Some basic rules of differentiation

- **R1(Constant Function Rule)** The derivative of the function $f(x) = c$ is zero
- **R2 (Power function rule)** The derivative of the function x^N is Nx^{N-1}
- **R3 (Multiplicative Constant Rule)** The derivative of $y = kf(x)$ is $kf'(x)$

1

Basic Rules

- Examples of R1-R3
  #1. If $y = 4$, then $dy/dx = 0$.
  #2. If $y = 3x^2$, then $dy/dx = 6x$.
- **R4 (Sum-difference Rule)** $g(x) = \Sigma_i f_i(x) \Rightarrow g'(x) = \Sigma_i f_i'(x)$.
- Example of R4
  If $y = x^2 - 3x^3 + 7$, then $dy/dx = 2x - 9x^2$.

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Basic Rules

- **R5 (Product Rule)** h(x) = f(x)g(x) ⇒ h'(x) = f'(x)g(x) + f(x)g'(x)

Example of R5

If y = (3x+4)(x^2 - 4x^3), then dy/dx = 3(x^2 - 4x^3) + (3x+4)(2x - 12x^2)

- **R6 (Quotient Rule)** h(x) = f(x)/g(x) ⇒ h'(x) = [f'(x)g(x) – g'(x)f(x)]/[g(x)]^2

Example of R6

If y = (2x - 4)/(x^4 + 3x), then dy/dx = [2(x^4 + 3x) - (4x^3 +3)(2x - 4)]/ (x^4 + 3x)^2

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Basic Rules

- **R7 (Chain Rule)** If z = f(y) is a differentiable function of y and y = g(x) is a differentiable function of x, then the composite function f ο g or h(x) = f[g(x)] is a differentiable function of x and

$$h'(x) = f'[g(x)] \cdot g'(x) = \frac{df}{dg}\frac{dg}{dx}$$

Example of R7

If h(x) = (g(x) + 3x)^2, then h'(x) = 2(g(x) +3x)(g'(x) + 3)

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Basic Rules

- Functions which are one-to-one can be inverted. If y = f(x), where f is one-to-one, then it is possible to solve for x as a function of y.
- We write this as $x = f^{-1}(y)$.
- For example if y = a + bx, then the inverse function is x = -a/b + (1/b)y.

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Basic Rules

- **R8 (Inverse Function Rule)** Given y = f(x) and $x = f^{-1}(y)$, we have $f^{-1'}(y) = 1/f'(x)$.

Examples of R8

If y = a + bx, then dy/dx = b and $f^{-1'}(y) = 1/b$.
If $y = x^2$, x > 0, then dy/dx = 2x and $f^{-1}(y) = (y)^{1/2}$. In this case $f^{-1'}(y) = 1/2x = (1/2)(y)^{-1/2}$.

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Basic Rules

- **R9 (Exponential Function Rule)** Let y = $e^{f(x)}$, then dy/dx = $f'(x)e^{f(x)}$.

Examples of R9

If y = e^{3x}, then dy/dx = 3 e^{3x}. If y = e^x, then dy/dx = e^x.

- **R10 (Log function Rule)** Let y = ln f(x), then dy/dx = f'(x)/f(x).

Examples of R10

If y = ln (2x+3), then dy/dx = 2/(2x+3). If y = ln x, then dy/dx = 1/x.

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Higher Order Derivatives: The Second Derivative

- If a function is differentiable, then its derivative function may itself be differentiable. If so, then the derivative of the derivative is called the second derivative of the function.
- The sign of the second derivative tells us about the curvature (concavity versus convexity) of the function.
- The second derivative is written as d^2y/dx^2 or f''(x).

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Computation

- We merely differentiate the derivative function.
- For example,

If $y = ax^2 + bx$, then $f'(x) = 2ax + b$ and $f''(x) = 2a$.

If $y = x^3$ then $f' = 3x^2$ and $f'' = 6x$

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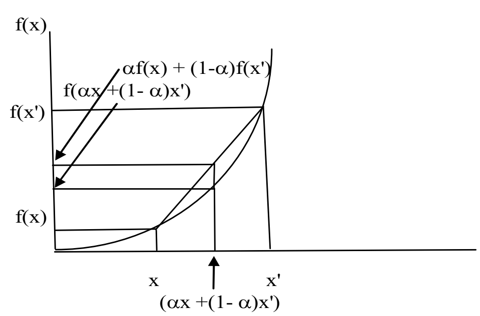
Convex or concave function

- A function f(x) is strictly convex (concave) if for all x, x', $f(\alpha x + (1-\alpha)x') < (>) \alpha f(x) + (1-\alpha)f(x')$, for $\alpha \in (0,1)$.



