3 + 2K$, and $MP_K = 2L$, can you write the equation which would determine the cost minimal amounts of labor and capital to produce 32 units of output? As an optional exercise compute these amounts.

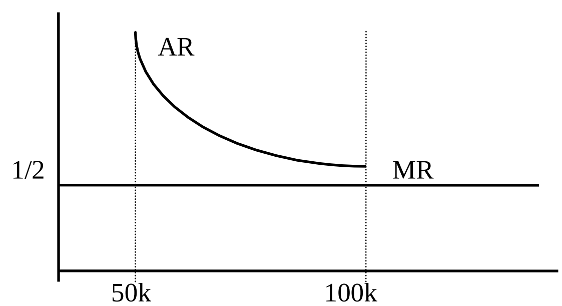
Answers to Practice Problems

1. $\Delta y = a + bx'' - a - bx' = b(x'' - x')$. Thus $\Delta y/\Delta x = b$.

2. The average function is given by $a/x + b$. The point elasticity function is given by $b/(a/x + b)$. the graphs are follows:



3. a. MR = ½ and AR = 100,000/Q + ½.



b. Elasticity = (2500/5000).(52,500/126,250) = .21.

c. Set marg rev = marg cost. Thus,

MR = ½ = Q/150,000 = MC.
Solving for Q, Q = 75,000.

4. R is maximized where marginal revenue is zero. Thus, $10 – 2q = 0$ implies that $q = 5$.

5. See notes above for a and b.

6. The relevant equilibrium conditions for cost minimization subject to an output constraint are that

$$MP_L/MP_K = 1/1 \text{ and } 32 = 2LK + 3L.$$

That is, the ratio of the marginal products should equal the input price ratio and the target output should be achieved. In this case, we have

$(3 + 2K) = 2L$ and $32 = 2LK + 3L$.

Multiply the first equation by L and we have

$$3L + 2LK = 2L^2.$$

Substituting $3L + 2LK = 32$ and solving for L, we obtain $L^2 = 16$ and $L = 4$. From the output constraint, it follows that $K = 2.5$.