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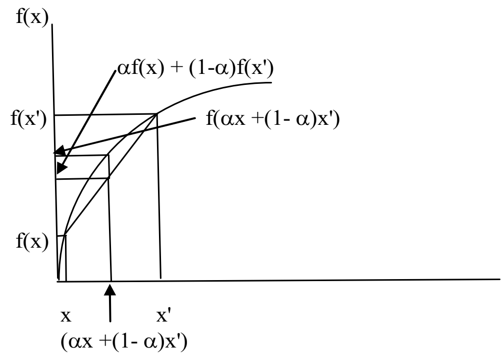
Concave Case



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Derivative Condition:Concavity or Convexity

- By visual inspection, a differentiable (strict) convex function has an increasing first derivative function and a (strict) concave function has a decreasing first derivative function.
- Thus, it is true that if the second derivative is positive, then the first derivative is increasing and the function is strictly convex.
- A negative second derivative is sufficient for strict concavity.

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Second derivative test

Examples.

The function $y = x^2$ ($x > 0$) is strictly convex and we have that $d^2y/dx^2 = 2 > 0$.

The function $y = x^{1/2}$ ($x > 0$) is strictly concave. We have that $d^2y/dx^2 = (-1/4)x^{-3/2} < 0$.

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Partial Derivatives

- Consider a function of n independent variables. It would be of the form

$$y = f(x_1, \ldots, x_n),\ f: R^n \to R.$$

Def. The *partial derivative* of the function $f(x_1, x_2, \ldots, x_n)$, $f: R^n \to R$, at a point $(x_1^o, x_2^o, \ldots, x_n^o)$ *with respect to x_i* is given by

$$\lim_{\Delta x_i \to 0} \Delta y / \Delta x_i = \frac{f(x_1^o, \ldots, x_i^0 + \Delta x_i, \ldots, x_N^0) - f(x_1^0, \ldots, x_N^0)}{\Delta x_i}$$

The notation for a partial derivative is $f_i(x_1, \ldots, x_n)$ or $\partial f / \partial x_i$.

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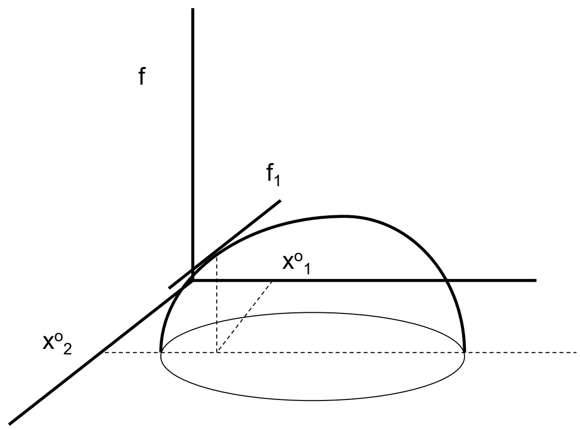
Illustration of f_1



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Mechanics of Computation

- When differentiating with respect to x_i, regard all other independent variables as constants
- Use the simple rules of differentiation for x_i.

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Examples

a. If $y = f(x_1,x_2) = x_1^3 x_2^2$, then $f_1 = 3x_1^2 x_2^2$, $f_2 = x_1^3 2x_2^1$.

b. If $y = f(x_1,x_2) = 2x_1 + x_1x_2$, then $f_1 = 2 + x_2$ and $f_2 = x_1$.

c. If $y = f(x_1,x_2) = x_1g(x_2)$, then $f_1 = g(x_2)$ and $f_2 = x_1g'(x_2)$.

d. If $y = f(x_1,x_2) = \ln (x_1 + 4x_2)$, then $f_1 = 1/(x_1 + 4x_2)$ and $f_2 = 4/(x_1 + 4x_2)$.

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Integration: Indefinite Integrals

- Here we are concerned with the inverse of the operation of differentiation. That is, the operation of searching for functions whose derivatives are a given function.
- Consider any arbitrary real-valued function defined on a subset X of the real line. By the antiderivative we mean any differentiable function F whose derivative is the given function f.

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Integration

- Hence, dF/dx = F′(x) = f(x).
- Clearly if F is an antiderivative of f then so is F + c, where c is a constant. F + c then represents the set of all antiderivative functions of f and this is called the indefinite integral of f.
- The indefinite integral is denoted as

$$\int f(x)\,dx = F(x) + c.$$

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Basic Rules

R1 (Power Function Rule) $\int x^N\,dx = \frac{1}{N+1}x^{N+1} + c.$

R2 (Multiplicative Constant Rule) $\int cf(x)\,dx = c\int f(x)\,dx.$

R3 (Sum Rule) $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx.$

R4 (Exponential Function Rule) $\int e^x\,dx = e^x + c.$

R5 (Logarithmic Function Rule) $\int \frac{1}{x}\,dx = \ln|x| + c.$

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Examples

#1 Let $f(x) = x^4$, then

$$\int x^4 dx = \frac{x^5}{5} + c.$$

check:

$$\frac{d(x^5/5)}{dx} = x^4.$$

#2 Let $f(x) = x^3 + 5x^4$, then

$$\int [x^3 + 5x^4]\, dx = \int x^3\, dx + 5\int x^4\, dx$$

$$= x^4/4 + 5(x^5/5) + c$$

$$= x^4/4 + x^5 + c$$

check.

$$\frac{d}{dx}(x^4/4 + x^5) = x^3 + 5x^4.$$

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Examples

#3 Let $f(x) = x^2 - 2x$

$$\int [x^2 - 2x]\, dx = \int x^2\, dx + \int -2x\, dx$$

$$= x^3/3 - 2\int x\, dx + c'$$

$$= x^3/3 - 2(x^2/2) + c$$

$$= x^3/3 - x^2 + c.$$

#4 Let $f(x) = e^{2x^1}$, then $\int e^{2x^1} dx = \frac{1}{2} e^{2x^1} + c.$

#5 Let $f(x) = \frac{2}{x}$, then $\int \frac{2}{x} dx = 2lnx + c.$

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Antiderivatives and Definite Integrals

- Let f(x) be continuous on an interval $X \subset R$, where $f: X \rightarrow R$. Let F(x) be an antiderivative of f, then $\int f(x)\,dx = F(x) + c$.
- Now choose $a, b \in X$ such that $a < b$. Form the difference

  $[F(b) + c] - [F(a) + c] = F(b) - F(a)$.
- $F(b) - F(a)$ is called the *definite integral of f from a to b*. The point a is termed the *lower limit of integration* and the point b, the *upper limit of integration.*

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Definite Integrals

- Notation: We would write

$$\int_a^b f(x)dx \equiv F(x)\Big]_a^b \equiv F(x)\Big|_a^b = F(b) - F(a)$$

- Examples

#1 Let $f(x) = x^3$, find

$$\int_0^1 x^3 dx = \frac{1}{4}x^4\Big|_0^1 = \frac{1}{4}(1)^4 - \frac{1}{4}(0)^4$$

$$= \frac{1}{4}.$$

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Examples

#2 f(x) = $2x\,e^{x^2}$, find $\int_3^5 f(x)dx$,

$$\int_3^5 2xe^{x^2} = e^{x^2}\Big|_3^5 = e^{25} - e^{9}$$

#3 f(x) = $2x + x^3$, find $\int_0^1 f(x)dx = (x^2 + \frac{1}{4}x^4)|_0^1 = \frac{5}{4}$

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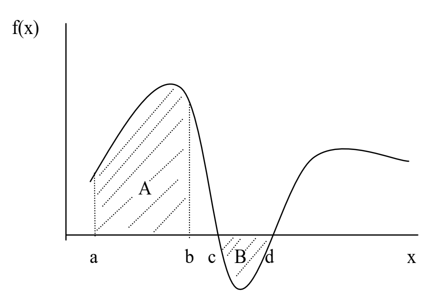
Illustration

The absolute value of the definite integral represents the area between f(x) and the x-axis between the points a and b.



A = $\int_a^b f(x)dx$ and the area B = $(-1)\int_c^d f(x)dx$.

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Differentiation of an Integral

- The following rule applies to the differentiation of an integral.

$$\frac{\partial}{\partial y}\int_{p(y)}^{q(y)} f(x,y)dx = \int_{p}^{q} f_y(x,y)dx + f(q,y)q'(y) - f(p,y)p'(y).$$

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Example

In Economics, we study a consumer's demand function in inverse form

p = p(Q),

where Q is quantity demanded and p denotes the maximum uniform price that the consumer is willing to pay for a given quantity level Q. We assume that p' is negative.

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Example

- The definite integral

$$\int_0^Q p(z)dz = TV(Q)$$

is called total value at Q. It gives us the maximum revenue that could be extracted from the consumer for Q units of the product.

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Example

- If a firm could extract maximum revenue from the consumer, its profit function would be

$$\int_0^Q p(z)dz - C(Q).$$

- Maximizing profit over Q choice implies

$p(Q) = C'(Q)$.

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Multiple Integrals

- In this section, we will consider the integration of functions of more than one independent variable.
- The technique is analogous to that of partial differentiation. When performing integration with respect to one variable, other variables are treated as constants.

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$$\int_c^d \int_a^b f(x,y)dxdy.$$

- We read the integral operators from the inside out. The bounds a,b refer to those on x, while the bounds c,d refer to y. Likewise, dx appears first and dy appears second.
- The integral is computed in two steps

#1. Compute $\int_a^b f(x,y)dx = g(y).$

#2. Compute $\int_c^d g(y)dy = \int_c^d \int_a^b f(x,y)dxdy$.

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Example

Example 1: Suppose that z = f(x,y). We wish to compute integrals of the form

$$\int_c^d \int_a^b f(x,y)dxdy.$$

Consider the example $f = x^2y$, where c = a = 0 and d = 2, b = 1. We have

$$\int_0^2 \int_0^1 x^2 ydxdy.$$

Begin by integrating with respect to x, treating y as a constant

$$\int_0^1 yx^2 dx = \frac{1}{3} yx^3 |_0^1 = \frac{1}{3} y.$$

Next, we integrate the latter expression with respect to y.

$$\int_0^2 \frac{1}{3} ydy = \frac{1}{3}\frac{1}{2} y^2 |_0^2 = \frac{1}{6} 4 = \frac{2}{3}.$$

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